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ON DECAY AND BLOW-UP OF SOLUTIONS FOR A SINGULAR NONLOCAL VISCOELASTIC PROBLEM WITH A NONLINEAR SOURCE TERM

Wenjun Liu — Yun Sun — Gang Li

Abstract. We consider a singular nonlocal viscoelastic problem with a nonlinear source term and a possible damping term. We prove that if the initial data enter into the stable set, the solution exists globally and decays to zero with a more general rate, and if the initial data enter into the unstable set, the solution with nonpositive initial energy as well as positive initial energy blows up in finite time. These are achieved by using the potential well theory, the modified convexity method and the perturbed energy method.

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1. Introduction

In this paper, we investigate the following one-dimensional viscoelastic problem with a nonlocal boundary condition:

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g(t-s)\,\dfrac{1}{x}(xu_x(x,s))_x\,ds + au_t = |u|^{p-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in (0,\ell),\ t \in (0,\infty), \\ u(\ell,t) = 0, \quad \displaystyle\int_0^\ell xu(x,t)\,dx = 0 \qquad \text{for } t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in [0,\ell], \end{cases} \tag{1.1}$$

where $a \geq 0$, $\ell < \infty$, $p > 2$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$.

This type of evolution problems, with nonlocal constraints, are generally encountered in heart transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. The nonlocal boundary conditions arise mainly when the data on the boundary cannot be measured directly, but their average values are known. We can refer to the works of Cahlon and Shi [4], Cannon [5], Choi and Chan [8], Ewing and Lin [9], Ionkin [10], Kamynin [11], Samarskii [33], Shi and Shilor [34], Wang et al. [36], and Wu et al. [37]. The first paper discussed second order partial differential equations with nonlocal integral conditions going back to Cannon [5]. In fact, most of the works on this topic were dedicated to classical solutions. Later, mixed problems with classical and nonlocal (integral) boundary conditions related to parabolic and hyperbolic equations received attention and have been extensively studied. Existence and uniqueness questions have been considered by Bouziani [3], Ionkin [10], Kamynin [11], Mesloub [25], Pulkina [32].

In the absence of the viscoelastic term (i.e., $g = 0$), Mesloub and Bouziani [23] studied the following equation:

$$v_{tt} - \frac{1}{x}v_x - v_{xx} = f(x,t), \quad x \in (0,\ell),\ t \in (0,T),$$

and obtained the existence and uniqueness of a strong solution. Later, Mesloub and Messaoudi [25] solved a three-point boundary-value problem for a hyperbolic equation with a Bessel operator and an integral condition based on an energy method. Then in [26] they considered a nonlinear one-dimensional hyperbolic problem with a linear damping term and established a blow-up result for large initial data and a decay result for small initial data.

In the presence of the viscoelastic term (i.e. $g \neq 0$), Mecheri et al. [22] studied the following equation:

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\,\frac{1}{x}(xu_x(x,s))_x\,ds + au_t = f(x,t), \quad 0 < x < 1,\ t > 0,$$