

## ON THE TAIL PRESSURE

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**ABSTRACT.** In this paper, we give two equivalent definitions of tail pressure involving open covers and establish a variational principle which exhibits the relationship between tail pressure and measure-theoretic tail entropy.

### 1. Introduction

Topological tail entropy quantifies the complexity of a dynamical system at arbitrarily small scales. It captures the entropy near any single orbit. This quantity was first introduced by Misiurewicz in [9] and was thoroughly studied by many others (e.g. see [1], [2], [5], [8]). (Historically, Misiurewicz and Buzzi called it the topological conditional entropy and local entropy respectively.) It is well known that the variational principle plays a fundamental role in ergodic theory and dynamical systems. In [6], Ledrappier obtained a variational principle of topological tail entropy, and Downarowicz ([5]) established a variational principle between the topological tail entropy and the entropy structure. Later, Burguet ([1]) presented a direct proof of Downarowicz's results and extended them to a noninvertible case. Recently, there appeared some works which study the tail entropy of dynamics of group actions (e.g. see [4], [15], [16]).

As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in thermodynamic formalism. This generalization was first done by Ruelle in [11] and next by many others (e.g.

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see [3], [10], [12], [14]). In [8], Li–Chen–Cheng extended the tail entropy to tail pressure for continuous transformations. In fact, they proved a tail variational principle which exhibits the relationship between the tail pressure and the tail entropy function and gave some applications of tail pressure.

In this paper, we give two equivalent definitions of tail pressure involving open covers and establish a variational principle which exhibits the relationship between the tail pressure and the measure-theoretic tail entropy. Let  $(X, d)$  be a compact metric space and  $T: X \rightarrow X$  be a homeomorphism. For  $\varepsilon > 0, n \in \mathbb{N}$  and  $x \in X$ , the Bowen's ball of order  $n$ , radius  $\varepsilon$  and center  $x$  is defined by

$$B(x, n, \varepsilon) = \{y \in X : d(T^k(x), T^k(y)) < \varepsilon, \text{ for all } k = 0, \dots, n-1\}.$$

Given  $K \subset X$ , a set  $E \subset X$  is said to be an  $(n, \varepsilon)$ -spanning subset for  $K$  if

$$K \subset \bigcup_{x \in E} B(x, n, \varepsilon),$$

and an  $(n, \varepsilon)$ -separated subset of  $K$  if for all  $x \neq y \in E$  there is  $0 \leq k \leq n-1$  such that  $d(T^k(x), T^k(y)) \geq \varepsilon$ .

Now we recall the concept of tail pressure which was defined by Li–Chen–Cheng in [8]. Let  $C(X, \mathbb{R})$  be the space of real-valued continuous functions of  $X$ . For  $f \in C(X, \mathbb{R})$ , denote by

$$(S_n f)(x) = \sum_{i=0}^{n-1} f(T^i(x)), \quad \text{for all } x \in X.$$

Let  $f \in C(X, \mathbb{R})$ ,  $n \in \mathbb{N}$ ,  $\varepsilon > 0, \delta > 0$  and  $x \in X$ . Write

$$Q_n(T, f, x, \delta, \varepsilon) = \inf \left\{ \sum_{y \in F} e^{(S_n f)(y)} : F \text{ is an } (n, \delta)\text{-spanning set for } B(x, n, \varepsilon) \right\}$$

and

$$P_n(T, f, x, \delta, \varepsilon) = \sup \left\{ \sum_{y \in E} e^{(S_n f)(y)} : E \text{ is an } (n, \delta)\text{-separated set of } B(x, n, \varepsilon) \right\}.$$

The *tail pressure*  $P^*(T, f)$  is defined by

$$P^*(T, f) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} \ln P_n(T, f, x, \delta, \varepsilon).$$

Write

$$Q^*(T, f) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} \ln Q_n(T, f, x, \delta, \varepsilon).$$

By Lemma 3.3 of [8],

$$(1.1) \quad P^*(T, f) = Q^*(T, f).$$

**2. Equivalent definitions of tail pressure**

In this section, we define the tail pressure involving open covers and prove that new definitions are equivalent to Li–Chen–Cheng’s definition.

**2.1. Tail pressure involving open covers.** For an open cover  $\mathcal{U}$  of  $X$ , a set  $K \subset X$ ,  $f \in C(X, \mathbb{R})$  and  $n \in \mathbb{N}$ , denote by

$$N_n(T, f, \mathcal{U}, K) = \inf \left\{ \sum_{A \in \mathcal{U}'} \sup_{x \in A} e^{(S_n f)(x)} : \mathcal{U}' \text{ is a finite subset of } \mathcal{U}^n \text{ and } K \subset \bigcup_{U \in \mathcal{U}'} U \right\},$$

here

$$\mathcal{U}^n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}.$$

DEFINITION 2.1. Let  $\mathcal{U}$  and  $\mathcal{V}$  be two open covers of  $X$ ,  $f \in C(X, \mathbb{R})$  and  $n \in \mathbb{N}$ . Define

$$\begin{aligned} p_n(T, f, \mathcal{U}|\mathcal{V}) &= \sup_{V \in \mathcal{V}^n} N_n(T, f, \mathcal{U}, V), \\ p(T, f, \mathcal{U}|\mathcal{V}) &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln p_n(T, f, \mathcal{U}|\mathcal{V}), \\ p(T, f|\mathcal{V}) &= \lim_{\delta \rightarrow 0} \sup_{\mathcal{U}} \{p(T, f, \mathcal{U}|\mathcal{V}) : \text{diam}(\mathcal{U}) \leq \delta\}, \\ p^*(T, f) &= \inf_{\mathcal{V}} p(T, f|\mathcal{V}), \end{aligned}$$

here  $\text{diam}(\mathcal{U}) = \sup\{\text{diam}(U) : U \in \mathcal{U}\}$  is the diameter of  $\mathcal{U}$ .

From the following lemma, we know that  $p(T, f, \mathcal{U}|\mathcal{V})$  is well-defined.

LEMMA 2.2. For any two open covers  $\mathcal{U}$  and  $\mathcal{V}$  of  $X$  and  $f \in C(X, \mathbb{R})$ , the limit  $\lim_{n \rightarrow \infty} (1/n) \ln p_n(T, f, \mathcal{U}|\mathcal{V})$  exists and equals  $\inf_n \{(1/n) \ln p_n(T, f, \mathcal{U}|\mathcal{V})\}$ .

PROOF. By the Subadditive Theorem (Theorem 4.9 of [13]), it is enough to prove

$$p_{n+m}(T, f, \mathcal{U}|\mathcal{V}) \leq p_n(T, f, \mathcal{U}|\mathcal{V}) \cdot p_m(T, f, \mathcal{U}|\mathcal{V}), \quad \text{for all } n, m \in \mathbb{N}.$$

For any fixed  $V \in \mathcal{V}^{n+m}$ , there are  $V_1 \in \mathcal{V}^n$  and  $V_2 \in T^{-n}\mathcal{V}^m$  such that  $V = V_1 \cap V_2$ . For any  $\tau > 0$ , take finite sets  $\mathcal{U}_1 \subset \mathcal{U}^n$  and  $\mathcal{U}_2 \subset \mathcal{U}^m$  which cover  $V_1$  and  $T^n(V_2)$  respectively, and satisfy

$$\sum_{A \in \mathcal{U}_1} \sup_{x \in A} e^{(S_n f)(x)} \leq (1 + \tau) \cdot N_n(T, f, \mathcal{U}, V_1)$$

and

$$\sum_{A \in \mathcal{U}_2} \sup_{x \in A} e^{(S_m f)(x)} \leq (1 + \tau) \cdot N_m(T, f, \mathcal{U}, T^n(V_2)).$$

Let  $\mathcal{U}_0 := \mathcal{U}_1 \vee T^{-n}\mathcal{U}_2$ . Then  $\mathcal{U}_0$  is a finite open cover of  $V$ . Moreover,

$$\begin{aligned} \sum_{A \in \mathcal{U}_0} \sup_{x \in A} e^{(S_{n+m}f)(x)} &= \sum_{A \in \mathcal{U}_1 \vee T^{-n}\mathcal{U}_2} \sup_{x \in A} e^{(S_n f)(x) + (S_m f)(T^n x)} \\ &\leq \left( \sum_{A \in \mathcal{U}_1} \sup_{x \in A} e^{(S_n f)(x)} \right) \cdot \left( \sum_{A \in T^{-n}\mathcal{U}_2} \sup_{x \in A} e^{(S_m f)(T^n x)} \right) \\ &= \left( \sum_{A \in \mathcal{U}_1} \sup_{x \in A} e^{(S_n f)(x)} \right) \cdot \left( \sum_{A \in \mathcal{U}_2} \sup_{x \in A} e^{(S_m f)(x)} \right) \\ &\leq (1 + \tau)^2 \cdot N_n(T, f, \mathcal{U}, V_1) \cdot N_m(T, f, \mathcal{U}, T^n(V_2)). \end{aligned}$$

So

$$\begin{aligned} N_{n+m}(T, f, \mathcal{U}, V) &\leq (1 + \tau)^2 \cdot N_n(T, f, \mathcal{U}, V_1) \cdot N_m(T, f, \mathcal{U}, T^n(V_2)) \\ &\leq (1 + \tau)^2 \cdot \sup_{V \in \mathcal{V}^n} N_n(T, f, \mathcal{U}, V) \cdot \sup_{V \in \mathcal{V}^m} N_m(T, f, \mathcal{U}, V) \\ &= (1 + \tau)^2 \cdot p_n(T, f, \mathcal{U}|\mathcal{V}) \cdot p_m(T, f, \mathcal{U}|\mathcal{V}). \end{aligned}$$

By the arbitrariness of  $V$  and  $\tau$ , we have

$$p_{n+m}(T, f, \mathcal{U}|\mathcal{V}) \leq p_n(T, f, \mathcal{U}|\mathcal{V}) \cdot p_m(T, f, \mathcal{U}|\mathcal{V}). \quad \square$$

For an open cover  $\mathcal{U}$  of  $X$ , a set  $K \subset X$ ,  $f \in C(X, \mathbb{R})$  and  $n \in \mathbb{N}$ , denote by

$$\begin{aligned} R_n(T, f, \mathcal{U}, K) &= \inf \left\{ \sum_{A \in \mathcal{U}'} \inf_{x \in A} e^{(S_n f)(x)} : \mathcal{U}' \text{ is a finite subset of } \mathcal{U}^n \text{ and } K \subset \bigcup_{U \in \mathcal{U}'} U \right\}. \end{aligned}$$

DEFINITION 2.3. Let  $\mathcal{U}$  and  $\mathcal{V}$  be two open covers of  $X$ ,  $f \in C(X, \mathbb{R})$  and  $n \in \mathbb{N}$ . Define

$$\begin{aligned} q_n(T, f, \mathcal{U}|\mathcal{V}) &= \sup_{V \in \mathcal{V}^n} R_n(T, f, \mathcal{U}, V), \\ q(T, f, \mathcal{U}|\mathcal{V}) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \ln q_n(T, f, \mathcal{U}|\mathcal{V}), \\ q(T, f|\mathcal{V}) &= \limsup_{\delta \rightarrow 0} \sup_{\mathcal{U}} \{q(T, f, \mathcal{U}|\mathcal{V}) : \text{diam}(\mathcal{U}) \leq \delta\}, \\ q^*(T, f) &= \inf_{\mathcal{V}} q(T, f|\mathcal{V}), \end{aligned}$$

and

$$\begin{aligned} \underline{q}(T, f, \mathcal{U}|\mathcal{V}) &= \liminf_{n \rightarrow \infty} \frac{1}{n} \ln q_n(T, f, \mathcal{U}|\mathcal{V}), \\ \underline{q}(T, f|\mathcal{V}) &= \limsup_{\delta \rightarrow 0} \sup_{\mathcal{U}} \{\underline{q}(T, f, \mathcal{U}|\mathcal{V}) : \text{diam}(\mathcal{U}) \leq \delta\}, \\ \underline{q}^*(T, f) &= \inf_{\mathcal{V}} \underline{q}(T, f|\mathcal{V}) \end{aligned}$$

THEOREM 2.4. For any  $f \in C(X, \mathbb{R})$ , it holds that  $p^*(T, f) = \underline{q}^*(T, f)$ .

PROOF. From the definitions, it is obvious that  $p^*(T, f) \geq \underline{q}^*(T, f)$ . We will prove  $p^*(T, f) \leq \underline{q}^*(T, f)$  as follows. For any  $\tau > 0$ , there is  $\delta_0 > 0$  such that

$$(2.1) \quad d(x, y) \leq \delta_0 \Rightarrow |f(x) - f(y)| \leq \tau, \quad \text{for all } x, y \in X.$$

Take an open cover  $\mathcal{V}$  of  $X$  and  $\delta \leq \delta_0$  arbitrarily. Let  $\mathcal{U}$  be an open cover of  $X$  with  $\text{diam}(\mathcal{U}) \leq \delta$ . For any  $A \in \mathcal{U}^n$ , there are  $x_0, y_0 \in \bar{A}$  such that

$$\sup_{x \in A} e^{(S_n f)(x)} = e^{(S_n f)(x_0)} \quad \text{and} \quad \inf_{x \in A} e^{(S_n f)(x)} = e^{(S_n f)(y_0)}$$

and hence, by (2.1),

$$\begin{aligned} \sup_{x \in A} e^{(S_n f)(x)} &= e^{(S_n f)(x_0)} = e^{(S_n f)(x_0) - (S_n f)(y_0)} \cdot e^{(S_n f)(y_0)} \\ &= e^{[f(x_0) - f(y_0)] + \dots + [f(T^{n-1}(x_0)) - f(T^{n-1}(y_0))]} \cdot e^{(S_n f)(y_0)} \\ &\leq e^{n\tau} \cdot \inf_{x \in A} e^{(S_n f)(x)}. \end{aligned}$$

So,

$$\begin{aligned} N_n(T, f, \mathcal{U}, V) &\leq e^{n\tau} \cdot R_n(T, f, \mathcal{U}, V), \quad \text{for all } V \in \mathcal{V}^n \\ &\Rightarrow p_n(T, f, \mathcal{U}|\mathcal{V}) \leq e^{n\tau} \cdot q_n(T, f, \mathcal{U}|\mathcal{V}) \\ &\Rightarrow p(T, f, \mathcal{U}|\mathcal{V}) \leq \underline{q}(T, f, \mathcal{U}|\mathcal{V}) + \tau. \end{aligned}$$

Note that the above inequalities hold for all open covers  $\mathcal{U}$  of  $X$  with  $\text{diam}(\mathcal{U}) \leq \delta \leq \delta_0$ . We have  $p(T, f|\mathcal{V}) \leq \underline{q}(T, f|\mathcal{V}) + \tau$  and hence  $p^*(T, f) \leq \underline{q}^*(T, f)$  by the arbitrariness of  $\tau$  and  $\mathcal{V}$ . □

**2.2. Equivalence of definitions of tail pressure.** We will prove that our definitions of tail pressure in the above subsection are equivalent to Li–Chen–Cheng’s one recalled in this subsection. Given  $f \in C(X, \mathbb{R})$ , and positive numbers  $\delta$  and  $\varepsilon$ , denote by

$$\begin{aligned} \underline{Q}(T, f, \delta, \varepsilon) &= \liminf_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} \ln Q_n(T, f, x, \delta, \varepsilon), \\ \underline{Q}(T, f, \varepsilon) &= \lim_{\delta \rightarrow 0} \underline{Q}(T, f, \delta, \varepsilon), \\ \underline{Q}^*(T, f) &= \lim_{\varepsilon \rightarrow 0} \underline{Q}(T, f, \varepsilon). \end{aligned}$$

LEMMA 2.5. *For any  $f \in C(X, \mathbb{R})$ ,  $\underline{q}^*(T, f) \leq \underline{Q}^*(T, f)$ .*

PROOF. Take an arbitrary open cover  $\mathcal{U}$  of  $X$  with the Lebesgue number bigger than  $2\delta$ . For any  $\varepsilon > 0$ , select an open cover  $\mathcal{V}$  of  $X$  with  $\text{diam}(\mathcal{V}) < \varepsilon$ . Then, for any  $n \in \mathbb{N}$  and  $V \in \mathcal{V}^n$ , there is  $x_V \in X$  such that  $V \subset B(x_V, n, \varepsilon)$ . For any  $V \in \mathcal{V}^n$  and any  $(n, \delta)$ -spanning set  $F$  for  $B(x_V, n, \varepsilon)$ , we have

$$R_n(T, f, \mathcal{U}, V) \leq \sum_{y \in F} e^{(S_n f)(y)}$$

since for each  $y \in F$ , there is an open cover  $U \in \mathcal{U}^n$  such that  $B(y, n, \delta) \subset U$ .

Then

$$R_n(T, f, \mathcal{U}, V) \leq Q_n(T, f, x_V, \delta, \varepsilon) \leq \sup_{x \in X} Q_n(T, f, x, \delta, \varepsilon).$$

The arbitrariness of  $V \in \mathcal{V}^n$  implies that

$$q_n(T, f, \mathcal{U}|\mathcal{V}) \leq \sup_{x \in X} Q_n(T, f, x, \delta, \varepsilon)$$

and hence  $\underline{q}(T, f, \mathcal{U}|\mathcal{V}) \leq \underline{Q}(T, f, \delta, \varepsilon) \leq \underline{Q}(T, f, \varepsilon)$ . Since the above inequalities hold for any open cover  $\mathcal{U}$ , it holds that

$$\underline{q}^*(T, f) \leq \underline{q}(T, f|\mathcal{V}) \leq \sup_{\mathcal{U}} \underline{q}(T, f, \mathcal{U}|\mathcal{V}) \leq \underline{Q}(T, f, \varepsilon).$$

Let  $\varepsilon \rightarrow 0$ . We complete the proof. □

LEMMA 2.6. *For any  $f \in C(X, \mathbb{R})$ ,  $P^*(T, f) \leq p^*(T, f)$ .*

PROOF. For any  $\tau > 0$ , there is an open cover  $\mathcal{V}$  of  $X$  such that  $p(T, f|\mathcal{V}) \leq p^*(T, f) + \tau$ . Choose  $\varepsilon > 0$  such that the Lebesgue number of  $\mathcal{V}$  is bigger than  $2\varepsilon$ . Write

$$P(T, f, \delta, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} \ln P_n(T, f, x, \delta, \varepsilon), \quad P(T, f, \varepsilon) = \lim_{\delta \rightarrow 0} P(T, f, \delta, \varepsilon).$$

If it holds that

$$(2.2) \quad P(T, f, \varepsilon) \leq p(T, f|\mathcal{V}),$$

then  $P^*(T, f) \leq P(T, f, \varepsilon) \leq p(T, f|\mathcal{V}) \leq p^*(T, f) + \tau$  and we complete the proof by the arbitrariness of  $\tau$ .

Now we prove (2.2). Let  $\delta > 0$  and  $\mathcal{U}$  be an open cover of  $X$  with  $\text{diam}(\mathcal{U}) \leq \delta$ . Since the Lebesgue number of  $\mathcal{V}$  is bigger than  $2\varepsilon$ , for each  $x \in X$  there is  $V_x \in \mathcal{V}^n$  such that  $B(x, n, \varepsilon) \subset V_x$ . Given an  $(n, \delta)$ -separated subset  $E$  of  $B(x, n, \varepsilon)$  and a finite set  $\mathcal{U}' \subset \mathcal{U}^n$  which covers  $V_x$ , we have

$$\sum_{y \in E} e^{(S_n f)(y)} \leq \sum_{A \in \mathcal{U}', x \in A} \sup e^{(S_n f)(x)},$$

because there is at most one point in  $E \cap U$  for each  $U \in \mathcal{U}'$ . Then

$$\begin{aligned} P_n(T, f, x, \delta, \varepsilon) &\leq N_n(T, f, \mathcal{U}, V_x), \quad \text{for all } x \in X, \\ &\Rightarrow P_n(T, f, x, \delta, \varepsilon) \leq \sup_{V \in \mathcal{V}^n} N_n(T, f, \mathcal{U}, V), \quad \text{for all } x \in X, \\ &\Rightarrow \sup_{x \in X} P_n(T, f, x, \delta, \varepsilon) \leq \sup_{V \in \mathcal{V}^n} N_n(T, f, \mathcal{U}, V), \\ &\Rightarrow P(T, f, \delta, \varepsilon) \leq p(T, f, \mathcal{U}|\mathcal{V}), \\ &\Rightarrow P(T, f, \delta, \varepsilon) \leq \sup_{\mathcal{U}} \{p(T, f, \mathcal{U}|\mathcal{V}) : \text{diam}(\mathcal{U}) \leq \delta\}, \\ &\Rightarrow P(T, f, \varepsilon) \leq p(T, f|\mathcal{V}). \end{aligned}$$

The last inequality is exactly (2.2). □

Now we can get the equivalence of the definitions of tail pressure.

**THEOREM 2.7.** *For any  $f \in C(X, \mathbb{R})$ , denote by*

$$\underline{P}^*(T, f) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} \ln P_n(T, f, x, \delta, \varepsilon).$$

Then

$$\underline{q}^*(T, f) = q^*(T, f) = \underline{Q}^*(T, f) = Q^*(T, f) = \underline{P}^*(T, f) = P^*(T, f) = p^*(T, f).$$

**PROOF.** By the definitions, it is obvious that

$$\underline{q}^*(T, f) \leq q^*(T, f), \quad \underline{P}^*(T, f) \leq P^*(T, f), \quad \underline{Q}^*(T, f) \leq Q^*(T, f).$$

Noting equality (1.1), Lemmas 2.5 and 2.6, we have

$$\underline{q}^*(T, f) \leq \underline{Q}^*(T, f) \leq Q^*(T, f) = P^*(T, f) \leq p^*(T, f).$$

It is easy to check that  $Q_n(T, f, x, \delta, \varepsilon) \leq P_n(T, f, x, \delta, \varepsilon)$  for all  $n \in \mathbb{N}$ ,  $f \in C(X, \mathbb{R})$ ,  $x \in X$  and  $\delta, \varepsilon > 0$ . So  $\underline{Q}^*(T, f) \leq \underline{P}^*(T, f)$  and then the conclusion of this theorem holds by Theorem 2.4.  $\square$

We remark that, in Theorem 3.4 of [8], Li–Chen–Cheng proved  $\underline{Q}^*(T, f) = Q^*(T, f) = \underline{P}^*(T, f) = P^*(T, f)$ . However, their proof depends on the existence of refining sequence of essential partitions (see the following section for its definition), which does not always exist. We give another proof of this theorem and do not require the existence of a refining sequence of essential partitions.

### 3. Variational principle of tail pressure

The term of a partition in this paper means a finite Borel partition of  $X$ . For two partitions  $\xi, \eta$  and a probability measure  $\mu$ , recall that the conditional entropy is defined by

$$H_\mu(X, \xi|\eta) = - \sum_{\substack{A \in \xi \\ B \in \eta}} \mu(A \cap B) \ln \frac{\mu(A \cap B)}{\mu(B)} = \sum_{B \in \eta} \mu(B) H_{\mu_B}(\xi|B),$$

here  $\mu_B$  is the conditional measure of  $\mu$  on  $B$ . It is easy to see that

$$(3.1) \quad H_\mu(X, \xi_1 \vee \xi_2|\eta_1 \vee \eta_2) \leq H_\mu(X, \xi_1|\eta_1) + H_\mu(X, \xi_2|\eta_2)$$

for partitions  $\xi_1, \xi_2, \eta_1, \eta_2$  of  $X$ . The following measure-theoretic tail entropy was first introduced in [15] for amenable group actions.

**DEFINITION 3.1.** Let  $\mu$  be a  $T$ -invariant probability measure on  $X$  and  $\xi, \eta$  be two finite partitions of  $X$ , we define

$$\begin{aligned} h_\mu(X, T, \xi|\eta) &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(X, \xi^n|\eta^n), & h_\mu(X, T|\eta) &= \sup_\xi h_\mu(X, T, \xi|\eta), \\ h_\mu(X, T, \varepsilon) &= \sup_{\text{diam}(\eta) \leq \varepsilon} h_\mu(X, T|\eta), & h_\mu^*(X, T) &= \lim_{\varepsilon \rightarrow 0} h_\mu(X, T, \varepsilon). \end{aligned}$$

We call  $h_\mu(X, T, \varepsilon)$  and  $h_\mu^*(X, T)$  the  $\varepsilon$ -local entropy and tail entropy of  $\mu$  respectively.

Let us note that  $h_\mu(X, T, \xi|\eta)$  is well-defined by (3.1) and the Subadditive Theorem (Theorem 4.9 of [13]).

A partition  $\xi$  of  $X$  is called *compact* if there is at most one element of  $\xi$  which is not compact. Denote by  $\mathcal{P}$  and  $\mathcal{P}_c$  the sets of all finite measurable partitions and all compact finite measurable partitions of  $X$  respectively.

LEMMA 3.2. For any partition  $\beta$  of  $X$  and  $T$ -invariant probability measure  $\mu$ ,

$$\sup_{\xi \in \mathcal{P}} h_\mu(X, T, \xi|\beta) = \sup_{\xi \in \mathcal{P}_c} h_\mu(X, T, \xi|\beta).$$

PROOF. It is sufficient to prove that  $\sup_{\xi \in \mathcal{P}} h_\mu(X, T, \xi|\beta) \leq \sup_{\xi \in \mathcal{P}_c} h_\mu(X, T, \xi|\beta)$ .

For any  $\xi = \{A_i\}_{i=1}^k \in \mathcal{P}$  and any  $\varepsilon > 0$ , one can choose  $\eta = \{B_j\}_{j=0}^k \in \mathcal{P}_c$  such that  $H_\mu(\xi|\eta) \leq \varepsilon$ . Then

$$\begin{aligned} h_\mu(X, T, \xi|\beta) &\leq h_\mu(X, T, \eta|\beta) + H_\mu(X, \xi|\eta) \\ &\leq h_\mu(X, T, \eta|\beta) + \varepsilon \leq \sup_{\eta \in \mathcal{P}_c} h_\mu(X, T, \eta|\beta) + \varepsilon \end{aligned}$$

and thus we complete the proof by the arbitrariness of  $\varepsilon$  and  $\xi$ . □

LEMMA 3.3. For any  $k \in \mathbb{N}$  and  $T$ -invariant probability measure  $\mu$ ,

$$kh_\mu^*(X, T) \leq h_\mu^*(X, T^k).$$

PROOF. For a partition  $\xi$ , we set

$$\eta(\xi) = \xi^k = \bigvee_{i=0}^{k-1} T^{-i}\xi.$$

Then

$$\begin{aligned} &H_\mu\left(X, \bigvee_{i=0}^{n-1} T^{-ki}\eta(\xi_1) \middle| \bigvee_{i=0}^{n-1} T^{-ki}\eta(\xi_2)\right) \\ &= \sum_{B \in \bigvee_{i=0}^{n-1} T^{-ki}\eta(\xi_2)} \mu(B) H_{\mu_B}\left(X, \bigvee_{i=0}^{n-1} T^{-ki}\eta(\xi_1) \middle| B\right) \\ &= \sum_{B \in \bigvee_{i=0}^{kn-1} T^{-i}\xi_2} \mu(B) H_{\mu_B}\left(X, \bigvee_{i=0}^{kn-1} T^{-i}\xi_1 \middle| B\right) = H_\mu(X, \xi_1^{kn} | \xi_2^{kn}). \end{aligned}$$

Thus,  $h_\mu(X, T^k, \eta(\xi_1)|\eta(\xi_2)) = kh_\mu(X, T, \xi_1|\xi_2)$  and hence

$$kh_\mu(X, T|\xi_2) = k \sup_{\xi_1} h_\mu(X, T, \xi_1|\xi_2) \leq \sup_{\eta} h_\mu(X, T^k, \eta|\eta(\xi_2)) = h_\mu(X, T^k|\eta(\xi_2)).$$

Note that  $\text{diam}(\eta(\xi)) \leq \text{diam}(\xi)$ , for all  $\xi \in \mathcal{P}$ . For any  $\xi_2$  with  $\text{diam}(\xi_2) \leq \varepsilon$ ,

$$kh_\mu(X, T|\xi_2) \leq h_\mu(X, T^k, \varepsilon).$$

This means that  $kh_\mu^*(X, T) \leq kh_\mu(X, T, \varepsilon) \leq h_\mu(X, T^k, \varepsilon)$ . By the arbitrariness of  $\varepsilon$ , we complete the proof.  $\square$

The following theorem is a variational principle which exhibits the connection between tail pressure and measure-theoretic tail entropy.

**THEOREM 3.4.** *Let  $T: X \rightarrow X$  be a homeomorphism on a compact metric space and  $f \in C(X, \mathbb{R})$ . Denote by  $\mathcal{M}(X, T)$  the set of all  $T$ -invariant probability measures. Then:*

(a) 
$$\sup_{\mu \in \mathcal{M}(X, T)} \left\{ h_\mu^*(X, T) + \int_X f d\mu \right\} \leq P^*(T, f).$$

(b) *If  $T$  has finite topological entropy and admits an infinite minimal factor, then*

$$\sup_{\mu \in \mathcal{M}(X, T)} \left\{ h_\mu^*(X, T) + \int_X f d\mu \right\} = P^*(T, f)$$

*and the supremum is achieved.*

Before presenting the proof of Theorem 3.4, we recall the concept of a *refining sequence of essential partitions* which will be used in the following. A sequence of partitions  $\{P_k\}$  of  $X$  is *refining*, if  $P_{k+1}$  is finer than  $P_k$  for all  $k$  and if the diameter of  $P_k$ ,  $d_k = \max_{p \in P_k} \{\text{diam}(p)\}$ , goes to zero when  $k$  goes to infinity. For a system  $(X, T)$ , a partition is said to be *essential* if the boundaries of its elements have zero measure for all  $T$ -invariant probability measures.

In [7], Lindenstrauss gave some conditions to insure the existence of a refining sequence of essential partitions.

**LEMMA 3.5** ([7]). *If  $T$  is a homeomorphism of finite topological entropy admitting an infinite minimal factor, then there exists a refining sequence of essential partitions.*

**PROOF OF THEOREM 3.4.** (a) Let  $\varepsilon > 0$  and  $\mu \in \mathcal{M}(X, T)$ . Take two partitions  $\eta$  and  $\xi = \{A_0, \dots, A_l\}$  of  $X$  such that  $\text{diam}(\eta) \leq \varepsilon$  and  $A_i$  are compact for  $i = 1, \dots, l$ .

For any  $\tau > 0$ , take  $\delta \in (0, \min\{d(A_i, A_j) : i, j = 1, \dots, l; i \neq j\}/2)$  satisfying

$$(3.2) \quad d(x, y) \leq \delta \Rightarrow |f(x) - f(y)| < \tau, \quad \text{for all } x, y \in X.$$

Then

$$\begin{aligned} H_\mu(X, \xi^n | \eta^n) + \int S_n f d\mu &= \sum_{B \in \eta^n} \mu(B) \left( H_{\mu_B}(X, \xi^n | B) + \int_B S_n f d\mu_B \right) \\ &\leq \sum_{B \in \eta^n} \mu(B) \left( \sum_{A \in \xi^n} \mu_B(A) (-\ln \mu_B(A) + \alpha(A \cap B)) \right) \\ &\leq \sum_{B \in \eta^n} \mu(B) \ln \left( \sum_{A \in \xi^n} e^{\alpha(A \cap B)} \right) \leq \sup_{B \in \eta^n} \ln \left( \sum_{A \in \xi^n} e^{\alpha(A \cap B)} \right), \end{aligned}$$

here  $\alpha(A \cap B) = \sup\{(S_n f)(x) : x \in A \cap B\}$ . For any  $A \in \xi^n$  and  $x \in X$ , if  $A \cap B(x, n, 2\varepsilon) \neq \emptyset$ , choose  $y_{A,x} \in A \cap B(x, n, 2\varepsilon)$  such that  $\alpha(A \cap B(x, n, 2\varepsilon)) = (S_n f)(y_{A,x})$ . Take an  $(n, \delta)$ -spanning set  $E_x$  for  $B(x, n, 2\varepsilon)$  such that

$$\sum_{y \in E_x} e^{(S_n f)(y)} \leq 2Q_n(T, f, x, \delta, 2\varepsilon) \leq 2P_n(T, f, x, \delta, 2\varepsilon).$$

By the choice of  $\delta$ , for each  $z \in E_x$ , there are at most  $2^n$   $y_{A,x}$  such that  $d(T^i(z), T^i(y_{A,x})) < \delta$  for  $i = 0, \dots, n - 1$ . So, noting (3.2),

$$\sum_{A \in \xi^n} e^{\alpha(A \cap B(x, n, 2\varepsilon)) - n\tau} \leq 2^n \cdot \sum_{z \in E_x} e^{(S_n f)(z)} \leq 2^{n+1} \cdot P_n(T, f, x, \delta, 2\varepsilon)$$

and hence

$$\ln \left( \sum_{A \in \xi^n} e^{\alpha(A \cap B(x, n, 2\varepsilon))} \right) \leq n\tau + (n + 1) \ln 2 + \ln P_n(T, f, x, \delta, 2\varepsilon).$$

Since for any  $B \in \eta^n$  there is  $x \in X$  such that  $B \subset B(x, n, 2\varepsilon)$ , it holds that

$$\sup_{B \in \eta^n} \ln \left( \sum_{A \in \xi^n} e^{\alpha(A \cap B)} \right) \leq \sup_{x \in X} \ln \left( \sum_{A \in \xi^n} e^{\alpha(A \cap B(x, n, 2\varepsilon))} \right).$$

Then

$$\begin{aligned} h_\mu(X, T, \xi | \eta) + \int f d\mu &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( H_\mu(X, \xi^n | \eta^n) + \int S_n f d\mu \right) \\ &\leq \tau + \ln 2 + P(T, f, \delta, 2\varepsilon) \end{aligned}$$

and hence

$$h_\mu(X, T, \xi | \eta) + \int f d\mu \leq \tau + \ln 2 + P(T, f, 2\varepsilon).$$

By Lemma 3.2, we have

$$h_\mu(X, T | \eta) + \int f d\mu \leq \tau + \ln 2 + P(T, f, 2\varepsilon),$$

for any  $\eta$  with  $\text{diam}(\eta) \leq \varepsilon$ . So,

$$h_\mu(X, T, \varepsilon) + \int f d\mu \leq \tau + \ln 2 + P(T, f, 2\varepsilon).$$

Letting  $\varepsilon \rightarrow 0$ ,

$$(3.3) \quad h_\mu^*(X, T) + \int f d\mu \leq \tau + \ln 2 + P^*(T, f).$$

Furthermore, by Theorem 3.5 (1) of [8],

$$(3.4) \quad P^*(T^k, S_k f) = kP^*(T, f), \quad \text{for all } k \in \mathbb{N}.$$

Combing (3.3) and (3.4) with Lemma 3.3, we have

$$k \left( h_\mu^*(X, T) + \int f d\mu \right) \leq \tau + \ln 2 + kP^*(T, f), \quad \text{for all } k \in \mathbb{N}.$$

Thus

$$h_\mu^*(X, T) + \int f d\mu \leq P^*(T, f).$$

(b) By Lemma 3.5, there exists a refining sequence of essential partitions  $\mathcal{P} = \{P_k\}_{k \in \mathbb{N}}$ . It is easy to check that

$$h_\nu^*(X, T) \geq \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} h_\nu(X, T, P_l|P_k), \quad \text{for all } \nu \in \mathcal{M}(X, T).$$

For natural numbers  $k$  and  $l$  with  $l > k$ , set

$$L(T, f, P_l|P_k) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{A \in P_k^n} \ln \left( \sum_{\substack{B \subset A \\ B \in P_l^n}} \inf_{x \in B} e^{(S_n f)(x)} \right).$$

By Theorem 3.2 of [8],  $\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} L(T, f, P_l|P_k) = P^*(T, f)$ . Furthermore, part (2) of the proof of Theorem 4.1 (line 12, p. 1414) of [8] implies that there is  $\mu \in \mathcal{M}(X, T)$  such that

$$h_\mu(X, T, P_l|P_k) + \int f d\mu \geq L(T, f, P_l|P_k).$$

This means that

$$h_\mu^*(X, T) + \int f d\mu \geq P^*(T, f). \quad \square$$

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