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DECENTLY REGULAR STEADY SOLUTIONS TO THE COMPRESSIBLE NSAC SYSTEM

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Abstract. We aim at proving existence of weak solutions to the stationary compressible Navier–Stokes system coupled with the Allen–Cahn equation – NSAC system. The model is studied in a bounded three dimensional domain with slip boundary conditions for the momentum equations and the Neumann condition for the Allen–Cahn model. The main result establishes existence of weak solutions with bounded densities. The construction is possible assuming sufficiently large value of the heat capacity ratio γ ($p \sim \varrho^\gamma$). As a corollary we obtain weak solutions for a less restrictive case losing pointwise boundedness of the density.

1. Introduction

Phase transition phenomena are an important subject of applied mathematics. Complexity of physical nature of these processes makes us to choose different models in order to obtain the best description in a concrete studied case. Here we want to concentrate our attention on phenomena with fuzzy phase interfaces. We consider the process of melting or freezing, at the level of almost constant critical temperature. We want to control densities of phase constituents, and

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here we arrive at the Allen–Cahn equation:

$$\varrho \frac{\partial f}{\partial c}(\varrho, c) - \Delta c = \varrho\mu. \tag{1.1}$$

The equation above gives us control of one of the phase constituents in terms of the chemical potential μ. In the chosen model this equation is coupled with the compressible Navier–Stokes system, which represents a well established and frequently studied model describing the flows of viscous compressible single constituted fluids.

The mathematical analysis of coupled systems, the compressible Navier–Stokes type and the phase separation, is in its infancy, see [1], [11], [17], [18], [7], [6], although the mathematical theory of each of them separately is quite developed (see e.g. [9], [14], [30]). In this article, we study existence of steady weak solutions. The thermodynamically consistent derivation of the model under consideration, which is a variant of a model proposed by Blesgen [2], was presented by Heida, Málek and Rajagopal in [15]. It is represented by the following system of partial differential equations for three unknowns, density of the fluid ϱ, velocity field $\mathbf{v}$, and concentration of one selected constituent c:

$$\operatorname{div}(\varrho\mathbf{v}) = 0, \tag{1.2}$$

$$\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) = \operatorname{div}\mathbb{T} + \varrho\mathbf{F}, \tag{1.3}$$

$$\operatorname{div}(\varrho c\mathbf{v}) = -\mu, \tag{1.4}$$

$$\varrho\mu = \varrho\frac{\partial f}{\partial c}(\varrho, c) - \Delta c, \tag{1.5}$$

where the stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(\nabla\mathbf{v}, \nabla c, \varrho, c) = \mathbb{S}(\nabla\mathbf{v}) - \left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\,\mathbb{I}\right) - p(\varrho, c)\mathbb{I},$$

with the thermodynamical pressure $p(\varrho, c) = \varrho^2\frac{\partial f}{\partial\varrho}$ and the viscous stress $\mathbb{S}$ satisfying the Stokes law for the Newtonian fluid

$$\mathbb{S}(\nabla\mathbf{v}) = \nu\left(\nabla\mathbf{v} + \nabla^T\mathbf{v}\right) + \eta\operatorname{div}\mathbf{v}\mathbb{I},$$

with viscosity coefficients $\nu > 0$, and $\eta \geq -2\nu/3$. The free energy is assumed to have the form with the so-called logarithmic potential

$$f(\varrho, c) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + \big(a_1 c + a_2(1-c)\big)\log\varrho + c\log c + (1-c)\log(1-c) + b(c)$$

with some positive γ, $a_1, a_2 \geq 0$, b a smooth bounded function with $|b'(c)| \leq C$. Moreover, we assume without loss of generality that $a_1 \geq a_2$ and we denote, for the sake of simplicity, $a = a_1 - a_2$, $d = a_1$ and $L(c) = c\log c + (1-c)\log(1-c)$. The logarithmic terms related to the entropy of the system assure that $c \in [0,1]$