Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 2, 2017, 565–575
DOI: 10.12775/TMNA.2016.090

UNIQUENESS OF POSITIVE AND COMPACTON-TYPE SOLUTIONS FOR A RESONANT QUASILINEAR PROBLEM

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Abstract. We study a one-dimensional p-Laplacian resonant problem with p-sublinear terms and depending on a positive parameter. By using quadrature methods we provide the exact number of positive solutions with respect to $\mu \in]0,+\infty[$. Specifically, we prove the existence of a critical value $\mu_1 > 0$ such that the problem under examination admits: no positive solutions and a continuum of nonnegative solutions compactly supported in $[0,1]$ for $\mu \in]0,\mu_1[$; a unique positive solution of compacton-type for $\mu = \mu_1$; a unique positive solution satisfying Hopf's boundary condition for $\mu \in]\mu_1,+\infty[$.

1. Introduction

We are concerned with the existence of positive solutions to the following two-point boundary value problem:

$$\text{(P}_\mu\text{)}\qquad \begin{cases} -(|u'|^{p-2}u')' = \lambda_p u^{p-1} - \mu u^{r-1} + u^{s-1} & \text{in }]0,1[,\\ u(0)=u(1)=0, \end{cases}$$

where $r,s,p \in]1,+\infty[$ with $r<s<p$, $\mu \in]0,+\infty[$ and

$$\lambda_p = (p-1)(2\pi)^p\biggl(p\sin\frac{\pi}{p}\biggr)^{-p}$$

2010 *Mathematics Subject Classification.* 34B08, 34B15, 34B18.
Key words and phrases. Quasilinear problem; resonant problem; positive solution; compacton-type solution; uniqueness.

is the first eigenvalue of the one-dimensional p-Laplacian with Dirichlet boundary conditions, i.e. the least eigenvalue of the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda u^{p-1} & \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

By a positive solution we mean any classical solution v to (P_μ) satisfying $v(0) = v(1) = 0$ and $v > 0$ in $]0,1[$. Among these solutions we are further interested in the so-called compactons, characterized by satisfying both Dirichlet and Neumann conditions at the endpoints of the interval.

Our approach relies on quadrature methods and was successfully adopted in the recent paper [1] to deal with a problem similar to (P_μ) with p-superlinear and p-sublinear terms. In that case the competition between the opposite trends of the nonlinearity resulted in the existence of (at least) three positive solutions and two distinct curves of compactons (Theorems 2.7 and 2.9 of [1], respectively). Here the situation is more delicate as the interaction occurs between a p-linear resonant term and two p-sublinear ones. The interesting fact we found out is that the presence of the resonance entails the uniqueness of positive solutions. Indeed one has the following exact description of the set of solutions of (P_μ) with respect to $\mu \in]0,+\infty[$ (see Theorem 2.3): there exists $\mu_1 > 0$ such that (P_μ) admits no positive solution for $\mu \in]0,\mu_1[$; a unique positive non-compacton solution for $\mu \in]\mu_1,+\infty[$ and a unique compacton for $\mu = \mu_1$. This represents the central result of the paper and is proved in the following section together with two crucial preliminary lemmas.

Before all, let us establish some notations and auxiliary results useful in the sequel. From now on we denote by F the primitive of the nonlinearity in (P_μ) vanishing at 0, i.e.

$$F(t) \overset{\text{def}}{=} \frac{\lambda_p}{p}\, t^p - \frac{\mu}{r}\, t^r + \frac{1}{s}\, t^s \quad \text{for all } t \in [0,+\infty[.$$

For all $i \in \mathbb{N} \cup \{0\}$, $F^{(i)}$ stands for the i-th derivative of F where, as customarily, $F^{(0)} \overset{\text{def}}{=} F$. Invoking Lemmas 2.1 and 2.2 of [1], one can easily prove the following properties.

Lemma 1.1. *There exist* $t_0, t_1, t_2 \in]0,+\infty[$, *with* $t_0 > t_1 > t_2$, *such that, for each* $i = 0,1,2$, *one has*

$$\begin{aligned} F^{(i)}(t) &< 0 \quad \text{if } t \in]0,t_i[, \\ F^{(i)}(t) &> 0 \quad \text{if } t \in]t_i,+\infty[. \end{aligned} \tag{1.1}$$

Lemma 1.2. *Let* t_0 *be as in Lemma* 1.1. *Then* $t_0 = t_0(\mu)$ *has the following properties*:

(a) t_0 *is increasing in* $]0,+\infty[$;