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EULER CHARACTERISTICS OF DIGITAL WEDGE SUMS AND THEIR APPLICATIONS

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Abstract. Many properties or formulas related to the ordinary Euler characteristics of topological spaces are well developed under many mathematical operands, e.g. the product property, fibration property, homotopy axiom, wedge sum property, inclusion-exclusion principle [48], etc. Unlike these properties, the digital version of the Euler characteristic has its own feature. Among the above properties, we prove that the digital version of the Euler characteristic has the wedge sum property which is of the same type as that for the ordinary Euler characteristic. This property plays an important role in fixed point theory for digital images, digital homotopy theory, digital geometry and so forth.

1. Introduction

Based on the research of the Euler characteristic for polyhedral surfaces, finite CW-complexes, more generally, for any topological space, we can define the n-th Betti number b_n as the rank of the n-th singular homology group [48]. The Euler characteristic can then be defined as the alternating sum. This quantity is well-defined if the Betti numbers are all finite and if they are zero beyond

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a certain index n_0. In modern mathematics, this notion arises from homology and, more abstractly, homological algebra. Nowadays it also contributes to the fields of pure and applied mathematics, see e.g. [9], [15], [20], [30]–[32], [42], [47].

In pure mathematics, homology groups are used for introducing the notion of Euler characteristic which naturally has the product property, fibration property, homotopy invariance and so forth. Besides, it is well known that any contractible space has a trivial homology, meaning that the 0-th Betti number is 1 and the others are 0. This case includes Euclidean subspaces of any dimension as well as the nD disk, etc.

Digital topology has a focus on studying digital topological properties of nD digital images, it contributed to the study of some areas of computer sciences, see e.g. [4], [43], [45], [46]. Thus establishment of a digital version of the Euler characteristic can be meaningful [3], [15], [19], [30]–[32], [47]. For a digital image $X \subset \mathbb{Z}^n$ with a k-adjacency, denoted by (X, k) in [45], [46], using the digital homology proposed in the papers [1], [8], the authors of [8] formulated a digital version of the Euler characteristic denoted by $\chi(X, k)$ (see Theorem 4.2 of the paper [8]), and further they studied its various properties. In Section 6 we will discuss some limitations of this quantity. Besides, the recent paper [14] observes that Euler characteristics for digital images do not have the product property, fibration property and homotopy invariance for a k-contractible digital image such as $SC_8^{2,4}$ (see Example 4.1 of [14]) and so forth. The study of non-k-contractible cases will be presented in Section 6 of the present paper.

At this moment we may pose the following query: under what operation do Euler characteristics for digital images have the same features as the ordinary Euler characteristic? Hence, for a digital wedge sum with a k-adjacency [16], [23], denoted by $(X \vee Y, k)$ (see Definition 4.1 of the present paper), we have the following question:

> In digital topology do we have a formula $\chi(X \vee Y, k)$
> which is of the same type as that in algebraic topology?

If we have a positive answer, then we can conclude that

$$\chi(X \vee Y, k) \text{ is equal to } \chi(X, k_1) + \chi(Y, k_2) - 1. \tag{1.1}$$

Property (1.1) can play an important role in fixed point theory for digital images and digital topology. In the present paper all digital images (X, k) are assumed to be non-empty and k-connected. Besides, we point out that the Euler characteristic introduced in [8] is not suitable for studying fixed point theory for digital images.

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 investigates some properties of digital homologies of several digital k-surfaces. Section 4 develops a formula calculating digital