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EXISTENCE OF SOLUTION FOR A KIRCHHOFF TYPE SYSTEM WITH WEIGHT AND NONLINEARITY INVOLVING A (p,q)-SUPERLINEAR TERM AND CRITICAL CAFFARELLI–KOHN–NIRENBERG GROWTH

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Abstract. We study a (p,q)-Laplacian system of Kirchhoff type equations with weight and nonlinearity involving a (p,q)-superlinear term, in which p may be different from q, and with critical Caffarelli–Kohn–Nirenberg exponent. Using the Mountain Pass Theorem, we obtain a nontrivial solution to the problem.

1. Introduction

This paper deals with existence of a nontrivial weak solution to the (p,q)-Laplacian system of Kirchhoff type equations

$$\begin{cases} L_p(u) = \lambda |x|^{-c} F_u(x,u,v) + \alpha |x|^{-\beta} |u|^{\alpha-2} u |v|^{\gamma} & \text{in } \Omega, \\ L_q(v) = \lambda |x|^{-c} F_v(x,u,v) + \gamma |x|^{-\beta} |u|^{\alpha} |v|^{\gamma-2} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where

$$L_p(u) = -\bigg[M_1\bigg(\int_\Omega |x|^{-a_1p}|\nabla u|^p\,dx\bigg)\bigg]\operatorname{div}(|x|^{-a_1p}|\nabla u|^{p-2}\nabla u),$$

$$L_q(v) = -\bigg[M_2\bigg(\int_\Omega |x|^{-a_2q}|\nabla v|^q\,dx\bigg)\bigg]\operatorname{div}(|x|^{-a_2q}|\nabla v|^{q-2}\nabla v);$$

$\Omega\subset\mathbb{R}^N$ is a bounded smooth domain with $N\geq 3$, $1<p<N$, $1<q<N$, $a_1<(N-p)/p$, $a_2<(N-q)/q$, $c\in\mathbb{R}$, $\alpha/p^*+\gamma/q^*=1$, where $p^*=Np/(N-d_1p)$ and $q^*=Nq/(N-d_2q)$ are the critical Caffarelli–Kohn–Nirenberg exponents with $d_i=1+a_i-b_i$, $a_i\leq b_i<a_i+1$, $i=1,2$, and $\beta=b_1p^*=b_2q^*$. Let $F\colon\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a measurable function in Ω, continuously differentiable in $\mathbb{R}\times\mathbb{R}$, where F_w is its partial derivative with respect to w, and $M_i\colon\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+$ be a continuous function, $i=1,2$.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\begin{cases} u_{tt}-M\bigg(\displaystyle\int_\Omega|\nabla u|^2\,dx\bigg)\Delta u=g(x,u) & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & \end{cases}\tag{1.2}$$

where $M(s)=a+bs$, $a,b>0$. It was proposed by Kirchhoff [14] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Due to the presence of terms $M_i(\int_\Omega |x|^{-a_i}|\nabla w|^r\,dx)$, $i=1,2$, the equations in (1.1) are no longer a pointwise identity, therefore it is often called a nonlocal problem. This phenomenon causes some mathematical difficulties, what makes the study of such class of problems particularly interesting.

In the last years many authors have studied the following nonlocal problem:

$$-M\bigg(\int_\Omega|\nabla u|^2\,dx\bigg)\Delta u=f(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega.\tag{1.3}$$

Problems of type (1.3) may be used to model several physical and biological problems, see [1] for more references. Many interesting results for problems of the Kirchhoff type have already been obtained, see for example [1], [5], [11], and the references therein. The study of Kirchhoff type equations has been extended to the case involving the p-Laplacian operator, see [7], [9], and [12]. Systems of Kirchhoff type equations were considered for example in [6] and [8].

To enunciate the main results, we shall pose some hypotheses on the functions M_1,M_2, and F. Hypotheses on the continuous functions $M_i\colon\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+$, $i=1,2$, are the following:

(M1) There exist $m_1>0$ and $m_2>0$ such that $M_i(t)\geq m_i$, for all $t\geq 0$, $i=1,2$.