Topological Methods in Nonlinear Analysis
Volume 48, No. 1, 2016, 213–233
DOI: 10.12775/TMNA.2016.043

EIGENVALUES, GLOBAL BIFURCATION AND POSITIVE SOLUTIONS FOR A CLASS OF NONLOCAL ELLIPTIC EQUATIONS

Guowei Dai

Abstract. In this paper, we shall study global bifurcation phenomenon for the following Kirchhoff type problem:

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda u + h(x,u,\lambda) & \text{in } \Omega,\\ u=0 & \text{on } \Omega. \end{cases}$$

Under some natural hypotheses on h, we show that $(a\lambda_1, 0)$ is a bifurcation point of the above problem. As an application of the above result, we shall determine the interval of λ, in which there exist positive solutions for the above problem with $h(x,u;\lambda) = \lambda f(x,u) - \lambda u$, where f is asymptotically linear at zero and asymptotically 3-linear at infinity. To study global structure of bifurcation branch, we also establish some properties of the first eigenvalue for a nonlocal eigenvalue problem. Moreover, we provide a positive answer to an open problem involving the case $a = 0$.

2010 *Mathematics Subject Classification.* 35B20, 35B32, 35H99, 35R15.

Key words and phrases. Bifurcation; eigenvalue; Kirchhoff type equation; positive solutions.

Research supported by NNSF of China (No. 11261052, 11401477), the Fundamental Research Funds for the Central Universities (No. DUT15RC(3)018) and Scientific Research Project of the Higher Education Institutions of Gansu Province (No. 2014A-009).

1. Introduction

In this paper, we study global bifurcation phenomenon for the following problem:

$$
(1.1) \qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda u + h(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $a, b > 0$ are real constants, λ is a parameter and $h\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variable and

$$
(1.2) \qquad \lim_{s\to 0} \frac{h(x,s,\lambda)}{s} = 0
$$

uniformly for almost every $x \in \Omega$ and λ on bounded sets. Moreover, we assume that h satisfies the following growth restriction:

(G) there exist $c > 0$ and $p \in (1, 2^*)$ such that

$$
|h(x,s,\lambda)| \le c(1 + |s|^{p-1})
$$

for almost every $x \in \Omega$ and λ on bounded sets, where

$$
2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N \le 2. \end{cases}
$$

Problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe transversal oscillations of a stretched string [27]. More precisely, Kirchhoff proposed a model given by the equation

$$
\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x,u),
$$

where ρ, ρ_0, h, E, L are constants, f is an external force, which extends the classical D'Alembert's wave equation by considering the effect of changing in length of the string during vibration. Problem (1.1) received much attention only after Lions [31] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [2], [4], [15], [14], [26]. Recently, many mathematicians were studying problem (1.1) by variational methods, see e.g. [5], [6], [32], [33], [35], [38], [41] and references therein.

The authors of [28] studied problem (1.1) with $h(x,s,\lambda) = \lambda f(x,s) - \lambda s$ by using the topological degree argument and variational method. Under some assumptions on f, they provided a positive answer to the existence of positive solutions to (1.1) for the cases $a, b > 0$ and $a > 0$, $b = 0$. They pointed out that the *case* $a = 0$ *and* $b > 0$ *is an open problem.* The study of Kirchhoff type