Topological Methods in Nonlinear Analysis
Volume 48, No. 1, 2016, 31–43
DOI: 10.12775/TMNA.2016.034

© 2016 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

AN EXISTENCE RESULT FOR A NONLINEAR BOUNDARY VALUE PROBLEM VIA TOPOLOGICAL ARGUMENTS

Khadijah Sharaf

Abstract. We investigate a nonlinear PDE related to the prescribing mean curvature problem on the boundary of the unit ball. We use variational and topological methods to prove the existence of at least one solution when the function to be prescribed satisfies at its critical points a non-degeneracy condition.

1. Introduction and main results

In this paper we consider the problem of existence of conformal scalar flat metric with prescribed boundary mean curvature on the unit n-dimensional ball. To be more specific, let $\mathbb{B}^n$ be the unit ball of $\mathbb{R}^n$, $n \geq 3$, with Euclidean metric g_0. Its boundary will be denoted by $\mathbb{S}^{n-1}$ and will be endowed with the standard metric still denoted by g_0. Let $H\colon \mathbb{S}^{n-1} \to \mathbb{R}$ be a given function, we study the problem of finding a conformal metric $g = u^{4/(n-2)}g_0$ (here u is a smooth positive function and the exponent $4/(n-2)$ is used to make the next equation simpler) such that $R_g = 0$ in $\mathbb{B}^n$ and $h_g = H$ on $\mathbb{S}^{n-1}$. Here R_g is the scalar curvature associated to the metric g in $\mathbb{B}^n$ and h_g is the mean curvature of g

2010 *Mathematics Subject Classification.* Primary: 58E05, 35J65; Secondary: 53C21, 35B40.

Key words and phrases. Conformal metric; boundary mean curvature; lack of compactness; critical points at infinity; stable and unstable manifolds; retracts by deformation.

on $\mathbb{S}^{n-1}$. This problem is equivalent to solving the following nonlinear boundary value equation:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{B}^n, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H u^{n/(n-2)} & \text{on } \mathbb{S}^{n-1}, \end{cases} \tag{1.1}$$

where ν is the outward unit normal vector on $\mathbb{S}^{n-1}$ with respect to the metric g_0, see e.g. [5].

Equation (1.1) enjoys a variational structure. A natural space to look in for solutions is the Sobolev space $H^1(\mathbb{B}^n)$. Recall that by the regularity result of P. Cherrier [12], a weak solution of (1.1) is indeed a smooth solution. In equation (1.1), the exponent is $N-1$, where $N = 2(n-1)/(n-2)$ is the critical case of the trace Soboev embedding $H^1(\mathbb{B}^n) \to L^q(\mathbb{S}^{n-1})$. In virtue of the lack of compactness of this embedding, the Euler–Lagrange functional J associated to our problem (see Section 2 for the definition of J) fails to satisfy the Palais–Smale condition; that is there exist non-compact sequences along which the functional is bounded and its gradient goes to zero. From the variational view point, it is the occurrence of critical points at infinity, that are the limit of the non-compact orbits of the gradient flow of J (see Definition 2.1 below for more precision). This prevents the use of standard variational methods to prove existence of solutions. Moreover, besides the obvious necessary condition that H should be positive somewhere, there is at least another obstruction to solving the problem, the so-called Kazdan–Warner condition [20].

Problem (1.1) has been studied by Escobar and Garcia [16] in dimension 3, who proved that blow-ups of solutions of subcritical approximations occur at one point and gave an index-count formula reminiscent to the one given by Bahri and Coron [8] and Chang, Gursky and Yang [10] for the prescribed scalar curvature on three dimensional sphere. We point out that the index formula of [16] has an equivalent in dimension 4, see [13] and [2]. However, the method cannot be generalized to higher dimension $n \geq 5$ under the non-degeneracy condition, since the corresponding index-count criteria, when taking into account all critical points at infinity is always equal to 1. There have been many works devoted to the existence results, trying to understand under what condition (1.1) is solvable. For details see [1]–[4], [11], [13]–[19], [22] and the references therein.

Motivated by the work of [1] and [4], and aiming to include a larger class of functions H in the existence results for (1.1), we develop in this paper a topological approach which enables us to provide sufficient conditions on H weaker than those of [2], [3], [11], [13], [16] to obtain solution of (1.1) for every dimension $n \geq 3$. Our method hinges on the theory of critical points at infinity of A. Bahri [6].