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ON THE CHAOS GAME OF ITERATED FUNCTION SYSTEMS

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Abstract. Every quasi-attractor of an iterated function system (IFS) of continuous functions on a first-countable Hausdorff topological space is renderable by the probabilistic chaos game. By contrast, we prove that the backward minimality is a necessary condition to get the deterministic chaos game. As a consequence, we obtain that an IFS of homeomorphisms of the circle is renderable by the deterministic chaos game if and only if it is forward and backward minimal. This result provides examples of attractors (a forward but no backward minimal IFS on the circle) that are not renderable by the deterministic chaos game. We also prove that every well-fibred quasi-attractor is renderable by the deterministic chaos game as well as quasi-attractors of both, symmetric and non-expansive IFSs.

1. Introduction

Within fractal geometry, iterated function systems (IFSs) provide a method for both generating and characterizing fractal images. An *iterated function system* (IFS) can also be thought of as a finite collection of functions which can be applied successively in any order. Attractors of this kind of systems are self-similar compact sets which draw any iteration of any point in an open neighbourhood of itself.

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There are two methods for generating an attractor: a *deterministic* algorithm, in which all the transformations are applied simultaneously, and a *random* algorithm, in which the transformations are applied one at a time in random order following the probability. A *chaos game*, popularized by Barnsley [3], is a simple algorithm implementing the random method. There are two different ways to run the chaos game that consists in taking a starting point and then choosing randomly a transformation on each iteration accordingly to the assigned probabilities. The latter starts by choosing a random order iteration and then applying this orbital branch anywhere in the basin of attraction. The first way of implementation is called the *probabilistic chaos game* [9], [7]. The second implementation is called the *deterministic chaos game* (also called the *disjunctive chaos game*) [27], [5], [12].

According to [7], every attractor of an IFS of continuous maps on a first-countable Hausdorff topological space is renderable by the probabilistic chaos game. By contract, we will see that this is not the case of the deterministic chaos game. Namely, we will provide necessary and sufficient conditions to get the deterministic chaos game. As an application we will obtain that an IFS of homeomorphisms of the circle is renderable by the deterministic chaos game if and only if it is forward and backward minimal which provides examples of attractors that are not renderable by the deterministic chaos game.

1.1. Iterated function systems. Let X be a Hausdorff topological space. We consider a finite set $\mathscr{F} = \{f_1, \ldots, f_k\}$ of continuous functions from X to itself. Associated with this set $\mathscr{F}$ we define the *semigroup* $\Gamma = \Gamma_{\mathscr{F}}$ generated by these functions, the *Hutchinson operator* $F = F_{\mathscr{F}}$ on the hyperspace $\mathscr{H}(X)$ of non-empty compact subsets of X

$$F\colon \mathscr{H}(X) \to \mathscr{H}(X), \qquad F(A) = \bigcup_{i=1}^{k} f_i(A)$$

and the *skew-product* $\Phi = \Phi_{\mathscr{F}}$ on the product space of $\Omega = \{1, \ldots, k\}^{\mathbb{N}}$ and X

$$\Phi\colon \Omega \times X \to \Omega \times X, \qquad \Phi(\omega, x) = (\sigma(\omega), f_{\omega_1}(x)),$$

where $\omega = \omega_1\omega_2 \ldots \in \Omega$ and $\sigma\colon \Omega \to \Omega$ is the lateral shift map. The action of the semigroup Γ on X is called the *iterated function system* generated by $f_1, \ldots, f_k$ (or, by the family $\mathscr{F}$ for short). Finally, given $\omega = \omega_1\omega_2 \ldots \in \Omega$ and $x \in X$,

$$f_\omega^n \stackrel{\text{def}}{=} f_{\omega_n} \circ \ldots \circ f_{\omega_1} \text{ for every } n \in \mathbb{N}, \quad \text{and} \quad O_\omega^+(x) = \{f_\omega^n(x) : n \in \mathbb{N}\}$$

are called, respectively, the *orbital branch* corresponding to ω (or the IFS-iteration driven by the sequence ω) and the ω-*fiberwise orbit* of x. We introduce now a number of different notions of invariant and minimal sets and next give the definition of an attractor. In what follows A denotes a closed subset of X.