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PERIODIC SOLUTIONS FOR THE NON-LOCAL OPERATOR $(-\Delta+m^2)^s - m^{2s}$ WITH $m \geq 0$

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Abstract. By using variational methods, we investigate the existence of T-periodic solutions to

$$\begin{cases} [(-\Delta_x+m^2)^s - m^{2s}]u = f(x,u) & \text{in } (0,T)^N, \\ u(x+Te_i)=u(x) & \text{for all } x\in\mathbb{R}^N,\ i=1,\dots,N, \end{cases}$$

where $s\in(0,1)$, $N>2s$, $T>0$, $m\geq 0$ and f is a continuous function, T-periodic in the first variable, verifying the Ambrosetti–Rabinowitz condition, with a polynomial growth at rate $p\in(1,(N+2s)/(N-2s))$.

1. Introduction

Recently, considerable attention has been given to fractional Sobolev spaces and corresponding non-local equations, in particular to the ones driven by the fractional powers of the Laplacian. In fact, this operator naturally arises in several areas of research and finds applications in optimization, finance, the thin obstacle problem, phase transitions, anomalous diffusion, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows and water waves. For more details and applications see [4], [6], [9], [12], [15], [16], [22], [26]–[28], [30] and references therein.

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The purpose of the present paper is to study T-periodic solutions to the problem

$$\begin{cases} [(-\Delta_x + m^2)^s - m^{2s}]u = f(x,u) & \text{in } (0,T)^N, \\ u(x+Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N,\ i = 1,\dots,N, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, (e_i) is the canonical basis in $\mathbb{R}^N$ and $f\colon \mathbb{R}^{N+1} \to \mathbb{R}$ is a function satisfying the following hypotheses:

(f1) $f(x,t)$ is T-periodic in $x \in \mathbb{R}^N$, that is $f(x+Te_i,t) = f(x,t)$.

(f2) f is continuous in $\mathbb{R}^{N+1}$.

(f3) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$.

(f4) There exist $1 < p < 2^\sharp_s - 1 = 2N/(N-2s) - 1$ and $C > 0$ such that

$$|f(x,t)| \leq C(1+|t|^p) \quad \text{for any } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

(f5) There exist $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu F(x,t) \leq t f(x,t) \quad \text{for } x \in \mathbb{R}^N \text{ and } |t| \geq r_0.$$

Here $F(x,t) = \int_0^t f(x,\tau)\,d\tau$.

(f6) $tf(x,t) \geq 0$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

We notice that (f2) and (f5) imply the existence of two constants $a, b > 0$ such that

$$F(x,t) \geq a|t|^\mu - b \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Then, since $\mu > 2$, $F(x,t)$ grows at a superquadratic rate and by (f5), $f(x,t)$ grows at a superlinear rate as $|t| \to \infty$. Here, the operator $(-\Delta_x + m^2)^s$ is defined through the spectral decomposition, by using the powers of the eigenvalues of $-\Delta + m^2$ with periodic boundary conditions.

Let $u \in \mathcal{C}^\infty_T(\mathbb{R}^N)$, that is u is infinitely differentiable in $\mathbb{R}^N$ and T-periodic in each variable. Then u has a Fourier series expansion

$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}}, \quad x \in \mathbb{R}^N,$$

where

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x) e^{-\imath\omega k\cdot x}\,dx, \quad k \in \mathbb{Z}^N,$$

are the Fourier coefficients of u. The operator $(-\Delta_x + m^2)^s$ is defined by setting

$$(-\Delta_x + m^2)^s u = \sum_{k \in \mathbb{Z}^N} c_k (\omega^2|k|^2 + m^2)^s \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}}.$$

For

$$u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}} \quad \text{and} \quad v = \sum_{k \in \mathbb{Z}^N} d_k \frac{e^{\imath\omega k\cdot x}}{\sqrt{T^N}},$$