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HARMONIC AND SUBHARMONIC SOLUTIONS FOR SUPLINEAR DUFFING EQUATION WITH DELAY

Zhibo Cheng — Jingli Ren

Abstract. We study the existence of harmonic and subharmonic solutions for the suplinear Duffing equation with delay. Our proofs are based on the twisting theorem due to W.Y. Ding.

1. Introduction

Consider the existence of harmonic and subharmonic solutions of the Duffing equation with delay

$$x''(t) + g(x(t-\tau)) = p(t), \tag{1.1}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, τ is a positive constant, $p\colon \mathbb{R} \to \mathbb{R}$ is continuous with $T > 0$ the minimal period.

The periodic problem of Duffing equations ($\tau = 0$, that is, without delay) has been widely studied lately because of its significance for the applications as well as for its intrinsic interest as a good model for testing the effectiveness of various technical tools of nonlinear analysis [1]–[3], [5]–[15]. For example, the oscillation problem of a spherical thick shell made of an elastic material can also be modeled

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by Duffing equations [12]. The focusing system of an electron beam immersed in a periodic magnetic field can be also modeled by this kind equation [3].

In [2], Ding, Iannacci, Zanolin study the Duffing equation (without delay)

$$x'' + g(x) = p(t) \tag{1.2}$$

for the so-called semilinear case, i.e., when $0 < A \leq g(x)/x \leq B < \infty$. Conditions of nonresonance type are developed under which there exist solutions of period nT, where n is a positive integer. Results are first obtained in terms of certain properties of the period of solutions of the unforced equation

$$x'' + g(x) = 0.$$

Then by relating properties of g to periods of solutions of this unforced equation, some results for such periodic solutions of the forced equation are obtained directly in terms of properties of g.

In [13], Qian deals with existence of multiple periodic solutions for (1.2). The case with jumping nonlinearity, i.e.

$$g(u) = a\max\{u,0\} - b\max\{-x,0\} + h(u),$$

where a,b are nonnegative real constants and $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, is considered. An interesting summary concerning both main existing results and the employed tools is presented in the introduction. Under suitable conditions on the growth at infinity of the function h (conditions which are more general than similar ones in the quoted paper by Dancer) and assuming that

$$\lim_{x\to\infty} x^{-2}\int_0^x g(u)du = \frac{a}{2},$$
$$\lim_{x\to-\infty} x^{-2}\int_0^x g(u)du = \frac{b}{2}\liminf_{|x|\to\infty} x^{-1}g(x) \geq \alpha > 0,$$

the author obtained sufficient conditions for existence both of 2π-periodic solutions and of infinitely many subharmonic solutions with arbitrarily large amplitude. The arguments employed are based on suitable estimates for the successor map and a generalization of the Poincaré–Birkhoff twist theorem.

In the above papers, the authors investigated the Duffing equation (without delay). However, the study of the delay Duffing equation is relatively infrequent. Motivated by [2], [8], [13], this paper aims to study further the existence of harmonic and subharmonic solutions of (1.1). Assume that g satisfies the following condition:

(S_p) (superlinear case) $g(x(t-\tau))/x(t-\tau) \to +\infty$ as $|x(t-\tau)| \to +\infty$.

By using the phase-plane analysis method and the twisting theorem [4], we obtain the following results.

Theorem 1.1. *Assume that condition* (S_p) *holds. Then there is at least one harmonic solution for* (1.1).

Theorem 1.2. *Assume that condition* (S_p) *holds. Then for an arbitrary integer $m > 1$, there is at least one mT-periodic solution* (*subharmonic solution*) *for* (1.1).

2. Basic lemmas for delay Duffing equation

Firstly, we apply the twisting theorem for the nonarea-preserving Poincaré map proposed by Ding [4].

Let A be an annular region: $R_1 \le r \le R_2$, $0 < R_1 < R_2$; and $\mathcal{D}(R), \mathcal{D}[R]$ be, respectively, an open and closed discs in the plane (i.e. two-dimensional balls), with center at the origin O and radius $R > 0$. F is called a twist in A if the map $F\colon A \to \mathbb{R} - \{0\}$ has the form $r^* = f(r,\theta)$, $\theta^* = \theta + g(r,\theta)$, where f and g are continuous on A and T-periodic in θ, and $g(R_1,\theta) \cdot g(R_2,\theta) < 0$ ("twist condition").

Lemma 2.1. *Assume that $F\colon \mathcal{D}[R_2] \to \mathbb{R}^2$ is a continuous map. If F is twist in A, then F has at least one fixed point in $\mathcal{D}(R_2)$.*

Lemma 2.2. *A periodic solution of* (1.1) *is either a harmonic solution or a subharmonic solution.*

Proof. Let $x = \psi(t)$ be a periodic solution of (1.1) and $\kappa > 0$ its minimal period. Since

$$\psi''(t) + g(\psi(t-\tau)) = p(t),$$

we have

$$\psi''(t+\kappa) + g(\psi(t-\tau+\kappa)) = p(t+\kappa),$$

hence

$$\psi''(t) + g(\psi(t-\tau)) = p(t+\kappa), \quad \text{for all } t \in \mathbb{R},$$

thus $p(t+\kappa) \equiv p(t)$. Since T is the minimal period of p, there is $m \ge 1$ such that $\kappa = mT$. Hence, when $m = 1$, $x = \psi(t)$ is a harmonic solution of (1.1); when $m > 1$, $x = \psi(t)$ is a subharmonic solution of (1.1). $\square$

Consider the following system equivalent to (1.1):

$$x' = y, \quad y' = -g(x(t-\tau)) + p(t). \tag{2.1}$$

Denote by $(x(t), y(t)) = (x(t;x_0,y_0), y(t;x_0,y_0))$ the solution of (2.1) satisfying the initial value condition

$$x(0;x_0,y_0) = x_0, \qquad y(0;x_0,y_0) = y_0.$$

We assume that g satisfies the following condition:

(g_0) $\displaystyle\lim_{|x(t-\tau)|\to+\infty} g(x(t-\tau)) = +\infty.$

Obviously, superlinear condition (S_p) implies condition (g_0), but the inverse does not hold.

Lemma 2.3. *Assume that condition* (g_0) *holds. Then every solution* $(x(t), y(t))$ *of* (2.1) *is defined on the whole* t*-axis.*

Proof. Set

$$H(t) = \int_0^t g(x(s-\tau))y(s)\,ds, \qquad P(t) = \int_0^t p(s)y(s)\,ds.$$

Define the potential function

$$V(t) = V(x(t), y(t)) = \frac{1}{2}y^2(t) + H(t) - P(t).$$

Then

$$\begin{aligned} V'(t) &= y(t)y'(t) + g(x(t-\tau))y(t) - p(t)y(t) \\ &= y(t)(y'(t) + g(x(t-\tau)) - p(t)) = 0. \end{aligned} \tag{2.2}$$

From (2.2), we have $V(t) = C$, here C is a constant. Thus

$$|V'(t)| \leq V(t) + M_2, \tag{2.3}$$

where $M_2 \geq |C|$. From (2.3) we have that, for $t \in [t_0, t_0 + \tau)$ with $t_0 \in \mathbb{R}$, $\tau > 0$,

$$V(t) \leq V(t_0)e^{\tau} + M_2 e^{\tau} \tag{2.4}$$

which implies that there is no blow-up for solution $(x(t), y(t))$ in any finite interval $[t_0, t_0 + \tau)$. □

Since g is locally Lipschitz continuous, from Lemma 2.3, we know that $(x(t), y(t))$ exists uniquely on the whole t-axis. Define a function $\mathcal{R}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$,

$$\mathcal{R}(x, y) = x^2 + y^2.$$

Lemma 2.4. *Assume that condition* (g_0) *holds. Then there exists a positive constant* $m \in \mathbb{N}^+$ *such that* $\mathcal{R}(x(t), y(t)) \to +\infty$ *for* $t \in [0, mT]$ *as* $\mathcal{R}(x_0, y_0) \to +\infty$.

Proof. From (2.4), it is easy to see that

$$(V(0) + A_1)e^{-T} \leq V(t) \leq (V(0) + A_1)e^{T},$$

for $t \in [0, mT]$, which shows that $\mathcal{R}(x(t), y(t)) \to +\infty$ uniformly for $t \in [0, T]$ as $\mathcal{R}(x_0, y_0) \to +\infty$. □

Remark 2.5. For a fixed constant $R_1 > 0$, there is a constant $R_2 \geq R_1$ such that

$$\begin{aligned} \mathcal{R}(x_0, y_0) \leq R_1 \quad &\Rightarrow \quad \mathcal{R}(x(t), y(t)) \leq R_2, \quad \text{for } t \in [0, mT]; \\ \mathcal{R}(x_0, y_0) \geq R_2 \quad &\Rightarrow \quad \mathcal{R}(x(t), y(t)) \geq R_1, \quad \text{for } t \in [0, mT]. \end{aligned}$$

From Lemma 2.4 we know that if $\mathcal{R}(x_0, y_0)$ is large enough, then $x^2(t) + y^2(t) \neq 0$, $t \in [0, mT]$, with T given in Lemma 2.4. Hence we can make a polar coordinate transformation $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$ when $\mathcal{R}(x_0, y_0)$ is large enough. Under this transformation, (2.1) becomes

$$\begin{cases} \dot{r} = [r\cos\theta - g(r(t-\tau)\cos\theta(t-\tau)) - p(t)]\sin\theta, \\ \dot{\theta} = -\left[\sin^2\theta + \dfrac{1}{r}(g(r(t-\tau)\cos\theta(t-\tau)) - p(t))\cos\theta\right]. \end{cases} \tag{2.5}$$

Denote by $(r(t), \theta(t)) = (r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ the solution of (2.5) satisfying $r(0) = r_0$ and $\theta(0) = \theta_0$. Let $\mathcal{D} = \{(r, \theta) \mid r > 0,\ \theta \in \mathbb{R}\}$, then the Poincaré map $\mathcal{P} \colon \mathcal{D} \to \mathcal{D}$, defined by

$$\mathcal{P} \colon (r_0, \theta_0) \mapsto (r_1, \theta_1) = (r(T; r_0, \theta_0), \theta(T; r_0, \theta_0)),$$

is continuous on $\mathcal{D}$ and it is a homeomorphism from $\mathcal{D}$ into itself. Clearly, fixed points of the map $\mathcal{P}$ correspond to periodic solutions of (2.1). Next, we will try to prove that $\mathcal{P}$ has a fixed point.

LEMMA 2.6. *Assume that* (g_0) *holds. Then there exists a positive constant* ρ_0 *such that, for* $\mathcal{R}(x_0, y_0) \geq \rho_0$, $\theta'(t) < 0$, $t \in [0, mT]$.

PROOF. It follows from (g_0) that there exists a positive constant N such that

$$g(x(t-\tau)) - p(t) > 0, \quad |x(t-\tau)| \geq N, \quad t \in [0, mT].$$

STEP 1. If $x(t-\tau) \geq N$, we know that $\cos\theta(t-\tau) > 0$, and $y(t) \geq 0$ or $y(t) \leq 0$.

Case 1. If $y(t) \geq 0$, we know that $x(t) \geq x(t-\tau) \geq N$. So, $\cos\theta(t) > 0$. Then,

$$\frac{d\theta}{dt} = -\left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)}\cos\theta\right] < 0.$$

Case 2. If $y(t) \leq 0$, we know that $-\pi/2 \leq \theta(t) \leq 0$. So, $\cos\theta(t) > 0$. Then,

$$\frac{d\theta}{dt} = -\left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)}\cos\theta\right] < 0.$$

STEP 2. Similarly, if $x(t-\tau) \leq -N$, we have

$$\frac{d\theta}{dt} = -\left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)}\cos\theta\right] < 0.$$

STEP 3. If $|x(t-\tau)| \leq N$, for y sufficiently large (then r is sufficiently large), we have $\sin^2\theta > 1/2$.

$$\sin^2\theta > \frac{1}{2}, \quad \frac{|g(x(t-\tau)) - p(t)|}{r(t)} \leq \frac{1}{4}, \quad t \in [0, mT].$$

So, we get

$$\frac{d\theta}{dt} = -\left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)}\cos\theta\right] < 0.$$

Summing up the above cases, we have $d\theta/dt < 0$ whenever $r \gg 1$. $\square$

Lemma 2.6 implies that $\theta(t)$ decreases strictly when r is large enough. Denote by $\tau(r_0, \theta_0)$ the time its takes for the solution $(r(t), \theta(t))$ to make one turn around the origin.

Next, we proof a basic lemma. It has played a very important role in the harmonic and subharmonic solutions of delay Duffing equation (1.1).

Lemma 2.7. *Assume that condition (S_p) holds, let $m \in \mathbb{N}^+$. Then for an arbitrary large integer $N \in \mathbb{N}^+$, there exists a large enough constant $\Lambda_0 > 0$ such that*

$$\theta(mT; r_0, \theta_0) - \theta_0 < -2N\pi$$

for $(x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \{(x, y) \mid \mathcal{R}(x, y) = \Lambda,\ \Lambda \geq \Lambda_0\}$.

Proof. Let $z = (x, y)$ and set Γ_{z_0} be such that $z = z(t) = (x(t), y(t))$, $t \in \mathbb{R}$, is a solution of (2.1) satisfying the initial value condition $z(0) = z_0 = (x_0, y_0)$.

Without loss of generality, there exist a large enough constant c_0 and a positive constant $\varepsilon \ll 1$ such that $R_1 := R_1(\varepsilon) > ((2c_0)^2 + \varepsilon^2 c_0^2)/\varepsilon^2$. Furthermore, let $R_2 := R_2(\varepsilon) > R_1$. From Remark 2.5, we know that $\mathcal{R}(x, y) \geq R_1$ if $\mathcal{R}(x_0, y_0) \geq R_2$.

When $0 \leq t \leq mT$, we choose $0 = t_0 < t_1 < \ldots < t_6$ so that

- $D_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x(t - \tau)| \leq c_0,\ y(t) \geq 0,\ t \in [t_0, t_1]\}$;
- $D_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \geq c_0,\ 0 \leq y(t) < +\infty,\ t \in [t_1, t_2]\}$;
- $D_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \geq c_0,\ -\infty < y \leq 0,\ t \in [t_2, t_3]\}$;
- $D_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x(t - \tau)| \leq c_0,\ y \leq 0,\ t \in [t_3, t_4]\}$;
- $D_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \leq -c_0,\ -\infty < y(t) \leq 0,\ t \in [t_4, t_5]\}$;
- $D_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \leq -c_0,\ 0 \leq y(t) < +\infty,\ t \in [t_5, t_6]\}$.

From Lemma 2.6, we know that there exist t_i, $i = 0, \ldots, 6$, such that t_i satisfy the above properties for large enough $\mathcal{R}(x_0, y_0)$. Next, we shall estimate $t_1 - t_0$, $t_2 - t_1$, $t_3 - t_2$, $t_4 - t_3$, $t_5 - t_4$, $t_6 - t_5$, respectively.

Step 1. Estimate of $t_1 - t_0$, $t_4 - t_3$. From the first equation in (2.1) and the choice of R_1, we have

$$x'(t - \tau) = y(t - \tau) \geq (R_1 - c_0^2)^{1/2} \geq \frac{2c_0}{\varepsilon},$$

and thus we obtain

$$2c_0 \geq x(t_1 - \tau) - x(t_0 - \tau) = \int_{t_0}^{t_1} x'(t - \tau)\, dt \geq \frac{2c_0(t_1 - t_0)}{\varepsilon}.$$

Hence $t_1 - t_0 \leq \varepsilon$. With obvious changes in the proof, we can obtain the estimate $t_4 - t_3 \leq \varepsilon$.

Step 2. Estimate of t_2-t_1, t_6-t_5. For $x^2+y^2=r^2$, $0\le y<+\infty$, we know that there exists large enough B such that $0\le y(t)\le B<+\infty$. From (S_p), we know that there exists large enough K such that

$$\frac{g(x(t-\tau))}{x(t-\tau)}\ge\frac{K(c_0+B\tau)}{c_0}:=K_1,\quad |x(t-\tau)|\ge c_0.$$

Let $t\in[t_2,t_1]$.

$$t_1-t_0=\int_{\theta(t_1)}^{\theta(t_0)}\frac{d\theta}{\sin^2\theta+(g(x(t-\tau))-p(t)/r(t))\cos\theta}.$$

As $x(t)$ is increasing and $0\le y(t)\le B<+\infty$, we have $x(t)\ge x(t-\tau)\ge c_0$ and

$$x(t)-x(t-\tau)=\int_{t-\tau}^{t}x'(s)\,ds\le B\tau,$$

i.e.

$$\frac{x(t)}{x(t-\tau)}-1\le\frac{B\tau}{x(t-\tau)}\le\frac{B\tau}{c_0}.$$

So,

$$\frac{x(t-\tau)}{x(t)}\ge\frac{c_0}{c_0+B\tau}. \tag{2.6}$$

From (2.6), for $r(t)$ large enough we have

$$\begin{aligned}
\sin^2\theta\ &+\frac{g(r(t-\tau)\cos\theta(t-\tau))-p(t)}{r(t)}\\
&=\sin^2\theta+\frac{g(r(t-\tau)\cos\theta(t-\tau))}{r(t-\tau)\cos\theta(t-\tau)}\cdot\frac{r(t-\tau)\cos\theta(t-\tau)}{r(t)\cos\theta(t)}\cos^2\theta-\frac{p(t)}{r(t)}\cos\theta\\
&\ge\sin^2\theta+K_1\cdot\frac{c_0}{c_0+B\tau}\cos^2\theta-\frac{p(t)}{r(t)}\cos\theta\\
&=\sin^2\theta+K\cos^2\theta-\frac{p(t)}{r(t)}\cos\theta\\
&=(\sin^2\theta+K\cos^2\theta)\left(1-\frac{p(t)}{r(t)(\sin^2\theta+K\cos^2\theta)}\cos\theta\right)\\
&=\sin^2\theta+K\cos^2\theta+o(1).
\end{aligned}$$

So,

$$\begin{aligned}
t_1-t_0&\le\int_0^{\pi/2}\frac{d\theta}{\sin^2\theta+K\cos^2\theta}+o(1)\\
&=\frac{1}{\sqrt{K}\arctan(\tan\theta/\sqrt{K})}\bigg|_0^{\pi/2}+o(1)=\frac{\pi}{2\sqrt{K}}+o(1)\ll1.
\end{aligned}$$

Similarly, we can estimate

$$t_6-t_5\le\frac{\pi}{2\sqrt{K}}+o(1)\ll1.$$

Step 3. Estimate of $t_3 - t_1$, $t_5 - t_4$.

$$t_3 - t_2 = \int_{\theta(t_3)}^{\theta(t_2)} \frac{d\theta}{\sin^2\theta + (g(x(t-\tau)) - p(t)/r(t))\cos\theta}.$$

Since $y(t) \le 0$, we have $0 < x(t) \le x(t-\tau)$, so there exists a constant $\sigma > 1$ such that

$$\frac{x(t-\tau)}{x(t)} \ge \sigma. \tag{2.7}$$

From (2.7), for $r(t)$ large enough we have

$$\begin{aligned}
&\sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau)) - p(t)}{r(t)} \\
&= \sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau))}{r(t-\tau)\cos\theta(t-\tau)} \cdot \frac{r(t-\tau)\cos\theta(t-\tau)}{r(t)\cos\theta(t)}\cos^2\theta - \frac{p(t)}{r(t)}\cos\theta \\
&\ge \sin^2\theta + K_1\sigma\cos^2\theta - \frac{p(t)}{r(t)}\cos\theta \\
&= \sin^2\theta + K_1\sigma\cos^2\theta - \frac{p(t)}{r(t)}\cos\theta \\
&= (\sin^2\theta + K_1\sigma\cos^2\theta)\bigg(1 - \frac{p(t)}{r(t)(\sin^2\theta + K_1\sigma\cos^2\theta)}\cos\theta\bigg) \\
&= \sin^2\theta + K_1\sigma\cos^2\theta + o(1).
\end{aligned}$$

So,

$$\begin{aligned}
t_3 - t_2 &\le \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta + K_1\sigma\cos^2\theta} + o(1) \\
&= \frac{1}{\sqrt{K_1\sigma}\arctan(\tan\theta/\sqrt{K_1\sigma})}\bigg|_0^{\pi/2} + o(1) = \frac{\pi}{2\sqrt{K_1\sigma}} + o(1) \ll 1.
\end{aligned}$$

Similarly, we can estimate

$$t_5 - t_4 \le \frac{\pi}{2\sqrt{K_1\sigma}} + o(1) \ll 1.$$

Furthermore,

$$\begin{aligned}
\tau(r_0,\theta_0) &= (t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + (t_4 - t_3) + (t_5 - t_4) + (t_6 - t_5) \\
&< 2\varepsilon + \frac{\pi}{\sqrt{K_1\sigma}} + \frac{\pi}{\sqrt{K}} + o(1) \ll 1.
\end{aligned}$$

Let constants $s_1 < s_2$ satisfy $\theta(s_2; r_0, \theta_0) - \theta(s_1; r_0, \theta_0) = -2\pi$, then $s_2 - s_1$ is the time needed for the path curve Γ_{z_0} to make one turn countclockwise around the origin O.

According to the above discussion, for an arbitrary small constant $\varepsilon > 0$, there exists sufficiently large $\mathcal{R}(x_0, y_0)$ such that the time needed for the path curve Γ_{z_0} to make one turn countclockwise around the origin O satisfies $0 < s_2 - s_1 < 7\varepsilon$. Since ε may be arbitrary small, in the time region $[0, mT]$ for the path curve Γ_{z_0} to make a turn countclockwise around the origin O of cycle may be arbitrary large. So, we get our result $\theta(mT; r_0, \theta_0) - \theta_0 < -2N\pi$. $\square$

3. Proof of Theorem 1.1

Let $z(t) = (x(t), y(t))$ satisfies (2.1) with the initial condition $z_0 = (x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0) = (r_0, \theta_0)$. Then $\theta(mT; r_0, \theta_0) = \theta(mT, z_0)$. Take $m = 1$ in Lemma 2.7 and consider

$$\Delta_1(z_0) = \theta(T, z_0) - \theta(0, z_0).$$

It is continuous at z_0. Firstly, we take an appropriately large constant a_1. Then there exists a positive integer K_1 such that $\inf \Delta_1(z_0) > -2K_1\pi$. So

$$\theta(T, z_0) - \theta(0, z_0) > -2K_1\pi, \quad |z_0| = a_1. \tag{3.1}$$

On the other hand, from Lemma 2.7, there exists a constant $b_1 > a_1$ such that

$$\theta(T, z_0) - \theta(0, z_0) < -2K_1\pi, \quad |z_0| = b_1. \tag{3.2}$$

Secondly, consider the Poincaré map associated to (2.1)

$$\mathcal{P}\colon \langle r_0, \theta_0\rangle \to \langle r(T, z_0), \theta(T, z_0)\rangle,$$

in the annular region $\mathcal{A}_1$ such that $a_1^* \le |z| \le b_1^*$. From (3.1) and (3.2), it is twist in $\mathcal{A}_1$. Therefore, by Lemma 2.1, there exists at least one fixed point for the map $\mathcal{P}$, i.e. $\zeta = \langle \varphi, \phi\rangle \in \mathcal{D}(b_1^*)$, with

$$\theta(T, \zeta) - \theta(0, \zeta) = -2K_1\pi. \tag{3.3}$$

Then $z = (t, \zeta)$ is a T-periodic solution of (2.1). Set $z = z(t) = z(t, \zeta)$. Therefore, there is at least one T-periodic solution $z = z(t)$, $z(0) = \zeta \in \mathcal{D}(b_1^*)$. Consequently, (1.1) has at least one harmonic solution.

4. Proof of Theorem 1.2

We will follow the same strategy and notations as the proof of Theorem 1.1. Consider $\Delta_2(z_0) = \theta(mT, z_0) - \theta(0, z_0)$. It is continuous about z_0. Firstly, we take appropriate large constant a_1. Obviously, there exists a prime positive integer q_1 such that $\inf\limits_{|z_0| = a_1} \Delta_2(z_0) > -2q_1\pi$. So, we have

$$\theta(mT, z_0) - \theta(0, z_0) > -2q_1\pi, \quad |z_0| = a_1. \tag{4.1}$$

On the other hand, from Lemma 2.7, there exists a constant $b_1 > a_1$ such that

$$\theta(mT, z_0) - \theta(0, z_0) < -2q_1\pi, \quad |z_0| = b_1. \tag{4.2}$$

Secondly, consider the m-th iteration of the Poincaré map associated to (2.1)

$$\mathcal{P}^m\colon \langle r_0, \theta_0\rangle \to \langle r(mT, z_0), \theta(mT, z_0)\rangle,$$

in the annular region $a_1^* \leq |z| \leq b_1^*$. From (4.1) and (4.2), it is twist in this annular region. Therefore, by Lemma 2.1, there exists at least one fixed point for the map $\mathcal{P}^m$, $\zeta = \langle \varphi, \phi \rangle \in \mathcal{D}(b_1^*)$, with

$$\theta(mT, \zeta) - \theta(0, \zeta) = -2q_1\pi. \tag{4.3}$$

Obviously, $z = z(t, \zeta)$ is an mT-periodic solution of (2.1). Since $\theta(t, \zeta)$ are decreasing for $t \in [0, mT]$, then $z = z(t, \zeta)$ turns clockwise q_1 times around the origin on $[0, mT]$.

Now we prove that the minimal period of $z = z(t, \zeta)$ is mT. Assume on the contrary that the minimal period of $z = z(t, \zeta)$ is nT, $0 < n < m$. Then ζ is an n-periodic point of $\mathcal{P}^m$ and n is the minimal period.

Let $m = sn + q$, here s and q are integers such that $s \geq 1$ and $0 \leq q \leq n$. As $\mathcal{P}^m(\zeta) = \zeta$ and $\mathcal{P}^n(\zeta) = \zeta$, we have $\mathcal{P}^q(\zeta) = \zeta$. Since ζ is an n-periodic point of $\mathcal{P}$ and n is the minimal period, q must be 0. So, we get $m = sn$ and $s > 1$.

On the other hand, since $\zeta \in \mathcal{D}(b_1^*)$ is an n-periodic point of $\mathcal{P}$ and n is the minimal period, $z = z(t, \zeta)$ turns clockwise N_0 times around the origin on $[0, nT]$. From Lemma 2.7, we know that for a_1^* sufficiently large there is $N_0 > 1$.

So, the periodic solution $z = z(t, \zeta)$ turns clockwise sN_0 times around the origin on $[0, mT]$. Thus, $q_1 = sN_0$, $N_0 > 1$ and $s > 1$, what contradicts to q_1 being prime. Therefore, the minimal period mT.

The end of the proof is the same as for Theorem 1.1. Consequently, (1.1) has at least one subharmonic solution.

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Zhibo Cheng
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo 454003, P.R. CHINA

E-mail address: czbo@hpu.edu.cn

Jingli Ren
School of Mathematics and Statistics
Zhengzhou University
Zhengzhou 450001, P.R. CHINA