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ON A CLASS OF QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTIAL GROWTH ON THE WHOLE SPACE

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Abstract. In this paper we prove a kind of weighted Trudinger–Moser inequality which is employed to establish sufficient conditions for the existence of solutions to a large class of quasilinear elliptic differential equations with critical exponential growth. The class of operators considered includes, as particular cases, the Laplace, p-Laplace and k-Hessian operators when acting on radially symmetric functions.

1. Introduction

In this paper we deal with a general class of quasilinear operators in radial form which includes perturbations of p-Laplace and k-Hessian operators. Let us first consider the following p-Laplace equation:

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \quad u_{|\partial\Omega} = 0. \tag{1.1}$$

In the seminal work [20], Gidas, Ni and Nirenberg proved that all positive solutions $u \in C^2$ of the above problem are necessarily radially symmetric when $p = 2$, $f \in C^1$ and $\Omega = B_R$ is the open ball with center 0 and radius $R > 0$ in $\mathbb{R}^N$, $N \geq 2$. Also, in [21], they proved symmetry of solutions when $\Omega = \mathbb{R}^N$, $N \geq 3$, is the whole space. This kind of results for $p \neq 2$ was established by

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Felmer et al. in [19] and Damascelli et al. in [7], [8]. In view of this, if $\Omega = B_R$, for a wide class of nonlinearities f we can reduce problem (1.1) to the following:

$$r^{1-N}(r^{N-1}|u'|^{p-2}u')' + f(u) = 0 \quad \text{in } (0,R), \quad u'(0) = u(R) = 0. \tag{1.2}$$

Another interesting problem investigated in this paper concerns the k-Hessian equation

$$S_k(D^2u) + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \quad u_{|\partial\Omega} = 0, \tag{1.3}$$

where $1 \leq k \leq N$ and $S_k(D^2u)$ is the the sum of all principal $k \times k$ minors of the Hessian matrix D^2u, see [28]. For instance, $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det(D^2u)$ is the Monge–Ampère operator. As noted in [22], when $\Omega = B_R$ is an open ball in $\mathbb{R}^N$ and f satisfies suitable conditions, the Alexandrov–Serrin moving plane method [27] used in [20] extends to (1.3) (see [11] for the Monge–Ampère case) reducing it to following equation:

$$r^{1-N}(r^{N-k}|u'|^{k-1}u')' + f(u) = 0 \quad \text{in } (0,R), \quad u'(0) = u(R) = 0. \tag{1.4}$$

Therefore, under the previous discussion, for a wide class of functions f all of the above problems are special cases of a more general family of problems

$$\begin{cases} r^{-\theta}(r^{\alpha}|u'|^{p-2}u')' + f(r,u) = 0 & \text{for } r \in (0,R), \\ u > 0 & \text{for } r \in (0,R), \\ u'(0) = u(R) = 0, \end{cases} \tag{1.5}$$

where certain conditions are to be imposed on the parameters α, p and θ. In recent years, several authors [5], [10], [17], [23], [24] have studied this class of problems under different conditions on parameters α, p and θ and on the nonlinearity f. In [5], de Figueiredo et al. introduced suitable function spaces to study problem (1.5) variationally. In particular, a critical exponent was found which allows to treat the Brezis–Nirenberg type problem [4]. More recently, in [17] the existence of non-trivial solution was established when f has critical exponential growth that represents the counterpart to [5].

All foregoing results on problem (1.5) were established for the bounded case $R < \infty$. The main goal of this article is to study the class of problems (1.5) for critical exponential growth on the whole space, that is, $R = +\infty$. In order to formulate our results, let us present the framework for the function space setting suitable to study these problems. Let $X^{1,p}_R(\alpha,\theta)$, or more simply X_R, be the weighted Sobolev spaces defined as follows: For $0 < R \leq \infty$ and $\theta \geq 0$, let $L^q_\theta = L^q_\theta(0,R)$ be the weighted Lebesgue space defined as the set of all measurable