MULTIPLICITY THEOREMS
FOR RESONANT AND SUPERLINEAR
NONHOMOGENEOUS ELLIPTIC EQUATIONS

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ABSTRACT. We consider nonlinear elliptic equations driven by the sum of a p-Laplacian ($p > 2$) and a Laplacian. We consider two distinct cases. In the first one, the reaction $f(z, \cdot)$ is $(p-1)$-linear near $\pm\infty$ and resonant with respect to a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We prove a multiplicity theorem producing three nontrivial solutions. In the second case, the reaction $f(z, \cdot)$ is $(p-1)$-superlinear but does not satisfy the Ambrosetti–Rabinowitz condition. We prove two multiplicity theorems. In the first main result we produce six nontrivial solutions all with sign information and in the second theorem we have five nontrivial solutions. Our approach uses variational methods combined with the Morse theory, truncation methods, and comparison techniques.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Dirichlet problem:

$$(1.1) \qquad -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0, \quad 2 < p < \infty.$$

Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

If $p = 2$, then $\Delta_2 = \Delta =$ the Laplace differential operator.

The reaction f is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). Additional regularity conditions on $f(z, \cdot)$ are introduced in order to produce extra solutions. We consider two distinct cases. In the first (Section 3), we assume that $f(z, \cdot)$ is $(p-1)$-linear near $\pm\infty$ and interacts with a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (resonant problem). We prove a multiplicity theorem, producing three nontrivial solutions two of which have constant sign. In the second case (Section 4), we deal with reaction $f(z, \cdot)$ which is $(p-1)$-superlinear near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). We also assume that $f(z, \cdot)$ has z-dependent zeros of constant sign. We prove a multiplicity theorem producing six nontrivial solutions, four of constant sign and two nodal (sign changing).

Recently nonhomogeneous nonlinear equations driven by the sum of a p-Laplacian and a Laplacian (a $(p,2)$-equation for short), were studied by Cingolani and Degiovanni [7], Papageorgiou and Rădulescu [21], Papageorgiou and Smyrlis [22] and Sun [24]. All these works deal with equations that have a $(p-1)$-linear reaction and either they do not allow resonance (see [7]) or the resonance is with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see [21], [22], [24]). Recall that for $p \neq 2$, we do not have a complete knowledge of the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$, the eigenspaces are not linear subspaces and we do not have a direct sum decomposition of $W_0^{1,p}(\Omega)$ in terms of them. All these facts, make problems resonant at higher parts of the spectrum difficult to deal with. On the other hand, problems with reactions which have zeros of constant sign, were investigated only in the context of p-Laplacian equations, by Bartsch, Liu and Weth [5] (constant zeros) and by Iturriaga, Massa, Sanchez and Ubilla [17] (variable zeros for a class of parametric equations). None of the aforementioned works produces six nontrivial solutions all with sign information.

Our approach is a combination of variational methods based on the critical point theory together with the Morse theory (critical groups) and truncation and comparison techniques. In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the sequel.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the Cerami condition (the C-condition for short), if the following holds: