Topological Methods in Nonlinear Analysis
Volume 48, No. 1, 2016, 257–282
DOI: 10.12775/TMNA.2016.049

GLOBAL PHASE PORTRAITS OF KUKLES DIFFERENTIAL SYSTEMS WITH HOMOGENOUS POLYNOMIAL NONLINEARITIES OF DEGREE 5 HAVING A CENTER

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(*Submitted by J. Llibre*)

Abstract. We provide 22 different global phase portraits in the Poincaré disk of all centers of the so-called Kukles polynomial differential systems of the form $\dot{x} = -y$, $\dot{y} = x + Q_5(x, y)$, where Q_5 is a real homogeneous polynomial of degree 5 defined in $\mathbb{R}^2$.

1. Introduction and statement of the main result

Consider a system of the form

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + Q_n(x, y),\end{aligned}\tag{1.1}$$

where the dot denotes the derivative with respect to the independent variable t and Q_n is a real homogeneous polynomial of degree n. A system of this form was

2010 *Mathematics Subject Classification.* Primary: 34C05.

Key words and phrases. Centers; Kukles; polynomial vector fields; phase portrait; Poincaré disk.

The final version of this paper will be submitted for publication elsewhere.

The first author was supported in part by the MINECO/FEDER grant MTM2008-03437, MINECO grant MTM2013-40998-P, the AGAUR grant 2014SGR-568, the ICREA Academia, grants FP7-PEOPLE-2012-IRSES 318999 and 316338, the MINECO/FEDER grant UNAB13-4E-1604, and the CAPES grant 88881.030454/2013-01 from the program CSF-PVE.

The second author was supported by CAPES.

called a *Kukles polynomial differential system* by Giné in [4]. In [11], Volokitin and Ivanov raised the following open question about such differential systems:

Open Question. *Is it true that a differential system of the form* (1.1) *with polynomial nonlinearity* Q_n *of degree higher than* 2 *has a center at the origin if and only if its vector field is symmetric with respect to one of the coordinate axes*?

For degrees $n = 2, 3$ the authors of the open problem knew that the answer is positive. A positive answer to this question for degrees $n = 4, 5$ was given in [4]. In [5], Giné, Llibre and Valls proved that this conjecture holds for $n \geq 5$ odd, and in [6] they proved the conjecture for $n \geq 6$ even.

More precisely, in [5] the authors proved that for a planar differential system of the form (1.1), the origin is a center if and only if its vector field is symmetric with respect to one of the coordinate axes.

In this work we use this information to classify topologically the global phase portraits of differential systems

$$\begin{aligned} \dot{x} &= - y, \\ \dot{y} &= x + ax^5 + bx^3y^2 + cxy^4, \end{aligned} \tag{1.2}$$

in the Poincaré disk, where a, b, c are real parameters. Note that systems (1.2) are invariant under the change of coordinates $(t, x, y) \mapsto (-t, -x, y)$. That is, the phase portrait is symmetric with respect to the y-axis.

In [1], [8], [12]–[14], there were classified the global phase portraits of linear systems with homogeneous nonlinearities of degree 3, so in particular the phase portraits of the Kukles systems (1.1) of degree 3. Moreover, in [2] and [7], there have been classified the global phase portraits of the Kukles systems (1.1) of degree 4.

Our main result is the following one.

Theorem 1.1. *The Poincaré compactification of system* (1.2) *is topologically equivalent to the Poincaré compactification of one of the following systems*:

(i) $\dot{x} = -y$, $\dot{y} = x$,
(ii) $\dot{x} = -y$, $\dot{y} = x + Axy^4$,
(iii) $\dot{x} = -y$, $\dot{y} = x + Ax^3y^2$,
(iv) $\dot{x} = -y$, $\dot{y} = x + Ax^3y^2 + Cxy^4$,
(v) $\dot{x} = -y$, $\dot{y} = x + Ax^5$,
(vi) $\dot{x} = -y$, $\dot{y} = x + Ax^5 + Cxy^4$,
(vii) $\dot{x} = -y$, $\dot{y} = x + Ax^5 + Bx^3y^2$,
(viii) $\dot{x} = -y$, $\dot{y} = x + Ax^5 + Bx^3y^2 + Cxy^4$,

for an appropriate choice of $A \in \{-1, 1\}$, $B, C \in \mathbb{R} \setminus \{0\}$. *The phase portraits in the Poincaré disk of systems* (i)–(viii) *are topologically equivalent to one of* 22