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NASH EQUILIBRIUM FOR BINARY CONVEXITIES

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Abstract. This paper is devoted to Nash equilibrium for games in capacities. Such games with payoff expressed by the Choquet integral were considered by Kozhan and Zarichnyi (2008) and existence of Nash equilibrium was proved. We also consider games in capacities but with expected payoff expressed by the Sugeno integral. We prove existence of Nash equilibrium in a general context of abstract binary (non-linear) convexity and then we obtain an existence theorem for games in capacities.

1. Introduction

The classical Nash equilibrium theory is based on fixed point theory and was developed in the frame of linear convexity with mixed strategies of a player being probability (additive) measures on a set of pure strategies. In last decades the interest in Nash equilibria in more general frames is rapidly growing. For instance, Briec and Horvath proved in [1] existence of a Nash equilibrium point for B-convexity and MaxPlus convexity which are non-linear. Let us remark that MaxPlus convexity is related to idempotent (Maslov) measures in the same sense as linear convexity is related to probability measures.

We can use additive measures only when we know precisely probabilities of all events considered in a game. However, this is not the case in many modern economic models. The decision theory under uncertainty considers a model when probabilities of states are either not known or imprecisely specified. Gilboa [5]

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and Schmeidler [17] axiomatized expectations expressed by Choquet integrals attached to non-additive measures called capacities, as a formal approach to decision-making under uncertainty.

An alternative to the so-called Choquet expected utility model is the qualitative decision theory. The corresponding expected utility is expressed by the Sugeno integral. See the papers [3], [4], [2], [16] and others for more details about the qualitative decision theory and motivation of using the Sugeno integral.

Kozhan and Zarichnyi introduced in [7] a notion of Nash equilibrium of a game where players are allowed to form non-additive beliefs about opponent's decision but also to play their mixed non-additive strategies. Such game was called by the authors as the game in capacities. The expected payoff function was defined using the Choquet integral. Kozhan and Zarichnyi proved an existence theorem using a linear convexity on the space of capacities which is preserved by the Choquet integral. The problem of existence of Nash equilibrium for other functors was stated in [7].

In this paper, following [7], we introduce a concept of Nash equilibrium for a game in capacities. However, motivated by the qualitative approach, we consider an expected payoff function defined by the Sugeno integral. In order to prove an existence theorem for this particular case, we consider a more general framework which could unify all situations mentioned before and give us a method to prove theorems about existence of Nash equilibrium in different contexts. We use categorical methods and abstract convexity theory.

The notion of convexity considered in this paper is considerably broader than the classical one; in particular, it is not restricted to the context of linear spaces. Such convexities appeared in the process of studying different structures like partially ordered sets, semilattices, lattices, superextensions etc. We base our approach on the notion of topological convexity from [20] where the general convexity theory is covered from axioms to applications in different areas. Particularly, this book contains the Kakutani fixed point theorem for abstract convexity.

The above mentioned constructions of spaces of probability measures, idempotent measures and capacities are functorial and can be completed to monads (see [15], [22] and [11] for more details). A convexity structure on each $\mathbb{F}$-algebra for any monad $\mathbb{F}$ in the category of compact Hausdorff spaces and continuous maps was introduced in [12].

We prove a counterpart of the Nash theorem for an abstract convexity. Particularly, we consider binary convexities. These results are used to obtain a Nash type theorem for algebras of any L-monad with binary convexity. Since a capacity monad is an L-monad with binary convexity [13], we obtain as corollary the corresponding result for capacities.

2. Binary convexities

A family $\mathcal{C}$ of closed subsets of a compactum X is called a *convexity* on X if $\mathcal{C}$ is stable for intersection, and contains X and the empty set. The elements of $\mathcal{C}$ are called $\mathcal{C}$-*convex* (or simply convex). Although we follow the general concept of abstract convexity from [20], our definition is different. We consider only closed convex sets. Such structure is called a closure structure in [20]. Our definition is the same as in [21]. The whole family of convex sets in the sense of [20] could be obtained by the operation of union of up-directed families. In what follows, we assume that each convexity contains all singletons.

A convexity $\mathcal{C}$ on X is called T_2 if for any distinct $x_1, x_2 \in X$ there exist $S_1, S_2 \in \mathcal{C}$ such that $S_1 \cup S_2 = X$, $x_1 \notin S_2$ and $x_2 \notin S_1$. Let us remark that if a convexity $\mathcal{C}$ on a compactum X is T_2, then $\mathcal{C}$ is a subbase for closed sets. A convexity $\mathcal{C}$ on X is called T_4 (normal) if for any disjoint $C_1, C_2 \in \mathcal{C}$ there exist $S_1, S_2 \in \mathcal{C}$ such that $S_1 \cup S_2 = X$, $C_1 \cap S_2 = \emptyset$ and $C_2 \cap S_1 = \emptyset$.

Let $(X, \mathcal{C}), (Y, \mathcal{D})$ be two compacta with convexity structures. A continuous map $f\colon X \to Y$ is called a *CP map* (convexity preserving map) if $f^{-1}(D) \in \mathcal{C}$ for each $D \in \mathcal{D}$; f is called a *CC map* (convex-to-convex map) if $f(C) \in \mathcal{D}$ for each $C \in \mathcal{C}$.

By a *multimap* (set-valued map) of a set X into a set Y we mean a map $F\colon X \to 2^Y$ and we use the notation $F\colon X \multimap Y$. If X and Y are topological spaces, then a multimap $F\colon X \multimap Y$ is called *upper semi-continuous* (USC) provided for each open set $O \subset Y$ the set $\{x \in X \mid F(x) \subset O\}$ is open in X. It is well-known that a multimap with compact values is USC iff its graph is closed in $X \times Y$.

Let $F\colon X \multimap X$ be a multimap. We say that a point $x \in X$ is a *fixed point* of F if $x \in F(x)$. The following counterpart of the Kakutani theorem for an abstract convexity is a partial case of Theorem 3 from [21] (it also could be obtained combining Theorem 6.15, Chapter IV and Theorem 4.10, Chapter III from [20]).

Theorem 2.1. *Let $\mathcal{C}$ be a normal convexity on a compactum X such that all convex sets are connected and $F\colon X \multimap X$ is a USC multimap with values in $\mathcal{C}$. Then F has a fixed point.*

Let $\mathcal{C}$ be a family of subsets of a compactum X. We say that $\mathcal{C}$ is *linked* if the intersection of any two its elements is non-empty. A convexity $\mathcal{C}$ is called *binary* if the intersection of every its linked subsystem of $\mathcal{C}$ is non-empty.

Lemma 2.2. *Let $\mathcal{C}$ be a T_2 binary convexity on a continuum X. Then $\mathcal{C}$ is normal and all convex sets are connected.*

Proof. The first assertion of the lemma is proved in [15, Lemma 3.1]. Let us prove the second one. Consider any $A \in \mathcal{C}$. A retraction $h_A\colon X \to A$ is defined in [9] by the formula $h_A(x) = \cap\{C \in \mathcal{C} \mid x \in C \text{ and } C \cap A \neq \emptyset\}$. Hence A is connected. $\square$

Now we can reformulate Theorem 2.1 for binary convexities.

Theorem 2.3. *Let $\mathcal{C}$ be a T_2 binary convexity on a continuum X and let $F\colon X \multimap X$ be a USC multimap with values in $\mathcal{C}$. Then F has a fixed point.*

Let us recall the definition of Nash equilibrium. We consider an n-players game $f\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ with compact Hausdorff spaces of strategies X_i. The coordinate function $f_i\colon X \to \mathbb{R}$ is called the *payoff function* of i-th player. For $x \in X$ and $t_i \in X_i$ we use the notation $(x; t_i) = (x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)$. A point $x \in X$ is called a *Nash equilibrium point* if for each $i \in \{1, \dots, n\}$ and for each $t_i \in X_i$ we have $f_i(x; t_i) \leq f_i(x)$.

Now, let $\mathcal{C}_i$ be a convexity on X_i. We say that a function $f_i\colon X \to \mathbb{R}$ is *quasiconcave* with respect to the i-th variable if we have $(f_i^x)^{-1}([t; +\infty)) \in \mathcal{C}_i$ for all $t \in \mathbb{R}$ and $x \in X$ where $f_i^x\colon X_i \to \mathbb{R}$ is the function defined as follows: $f_i^x(t_i) = f_i(x; t_i)$ for $t_i \in X_i$.

Theorem 2.4. *Let $f\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ be a game with a normal convexity $\mathcal{C}_i$ defined on each compactum X_i such that all convex sets are connected, the function f is continuous and the function $f_i\colon X \to \mathbb{R}$ is quasiconcave with respect to the i-th variable for each $i \in \{1, \dots, n\}$. Then there exists a Nash equilibrium point.*

Proof. Fix any $x \in X$. For each $i \in \{1, \dots, n\}$ let

$$M_i^x = \Big\{t \in X_i \;\Big|\; f_i^x(t) = \max_{s \in X_i} f_i^x(s)\Big\}.$$

Then M_i^x is a closed subset X_i. Since the function $f_i\colon X \to \mathbb{R}$ is quasiconcave with respect to the i-th variable, $M_i^x \in \mathcal{C}_i$. Define a multimap $F\colon X \multimap X$ by the formula $F(x) = \prod_{i=1}^{n} M_i^x$ for $x \in X$.

Let us show that F is USC. Consider any point $(x, y) \in X \times X$ such that $y \notin F(x)$. Then there exists $i \in \{1, \dots, n\}$ such that $f_i^x(y_i) < \max_{s \in X_i}\{f_i^x(s)\}$. Hence we can choose $t_i \in X_i$ such that $f_i(x; y_i) < f_i(x; t_i)$. Since f_i is continuous, there exist a neighbourhood O_x of x in X and a neighbourhood O_{y_i} of y_i in Y_i such that for any $x' \in O_x$ and $y_i' \in O_{y_i}$ we have $f_i(x; y_i') < f_i(x; t_i)$. Put $O_y = (\mathrm{pr}_i)^{-1}(O_{y_i})$. Then for each $(x', y') \in O_x \times O_y$ we have $y' \notin F(x')$. Thus the graph of F is closed in $X \times Y$, hence F is upper semicontinuous.

We consider on X the family $\mathcal{C} = \Big\{\prod_{i=1}^{n} C_i \;\Big|\; C_i \in \mathcal{C}_i\Big\}$. It is easy to see that $\mathcal{C}$ forms a normal convexity on the compactum X such that all convex sets are connected. Then, by Theorem 2.1, F has a fixed point which is a Nash equilibrium point. $\square$

Now, the previous theorem and Lemma 2.2 imply the following corollary.

COROLLARY 2.5. *Let* $f\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ *be a game such that a* T_2 *binary convexity* $\mathcal{C}_i$ *is defined on each continuum* X_i*, the function* f *is continuous and the function* $f_i\colon X \to \mathbb{R}$ *is quasiconcave with respect to the* i*-th variable for each* $i \in \{1, \dots, n\}$*. Then there exists a Nash equilibrium point.*

3. L-monads and its algebras

By Comp we denote the category of compact Hausdorff spaces (compacta) and continuous maps. For each compactum X we denote by $C(X)$ the Banach space of all continuous functions on X with the usual sup-norm. In what follows, all spaces and maps are assumed to be in Comp except for $\mathbb{R}$ and maps in sets $C(X)$ with X compact Hausdorff.

We apply Corollary 2.5 to study games defined on the algebras of binary L-monads. We recall some categorical notions (see [8] and [19] for more details). We define them only for the category Comp. Let $F\colon \mathsf{Comp} \to \mathsf{Comp}$ be a covariant functor. A functor F is called *continuous* if it preserves the limits of inverse systems. In what follows, all functors are assumed to preserve monomorphisms, epimorphisms, weight of infinite compacta. We also assume that our functors are continuous. For a functor F which preserves monomorphisms and for an embedding $i\colon A \to X$ we shall identify the space FA and the subspace $F(i)(FA) \subset FX$.

A *monad* $\mathbb{T} = (T, \eta, \mu)$ in the category Comp consists of an endofunctor $T\colon \mathsf{Comp} \to \mathsf{Comp}$ and natural transformations $\eta\colon \mathrm{Id}_{\mathsf{Comp}} \to T$ (unity), $\mu\colon T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By $\mathrm{Id}_{\mathsf{Comp}}$ we denote the identity functor on the category Comp and T^2 is the superposition $T \circ T$.)

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category Comp. A pair (X, ξ), where $\xi\colon TX \to X$ is a map, is called a $\mathbb{T}$-*algebra* if $\xi \circ \eta X = \mathrm{id}_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X, ξ), (Y, ξ') be two $\mathbb{T}$-algebras. A map $f\colon X \to Y$ is called a *morphism of* $\mathbb{T}$-*algebras* if $\xi' \circ Tf = f \circ \xi$.

Let (X, ξ) be an $\mathbb{F}$-algebra for a monad $\mathbb{F} = (F, \eta, \mu)$ and let A be a closed subset of X. Denote by f_A the quotient map $f_A\colon X \to X/A$ (the equivalence classes are one-point sets $\{x\}$ for $x \in X \setminus A$ and the set A) and put $a = f_A(A)$. Denote $A^+ = (Ff_A)^{-1}(\eta(X/A)(a))$. Define the $\mathbb{F}$-*convex hull* $\mathrm{conv}_{\mathbb{F}}(A)$ of A as follows $\mathrm{conv}_{\mathbb{F}}(A) = \xi(A^+)$. Put additionally $\mathrm{conv}_{\mathbb{F}}(\emptyset) = \emptyset$. We define the family $\mathcal{C}_{\mathbb{F}}(X, \xi) = \{A \subset X \mid A \text{ is closed and } \mathrm{conv}_{\mathbb{F}}(A) = A\}$. The elements of the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ will be called $\mathbb{F}$-*convex*. It was shown in [12] that the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ forms a convexity on X, moreover, each morphism of $\mathbb{F}$-algebras is a CP map. Let us remark that the one-point sets are always $\mathbb{F}$-convex.

We do not know if the introduced convexities are T_2. In this section we consider a class of monads generating convexities which have this property. The class of L-monads was introduced in [12] and it contains many well-known monads in Comp like superextension, hyperspace, probability measure, capacity, idempotent measure, etc. For $\phi \in C(X)$ by $\max \phi$ (respectively, $\min \phi$) we denote $\max_{x\in X} \phi(x)$ $\Big($respectively, $\min_{x\in X} \phi(x)\Big)$ and π_ϕ or $\pi(\phi)$ denote the corresponding projection $\pi_\phi \colon \prod_{\psi\in C(X)} [\min \psi, \max \psi] \to [\min \phi, \max \phi]$. It was shown in [14] that for each L-monad $\mathbb{F} = (F, \eta, \mu)$ the space FX can be considered as a subset of the product $\prod_{\phi\in C(X)} [\min \phi, \max \phi]$, moreover, we have $\pi_\phi \circ \eta X = \phi$, $\pi_\phi \circ \mu X = \pi(\pi_\phi)$ for all $\phi \in C(X)$ and $\pi_\psi \circ Ff = \pi_{\psi\circ f}$ for all $\psi \in C(Y)$, $f \colon X \to Y$. We can consider these properties of L-monads as a definition [14].

We say that an L-monad $\mathbb{F} = (F, \eta, \mu)$ *weakly preserves preimages* if for each map $f \colon X \to Y$ and each closed subset $A \subset Y$ we have

$$\pi_\phi(\nu) \in \big[\min \phi(f^{-1}(A)), \max \phi(f^{-1}(A))\big]$$

for each $\nu \in (Ff)^{-1}(A)$ and $\phi \in C(X)$ [12]. It was shown in [12] that for each L-monad $\mathbb{F}$ which weakly preserves preimages the convexity $\mathcal{C}_{\mathbb{F}}(FX, \mu X)$ is T_2.

Lemma 3.1. *Let (X, ξ) be an $\mathbb{F}$-algebra for an L-monad $\mathbb{F} = (F, \eta, \mu)$ which weakly preserves preimages. Then the map $\xi \colon FX \to X$ is a CC map for convexities $\mathcal{C}_{\mathbb{F}}(FX, \mu)$ and $\mathcal{C}_{\mathbb{F}}(X, \xi)$, respectively.*

Proof. Consider any $B \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$. We should show that $\xi(B) \in \mathcal{C}_{\mathbb{F}}(X, \xi)$. Denote by $\chi \colon X \to X/\xi(B)$ the quotient map and put $b = \chi(\xi(B))$. Consider any $\mathcal{A} \in FX$ such that $F\chi(\mathcal{A}) = \eta(X/\xi(B))(b)$. We should show that $\xi(\mathcal{A}) \in \xi(B)$.

Consider the quotient map $\chi_1 \colon FX \to FX/B$ and put $b_1 = \chi_1(B)$. There exists a (unique) continuous map $\xi' \colon FX/B \to X/\xi(B)$ such that $\xi'(b_1) = b$ and $\xi' \circ \chi_1 = \chi \circ \xi$. Put $\mathcal{D} = F(\eta X)(\mathcal{A})$. We have $F\xi(\mathcal{D}) = \mathcal{A}$, hence $F\xi' \circ F\chi_1(\mathcal{D}) = F\chi \circ F\xi(\mathcal{D}) = F\chi(\mathcal{A}) = \eta(X/\xi(B))(b)$. Since F weakly preserves preimages, we have $F\chi_1(\mathcal{D}) = \eta(FX/B)(b_1)$. Since $B \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$, we have $\mu X(\mathcal{D}) \in B$. Hence, $\xi(\mathcal{A}) = \xi \circ F\xi(\mathcal{D}) = \xi \circ \mu(\mathcal{D}) \in \xi(B)$. $\square$

We call a monad $\mathbb{F}$ *binary* if $\mathcal{C}_{\mathbb{F}}(X, \xi)$ is binary for each $\mathbb{F}$-algebra (X, ξ).

Lemma 3.2. *Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. Then for each $\mathbb{F}$-algebra (X, ξ) the convexity $\mathcal{C}_{\mathbb{F}}(X, \xi)$ is T_2.*

Proof. Consider any two distinct points $x, y \in X$. Being a morphism of $\mathbb{F}$-algebras $(FX, \mu X)$ and (X, ξ) the map ξ is a CP map and we have $\xi^{-1}(x), \xi^{-1}(y) \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$. Since $\mathcal{C}_{\mathbb{F}}(FX, \mu)$ is T_2 and binary, it is normal, by Lemma 2.2. Hence we can choose $L_1, L_2 \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$ such that $L_1 \cup L_2 = FX$ and $L_1 \cap \xi^{-1}(x) = \emptyset$, $L_2 \cap \xi^{-1}(y) = \emptyset$. Then we have $\xi(L_1), \xi(L_2) \in \mathcal{C}_{\mathbb{F}}(X, \xi)$, by Lemma 3.1, $\xi(L_1) \cup \xi(L_2) = X$, $x \notin L_1$ and $y \notin L_2$. $\square$

Consider any L-monad $\mathbb{F} = (F, \eta, \mu)$. It is easy to check that for each segment $[a, b] \subset \mathbb{R}$ the pair $([a, b], \xi_{[a,b]})$ is an $\mathbb{F}$-algebra where $\xi_{[a,b]} = \pi_{\mathrm{id}_{[a,b]}}$. Consider a game $f\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ where for each compactum X_i there exists a map $\xi_i\colon FX_i \to X_i$ such that the pair (X_i, ξ_i) is an $\mathbb{F}$-algebra. We say that the function $f_i\colon X \to \mathbb{R}$ is a morphism of $\mathbb{F}$-algebras with respect to the i-th variable if for each $x \in X$ the function $f_i^x\colon X_i \to \mathbb{R}$ is a morphism of $\mathbb{F}$-algebras (X_i, ξ_i) and $([\min f_i^x, \max f_i^x], \xi_{[\min f_i^x, \max f_i^x]})$.

Theorem 3.3. *Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. Let $f\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ be a game such that an $\mathbb{F}$-algebra map $\xi_i\colon FX_i \to X_i$ is defined on each continuum X_i, the function f is continuous and the function $f_i\colon X \to \mathbb{R}$ is a morphism of $\mathbb{F}$-algebras with respect to the i-th variable for each $i \in \{1, \dots, n\}$. Then there exists a Nash equilibrium point.*

Proof. Since for each $x \in X$ the function $f_i^x\colon X_i \to \mathbb{R}$ is a morphism of $\mathbb{F}$-algebras, it is a CP map, hence quasiconcave. Now, our theorem follows from Lemma 3.2 and Corollary 2.5. $\square$

4. Pure and mixed strategies

Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. In this section we consider Nash equilibrium for free algebras $(FX, \mu X)$. The points of a compactum X are called *pure strategies* and the points of FX are called *mixed strategies*. Such approach is a natural generalization of the model from [7] where spaces of capacities were considered.

We consider a game $u\colon X = \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ with compact Hausdorff spaces of pure strategies $X_1, \dots, X_n$ and continuous payoff functions $u_i\colon \prod_{i=1}^{n} X_i \to \mathbb{R}$.

It is well-known how to construct the tensor product of two (or finite number) probability measures. This operation was generalized in [19] over any monad in the category Comp. More precisely, for any compacta $X_1, \dots, X_n$ a continuous map $\otimes\colon \prod_{i=1}^{n} FX_i \to F\left(\prod_{i=1}^{n} X_i\right)$ was constructed such that $\otimes$ is natural by each argument and for each i we have $F(p_i) \circ \otimes = \mathrm{pr}_i$ where $p_i\colon \prod_{j=1}^{n} X_j \to X_i$ and $\mathrm{pr}_i\colon \prod_{j=1}^{n} FX_j \to FX_i$ are natural projections.

We define the payoff functions $eu_i\colon FX_1 \times \dots \times FX_n \to \mathbb{R}$ by the formula $eu_i = \pi_{u_i} \circ \otimes$. Evidently, eu_i is continuous. Consider any $t \in \mathbb{R}$ and $\nu \in FX_1 \times \dots \times FX_n$. Then we have $(eu_i^\nu)^{-1}[t; +\infty) = \{\mu_i \in FX_i \mid eu_i(\nu; \mu_i) \geq t_i\} = l^{-1}(\pi_{u_i}^{-1}[t; +\infty) \cap \{\nu_i\} \times \dots \times FX_i \times \dots \times \{\nu_n\})$, where $l\colon FX_i \to \prod_{j=1}^{n} FX_j$

is an embedding defined by $l(\mu_i) = (\nu; \mu_i)$ for $\mu_i \in FX_i$. A structure of $\mathbb{F}$-algebra on the product $\prod_{j=1}^{n} FX_j$ of $\mathbb{F}$-algebras $(FX_i, \mu X_i)$ is given by a map $\xi\colon F\left(\prod_{i=1}^{n} FX_i\right) \to \prod_{i=1}^{n} FX_i$ defined by the formula $\xi = (\mu X_i \circ F(p_i))_{i=1}^{n}$. It is easy to check that the product of convex in FX_i sets is convex in $\prod_{i=1}^{n} FX_i$. Since $\mathbb{F}$ weakly preserves preimages, $\pi_{u_i}^{-1}[t; +\infty)$ is convex in $\prod_{i=1}^{n} FX_i$. It is easy to see that l is a CP map, hence the map eu_i is quasiconcave with respect to the i-th variable. Hence, using Corollary 2.5, we obtain the following theorem.

Theorem 4.1. *The game with payoff functions eu_i has a Nash equilibrium point provided each FX_i is connected.*

5. Games in capacities

We need a definition of capacity on a compactum X. We follow the terminology of [11]. A function c which assigns to each closed subset A of X a real number $c(A) \in [0, 1]$ is called an *upper-semicontinuous capacity* on X if the following three properties hold for any closed subsets F and G of X:

1. $c(X) = 1$, $c(\emptyset) = 0$,
2. if $F \subset G$, then $c(F) \le c(G)$,
3. if $c(F) < a$, then there exists an open set $O \supset F$ such that $c(B) < a$ for each compactum $B \subset O$.

We extend a capacity c to all open subsets $U \subset X$ by the formula $c(U) = \sup\{c(K) \mid K$ is a closed subset of X such that $K \subset U\}$.

It was proved in [11] that the space MX of all upper-semicontinuous capacities on a compactum X is a compactum as well, if the topology on MX is defined by the subbase that consists of all sets of the form $O_-(F, a) = \{c \in MX \mid c(F) < a\}$, where F is a closed subset of X, $a \in [0, 1]$, and $O_+(U, a) = \{c \in MX \mid c(U) > a\}$, where U is an open subset of X, $a \in [0, 1]$. Since all capacities here are upper-semicontinuous, in the sequel we call elements of MX simply capacities.

The assignment M extends to the capacity functor M in the category of compacta, if the map $Mf\colon MX \to MY$ for a continuous map of compacta $f\colon X \to Y$ is defined by the formula $Mf(c)(F) = c(f^{-1}(F))$, where $c \in MX$ and F is a closed subset of X. This functor was completed to the monad $\mathbb{M} = (M, \eta, \mu)$ (see [11]), where the components of the natural transformations are defined as follows: $\eta X(x)(F) = 1$ if $x \in F$ and $\eta X(x)(F) = 0$ if $x \notin F$; $\mu X(\mathcal{C})(F) = \sup\{t \in [0, 1] \mid \mathcal{C}(\{c \in MX \mid c(F) \ge t\}) \ge t\}$, where $x \in X$, F is a closed subset of X and $\mathcal{C} \in M^2(X)$.

The tensor product for capacities was considered in [7]. It is a continuous map $\otimes\colon MX_1 \times \ldots \times MX_n \to M(X_1 \times \ldots \times X_n)$. Note that, despite the space of capacities contains the space of probability measures, the tensor product of capacities in general does not extend the tensor product of probability measures.

Due to Zhou [23] we can identify the set MX with some set of functionals defined on the space $C(X)$ using the Choquet integral. We consider for each $\mu \in MX$ its value on a function $f \in C(X)$ defined by the formula

$$\mu(f) = \int f\, d\mu = \int_0^\infty \mu\{x \in X \mid f(X) \geq t\}\, dt + \int_{-\infty}^0 (\mu\{x \in X \mid f(X) \geq t\} - 1)\, dt.$$

Kozhan and Zarichnyi proved in [7] the existence of Nash equilibrium for a game in capacities $ef\colon \prod_{i=1}^n MX_i \to \mathbb{R}^n$ with expected payoff functions defined by

$$ef_i(\mu_1, \ldots, \mu_n) = \int_{X_1 \times \ldots \times X_n} f_i\, d(\mu_1 \otimes \ldots \otimes \mu_n).$$

Let us remark that the Choquet functional representation of capacities preserves the natural linear convexity structure on MX which was used in the proof of existence of Nash equilibrium in [7]. However, this representation does not preserve the capacity monad structure.

Another functional representation of capacities was introduced in [13] (see also [10] for similar result). It uses the Sugeno integral. This representation preserves the capacity monad structure. Let us describe such a representation. Fix any increasing homeomorphism $\psi\colon (0,1) \to \mathbb{R}$. We put additionally $\psi(0) = -\infty$, $\psi(1) = +\infty$ and assume that $-\infty < t < +\infty$ for each $t \in \mathbb{R}$. We consider for each $\mu \in MX$ its value on a function $f \in C(X)$ defined by the formula

$$\mu(f) = \int_X^{\mathrm{Sug}} f\, d\mu = \max\{t \in \mathbb{R} \mid \mu(f^{-1}([t, +\infty))) \geq \psi^{-1}(t)\}.$$

Let us remark that we use certain modification of the Sugeno integral. The original Sugeno integral [18] "ignores" the values of a function outside the interval $[0,1]$ and we introduce a "correction" homeomorphism ψ to avoid this problem. Now, following [7], we consider a game in capacities $sf\colon \prod_{i=1}^n MX_i \to \mathbb{R}^n$, but motivated by [3], we consider Sugeno expected payoff functions defined by

$$sf_i(\mu_1, \ldots, \mu_n) = \int_{X_1 \times \ldots \times X_n}^{\mathrm{Sug}} f_i\, d(\mu_1 \otimes \ldots \otimes \mu_n).$$

The question of existence of Nash equilibrium arises naturally. Since the Sugeno integral does not preserve linear convexity on MX, we cannot use methods from [7].

It is easy to see that MX is connected for each compactum X. Since the capacity monad $\mathbb{M}$ is a binary L-monad which weakly preserves preimages with

$\pi_\varphi(\nu) = \int_X^{\mathrm{Sug}} f\, d\nu$ for any $\nu \in MX$ and $\varphi \in C(X)$ [13], we obtain the following theorem as a consequence of Theorem 4.1.

Theorem 5.1. *A game in capacities* $sf\colon \prod_{i=1}^{n} MX_i \to \mathbb{R}^n$ *with Sugeno payoff functions has a Nash equilibrium point.*

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