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BIFURCATION AND MULTIPLICITY RESULTS FOR CRITICAL p-LAPLACIAN PROBLEMS

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(*Submitted by P. Drábek*)

Abstract. We prove a bifurcation and multiplicity result that is independent of the dimension N for a critical p-Laplacian problem that is an analog of the Brezis–Nirenberg problem for the quasilinear case. This extends a result in the literature for the semilinear case $p = 2$ to all $p \in (1, \infty)$. In particular, it gives a new existence result when $N < p^2$. When $p \neq 2$ the nonlinear operator $-\Delta_p$ has no linear eigenspaces, so our extension is nontrivial and requires a new abstract critical point theorem that is not based on linear subspaces. We prove a new abstract result based on a pseudo-index related to the $\mathbb{Z}_2$-cohomological index that is applicable here.

1. Introduction and main results

Elliptic problems with critical nonlinearities have been widely studied in the literature. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary.

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In the celebrated paper [4], Brézis and Nirenberg considered the problem

$$
\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

when $N \geq 3$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Among other things, they proved that this problem has a positive solution when $N \geq 4$ and $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω. Capozzi et al. [6] extended this result by proving the existence of a nontrivial solution for all $\lambda > 0$ when $N \geq 4$. The existence of infinitely many solutions for all $\lambda > 0$ was established by Fortunato and Jannelli [12] when $N \geq 4$ and Ω is a ball, and by Devillanova and Solimini [9] when $N \geq 7$ and Ω is an arbitrary bounded domain (see also Schechter and Zou [18]).

García Azorero and Peral Alonso [13], Egnell [10], and Guedda and Véron [14] studied the corresponding problem for the p-Laplacian

$$
\begin{cases} -\Delta_p\, u = \lambda\, |u|^{p-2}\, u + |u|^{p^*-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

when $1 < p < N$, where $\Delta_p\, u = \operatorname{div}(|\nabla u|^{p-2}\, \nabla u)$ is the p-Laplacian of u and $p^* = Np/(N-p)$. They proved that this problem has a positive solution when $N \geq p^2$ and $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω. Degiovanni and Lancelotti [8] extended their result by proving the existence of a nontrivial solution when $N \geq p^2$ and $\lambda > \lambda_1$ is not an eigenvalue, and when $N^2/(N+1) > p^2$ and $\lambda \geq \lambda_1$ (see also Arioli and Gazzola [1]). The existence of infinitely many solutions for all $\lambda > 0$ was recently established by Cao et al. [5] when $N > p^2 + p$ (see also Wu and Huang [19]).

On the other hand, Cerami et al. [7] proved the following bifurcation and multiplicity result for problem (1.1) that is independent of N and Ω. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty$ be the Dirichlet eigenvalues of $-\Delta$ in Ω, repeated according to multiplicity, let

$$
S = \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}
$$

be the best constant for the Sobolev imbedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ when $N \geq 3$, and let $|\cdot|$ denote the Lebesgue measure in $\mathbb{R}^N$. If $\lambda_k \leq \lambda < \lambda_{k+1}$ and

$$
\lambda > \lambda_{k+1} - \frac{S}{|\Omega|^{2/N}},
$$

and m denotes the multiplicity of λ_{k+1}, then problem (1.1) has m distinct pairs of nontrivial solutions $\pm\, u^\lambda_j$, $j = 1, \ldots, m$, such that $u^\lambda_j \to 0$ as $\lambda \nearrow \lambda_{k+1}$ (see [7, Theorem 1.1]).