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EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR KIRCHHOFF TYPE EQUATION WITH GENERAL CRITICAL GROWTH

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Abstract. We study the existence of positive ground state solutions for the nonlinear Kirchhoff type equation

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $a,b>0$ are constants, $f\in C(\mathbb{R},\mathbb{R})$ has general critical growth. We generalize a Berestycki–Lions theorem about the critical case of Schrödinger equation to Kirchhoff type equation via variational methods. Moreover, some subcritical works on Kirchhoff type equation are extended to the current critical case.

1. Introduction

We are concerned with the following Kirchhoff-type equation:

$$\text{(K)}\qquad \begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3, \end{cases}$$

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where $a, b > 0$ are constants, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R})$ satisfy some conditions to be made precise later.

We recall that u is said to be the ground state (or the least energy) solution of (K) if and only if u solves (K) and minimizes the functional associated with (K) among all possible nontrivial solutions. Almost sufficient and necessary conditions for the existence of ground state solutions to the following nonlinear elliptic equation:

$$\begin{cases} -\Delta u = h(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

are given by Berestycki and Lions in [6] when $N \geq 3$ and Berestycki et al. in [7] when $N = 2$. In particular, in [6], the following existence result is obtained.

THEOREM 1.1. *Suppose $N \geq 3$ and h satisfies the following conditions:*

(H_1) $h \in C(\mathbb{R}, \mathbb{R})$ *is odd*;

(H_2) $-\infty < \liminf_{s \to 0^+} h(s)/s \leq \limsup_{s \to 0^+} h(s)/s = -m < 0$;

(H_3) $-\infty \leq \limsup_{s \to 0^+} h(s)/s^l \leq 0$, *where* $l = (N+2)/(N-2)$;

(H_4) *there exists* $\zeta > 0$ *such that* $H(\zeta) := \int_0^\zeta h(s)\,ds > 0$.

Then (1.1) *possesses a positive radial ground state solution.*

This problem was studied in [6] in the space $H_r^1(\mathbb{R}^N)$ of radical symmetric functions, in which case the nonlinear term h is independent of $x \in \mathbb{R}^N$. More importantly, the imbedding of $H_r^1(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $r \in (2, 2^*)$. Note also that (H_3) implies that the nonlinear term has subcritical growth.

In [34], Zhang and Zou studied problem (1.1) for

$$h = h(x, u) = -V(x)u + f(u),$$

i.e. h has critical growth and depends on x. Under conditions

(f_1) $f \in C^1(\mathbb{R}^N, \mathbb{R})$;

(f_2) $f(t) = o(t)$ as $t \to 0^+$;

(f_3) $\lim_{t \to +\infty} f(s)/s^{(N+2)/(N-2)} = K > 0$;

(f_4) there exist $D > 0$ and $q \in (2, 2^*)$ such that $f(s) \geq Kt^{(N+2)/(N-2)} + Dt^{q-1}$ for all $t \geq 0$, where $2^* = 2N/(N-2)$;

(f_5) $|f'(t)| \leq C(1 + |t|^{4/(N-2)})$ for $t \geq 0$ and some $C > 0$.

they proved that (1.1) has a ground state solution if the potential V satisfies certaiin reasonable hypotheses. These results of Zhang and Zou can be regarded as a generalization of the Berestycki–Lions theorem to critical and non-radial case. Conditions (f_3) and (f_4) characterize equation (1.1) to be of critical growth. Azzollini in [4] studied a class of Kirchhoff equations and extended the Berestycki–Lions theorem to problem (K) by using minimizing arguments on a suitable