Topological Methods in Nonlinear Analysis
Volume 46, No. 2, 2015, 813–833
DOI: 10.12775/TMNA.2015.069

EQUATION WITH POSITIVE COEFFICIENT IN THE QUASILINEAR TERM AND VANISHING POTENTIAL

José F.L. Aires — Marco A.S. Souto

Abstract. In this paper we study the existence of nontrivial classical solution for the quasilinear Schrödinger equation:

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = f(u),$$

in $\mathbb{R}^N$, where $N \geq 3$, f has subcritical growth and V is a nonnegative potential. For this purpose, we use variational methods combined with perturbation arguments, penalization technics of Del Pino and Felmer and Moser iteration. As a main novelty with respect to some previous results, in our work we are able to deal with the case $\kappa > 0$ and the potential can vanish at infinity.

1. Introduction

In this article, we consider the following quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = f(u), \quad x \in \mathbb{R}^N \tag{1.1}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are continuous functions with V being a nonnegative function, f having a kind of subcritical growth at infinity and $\kappa > 0$ is a parameter.

2010 *Mathematics Subject Classification.* 35J20, 35J65.

Key words and phrases. Quasilinear Schrödinger equation, subcritical growth, vanishing potentials.

Marco A.S. Souto was partially supported by INCT-MAT, casadinho/PROCAD, CNPq/Brazil 552.464/2011-2 and 304.652/2011-3.

This equation arises in various branches of mathematical physics and has been the subject of extensive study in recent years. As it is well known, solutions of (1.1) are related to the existence of a standing wave solutions for quasilinear Schrödinger equation of the form:

$$i\partial_t z = -\Delta z + W(x)z - l(|z|^2)z + \frac{\kappa}{2}[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.2}$$

where $z\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W\colon \mathbb{R}^N \to \mathbb{R}$ is a given potential and l, ρ are real functions.

Quasilinear Schrödinger equations of the form (1.2) appear naturally in mathematical physics and have been derived as mathematical models of several physical phenomena corresponding to various types of the nonlinear term ρ. The case $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara in [21]. In the case $\rho(s) = (1+s)^{1/2}$, considering solutions of the form $z(t,x) = e^{-i\xi t}u(x)$ where ξ is some real constant, equation (1.2) models the self-channeling of a highpower ultra short laser in matter, see [13], [16] and references in [18]. It is clear that $z(t,x)$ solves (1.2) if and only if $u(x)$ solves (1.1) with $V(x) = W(x) - \xi$ and $f(u) = l(u^2)u$.

Taking into account the values of κ, we find in the literature several papers devoted to the existence of solutions for equation (1.1) when the potential V vanishes at infinity.

The semilinear case corresponding to $\kappa = 0$, that is,

$$-\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

has been studied extensively. See for example [3]–[7], [9]–[12], [14], [20] and the references therein. Among them, we recall the article due to Berestycki and Lions [12] that showed the existence of a positive solution in the case $V \equiv 0$, where the nonlinearity has a supercritical growth near the origin and subcritical growth at infinity. In [20] Ghimenti and Micheletti established existence of sign changing solutions. In [10] Benci, Grisanti and Micheletti established additional conditions on V which provide existence or non existence of the ground state solution. In the papers of Ambrosetti, Felli and Malchiodi [5], Ambrosetti and Wang [7], the nonlinearity $f(u)$ is replaced by a function $f(x,u)$ of the type $k(x)|u|^p$ where $k(x) \to 0$ as $|x| \to \infty$. In [3], Alves and Souto have introduced a new set of hypotheses on the potential V to show the existence of positive solution for (1.3) where f has a subcritical growth.

In the literature we also may cite the article due to Bastos, Miyagaki and Vieira [8] that has established the existence of positive solution for the following class of degenerate quasilinear elliptic problem

$$-\mathcal{L}u_{ap} + V(x)|x|^{-ap^*}|u|^{p-2}u = f(u), \quad \text{in } \mathbb{R}^N,$$

where $\mathcal{L}u_{ap} = -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, $1 < p < N, -\infty < a < (N-p)/p$, $a \leq e \leq a+1$, $d = 1+a-e$, and $p^* := p^*(a,e) = Np/(N-dp)$ denotes the Hardy–Sobolev critical exponent, V is a bounded nonnegative vanishing potential and f has subcritical growth at infinity.

When $\kappa < 0$, specifically $\kappa = -2$, we cite Aires and Souto [1]. Using the change of variables introduced by Colin and Jeanjean in [17] and by Liu, Wang and Wang in [24], jointly with some arguments of [3], [19], they proved the existence of nontrivial solution for equation (1.1) with f has a quasicritical growth and V is a nonnegative potential, which can vanish at infinity.

Recently, Shen and Wang in [25] and Yang, Wang and Abdelgadir in [26] introduced the changing of variables $s = G^{-1}(t)$ for $t \in [0,+\infty)$ and $G^{-1}(t) = -G^{-1}(-t)$ for $t \in (-\infty,0)$, where

$$G(s) = \int_0^s \sqrt{1-\kappa t^2}\,dt. \tag{1.4}$$

with $\kappa < 0$. Using variational methods they established the existence of nontrivial solutions for (1.1) with subcritical or critical growth and among other conditions on the potential $V(x)$, assumed that $\inf_{\mathbb{R}^N} V(x) > 0$.

In a pioneering work, for $\kappa > 0$ and $N \geq 3$, Alves, Wang and Shen in [2] used the method of changing of variables and Morse L^∞ estimates to show the existence of nontrivial solutions for the model (1.1), where $f(u) = |u|^{q-2}u$, $2 < q < 2^*$ or $f(u) = [1-1/(1+|u|^2)^3]u$. Moreover, they assumed that the potential $V\colon \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies

$$0 < V_0 \leq V(x) \leq V_\infty, \quad \text{for all } x \in \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty} V(x) = V_\infty.$$

In [15], Brüll, Lange and Köln studied the one-dimensional quasilinear Schrödinger equations

$$i\partial_t z = -\partial_x^2 z - |z|^{2p} z + \kappa\partial_x^2(|z|^2)z, \quad x \in \mathbb{R} \tag{1.5}$$

and

$$i\partial_t z = -\partial_x^2 z - \left[\mu + \frac{A}{(a+|z|^2)^3}\right] z + \kappa\partial_x^2(|z|^2)z, \quad x \in \mathbb{R}, \tag{1.6}$$

where $z = z(x,t)$ is the unknown wave function, κ is a real constant, $p > 0, \mu > 0$ and $A < 0$. Under some conditions on p, μ and A, they proved that if $0 < \kappa < \kappa_2$ (or $0 < \kappa < \kappa_3$) with some $\kappa_2, \kappa_3 > 0$, then (1.5) (or (1.6)) has a standing wave solution $v(x)$ with $v(x) > 0$, $v(-x) = v(x), v'(x) < 0$ for $x > 0$ and $\lim_{|x|\to\infty} v(x) = 0$. Moreover, this solution is unique up to translation.

Still for $\kappa > 0$, Lange, Poppenberg and Teisniann [22] studied the whole space Cauchy problem for quasilinear Schrödinger equation (1.2) with $W = 0$ and $\rho = 0$. When $N = 1$ and $z(0,x) = \phi(x)$, they obtained L^2- solutions for (1.2) with $\kappa|\phi(x)| \leq \delta < 1$. Moreover, for $2\kappa||\phi||_{W^{1,\infty}} < 1$, they also proved the existence of H^2-solutions for arbitrary space dimension. We refer to [22] for more details.

The main purpose of the present article is to show that, using some ideas of [1] jointly with some arguments of [2], it is possible to extend the results proved in the aforementioned papers to the case where the parameter $\kappa > 0$ and the potential V vanish at infinity.

Related to the function f, we assume that:

(f_1) $\limsup_{s\to 0^+} sf(s)/s^{2^*} < +\infty$, where $2^* = 2N/(N-2)$ and $N \geq 3$.

(f_2) $\lim_{s\to +\infty} sf(s)/s^{2^*} = 0$.

(f_3) There exists $\theta > 2$ such that $\theta F(s) \leq sf(s)$, for all $s > 0$.

The following theorem is our main result:

THEOREM 1.1. *Suppose that f satisfies* (f_1)–(f_3) *and V is a continuous nonnegative function that verifies the condition*:

(V_Λ) *there are $\Lambda > 0$ and $R > 1$ such that*

$$\frac{1}{R^4}\inf_{|x|\geq R}|x|^4V(x) \geq \Lambda.$$

Then, there exist constants $\kappa_0 > 0$ and $\Lambda^* = \Lambda^*(\theta,\ c_0) > 0$ such that (1.1) *possesses a nontrivial solution for all $\kappa \in [0,\kappa_0)$ and $\Lambda \geq \Lambda^*$.*

Note that (1.1) is the Euler–Lagrange equation associated to the natural energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1-\kappa u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\,dx - \int_{\mathbb{R}^N}F(u(x))\,dx. \tag{1.7}$$

It should be pointed out that we may not apply directly the variational method to study (1.1) since the o functional I is not well defined in general, because, $\int_{\mathbb{R}^N}\kappa u^2|\nabla u|^2\,dx$ is not finite, for all $u \in D^{1,2}(\mathbb{R}^N)$ and $\kappa \neq 0$. Beyond this difficulty is overcome we face another one: to ensure the positiveness of the term $1-\kappa t^2$.

In order to prove our main result, we first establish a nontrivial solution for a modified quasilinear Schrödinger equation. Precisely, we consider the existence of nontrivial solutions for the following quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(u), \quad x \in \mathbb{R}^N \tag{1.8}$$

with $g(t) = \sqrt{1-\kappa t^2}$ for $|t| < \sqrt{1/(3\kappa)}$ and $\kappa > 0$. Clearly, when the function $g(t) = \sqrt{1-\kappa t^2}$, equation (1.8) turns into (1.1).

The organization of this paper is as follows: In Section 2, using a change of variable as in references [2], [25] and [26] we reformulate the problem obtaining a semilinear one. In Section 3, we adapt a method explored by Del Pino and Felmer in [19] (see also [3]) to modify the reformulated problem and we show the existence of nontrivial solutions of a modified semilinear Schrödinger equation (3.6) via the mountain pass theorem. In Section 4, we provide an estimate involving the L^∞-norm of a solution of the modified equation. In Section 5 we prove Theorem 1.1.

Notation. In this paper we make use of the following notation:

- C, C_0, $C_1, \ldots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at origin with radius $R > 0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes the functions infinitely differentiable with compact support.
- For $1 \leq s \leq \infty$, we denote the usual norms in the space $L^s(\mathbb{R}^N)$ by

$$||u||_{L^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^s \, dx \right)^{1/s}.$$

- $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$ endowed with the norm $||\nabla u||_{L^2(\mathbb{R}^N)}$.
- S denotes the best constant that verifies

$$\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \leq S \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N).$$

- We denote the weak convergence in E and E' by $\rightharpoonup$ and the strong convergence by $\to$.
- ω_N denotes the volume of the unitary ball in $\mathbb{R}^N$.
- $[|x| \leq a] := \{x \in \mathbb{R}^N : |x| \leq a\}$, $a \in \mathbb{R}$.

2. Preliminaries

We start observing that V is nonnegative, we can introduce the subspace

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 \, dx < +\infty \right\}$$

of $D^{1,2}(\mathbb{R}^N)$ endowed with the norm

$$||u||^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) \, dx.$$

Since $V(x)$ is not suppose to be bounded from bellow by a positive constant, we can not have a continuous imbedding E into $L^q(\mathbb{R}^N)$, for $2 \leq q < 2^*$. Indeed, $q = 2^*$ is the unique $L^q(\mathbb{R}^N)$ space where it is possible to guarentee that $E \hookrightarrow L^q(\mathbb{R}^N)$, continuously.

Let us consider the function $g\colon [0, +\infty) \to \mathbb{R}$ given by

$$g(t) = \begin{cases} \sqrt{1 - \kappa t^2} & \text{if } 0 \leq t < \sqrt{\dfrac{1}{3\kappa}}, \\ \dfrac{1}{3\sqrt{2\kappa} t} + \sqrt{\dfrac{1}{6}} & \text{if } \sqrt{\dfrac{1}{3\kappa}} \leq t, \end{cases}$$

Setting $g(t) = g(-t)$ for all $t \leq 0$, it follows that $g \in C^1(\mathbb{R}, (\sqrt{1/6}, 1])$, g is an even function, increases in $(-\infty, 0)$ and decreases in $[0, +\infty)$.

Observe that (1.8) is the Euler–Lagrange equation associated to the natural energy functional

$$I_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \int_{\mathbb{R}^N} F(u(x))\,dx. \tag{2.1}$$

In what follows, let us define

$$G(t) = \int_0^t g(s)\,ds$$

and we note that the inverse function $G^{-1}(t)$ exists and it is an odd function. Moreover, $G, G^{-1} \in C^2(\mathbb{R})$.

In the following lemma we present some properties of the functions g and G^{-1}, which proofs can be found in [2].

LEMMA 2.1. *The functions g and G^{-1} satisfy the following properties*:

(a) $\lim_{t\to 0} \dfrac{G^{-1}(t)}{t} = 1$;
(b) $\lim_{t\to \infty} \dfrac{G^{-1}(t)}{t} = \sqrt{6}$;
(c) $t \leq G^{-1}(t) \leq \sqrt{6}t$, *for all* $t \geq 0$;
(d) $-\dfrac{1}{2} \leq \dfrac{t}{g(t)} g'(t) \leq 0$, *for all* $t \geq 0$.

At this moment, it is important to say that properties (a) and (b) of Lemma 2.1, together with (f_1) and (f_2) imply that there exists $c_0 > 0$ such that

$$|G^{-1}(s) f(G^{-1}(s))| \leq c_0|s|^{2^*} \quad \text{for all } s \in \mathbb{R}, \tag{2.2}$$

and from condition (g_3) it follows that

$$|F(G^{-1}(s))| \leq \frac{c_0}{\theta}|s|^{2^*} \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$

Now, setting the change of variables

$$v = G(u) = \int_0^u g(s)\,ds,$$

by $I_\kappa(u)$ we obtain the following functional

$$J_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2\,dx - \int_{\mathbb{R}^N} F(G^{-1}(v))\,dx, \tag{2.4}$$

which, due to Lemma 2.1 and the assumptions on the potential $V(x)$ and on the nonlinearity $F(s)$, is well defined in E and $J_\kappa \in C^1(E, \mathbb{R})$ with

$$J'_\kappa(v)\varphi = \int_{\mathbb{R}^N} \left[\nabla v \nabla \varphi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\,\varphi - \frac{f(G^{-1}(v))}{g(G^{-1}(v))}\,\varphi\right] dx, \tag{2.5}$$

for all $v, \varphi \in E$.

Note that if $v \in C^2(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$ is a critical point of the functional J_κ, then $u = G^{-1}(v)$ is a classical solution of (1.8) (see Alves, Wang and Shen in [2]).

Therefore, in order to find a nontrivial solutions of (1.8), it suffices to study the existence of nontrivial solutions of the following equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{2.6}$$

REMARK 2.2. Once assured the existence of a non-trivial solution v for the equation (2.6), then $u = G^{-1}(v)$ will be a nontrivial solution to (1.1) if the estimate $\sup\limits_{\mathbb{R}^N} |u| < \sqrt{1/(3\kappa)}$ holds.

3. The modified equation

In this section, we adapt a method explored by Del Pino and Felmer in [19] (see also [1], [3]) to modify the reformulated problem (2.6). Next, we show the existence of nontrivial solutions of a modified semilinear Schrödinger equation (3.6) via the mountain pass theorem.

To do this, we shall consider constants μ and R satisfying

$$\mu > \frac{\theta}{\theta - 2}(\mu > 1) \quad \text{and} \quad R > 1,$$

and the function

$$h(x,s) = \begin{cases} f(s) & \text{if } |x| \le R, \\ f(s) & \text{if } |x| > R \text{ and } f(s) \le \dfrac{V(x)}{\mu}\, s, \\ \dfrac{V(x)}{\mu}\, s & \text{if } |x| > R \text{ and } f(s) > \dfrac{V(x)}{\mu}\, s. \end{cases}$$

Set $H(x,s) = \int_0^s h(x,t)\,dt$. It is not difficulty to check that $h(x,s)$ satisfies, for all $s \in \mathbb{R}$, the following properties:

$$h(x,s) \le f(s), \qquad \text{for all } x \in \mathbb{R}^N, \tag{3.1}$$

$$h(x,s) \le \frac{V(x)}{\mu}\, s, \qquad \text{for all } |x| \ge R, \tag{3.2}$$

$$H(x,s) = F(s), \qquad \text{if } |x| \le R, \tag{3.3}$$

$$H(x,s) \le \frac{V(x)}{2\mu}\, s^2, \quad \text{if } |x| > R, \tag{3.4}$$

and

$$sh(x,s) - \theta H(x,s) \ge \left(\frac{2-\theta}{2}\right)\frac{V(x)}{\mu}\, s^2, \quad \text{for all } x \in \mathbb{R}^N. \tag{3.5}$$

Now, we study the existence of nontrivial solutions for the modified problem, i.e.

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N, \tag{3.6}$$

which corresponds to the critical points of the functional $\Phi_\kappa\colon E \to \mathbb{R}$ given by

$$\Phi_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2\,dx - \int_{\mathbb{R}^N} H(x,G^{-1}(v))\,dx. \tag{3.7}$$

Note that

$$\Phi_\kappa'(v_n)\varphi = \int_{\mathbb{R}^N}\bigg[\nabla v\nabla\varphi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi - \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}\varphi\bigg]\,dx, \tag{3.8}$$

for all $v,\varphi \in E$.

Remark 3.1. If a nontrivial solution v of (3.6) satisfies

$$f(G^{-1}(v)) \le \frac{V(x)}{k}G^{-1}(v) \quad \text{in } |x| \ge R,$$

then v also is an nontrivial solution of (2.6).

Now we prove that the functional Φ_κ has the mountain pass geometry.

Lemma 3.2. *Suppose that* (f$_1$)–(f$_3$) *are satisfied and that V is nonnegative. Then, there exist $\rho,\alpha > 0$, such that $\Phi_\kappa(v) \ge \alpha$ for $||v|| = \rho$. Moreover, there exists $e \in E$ such that $\Phi_\kappa(e) < 0$.*

Proof. From (3.1), (2.3), the Sobolev–Gagliardo–Nirenberg inequality and being V nonnegative, we have

$$\int_{\mathbb{R}^N} H(x,G^{-1}(v))dx \le C_1\bigg(\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|v|^2)\,dx\bigg)^{2^*/2},$$

from which it follows, using also the propriety (3) of the Lemma 2.1, that

$$\Phi_\kappa(v) \ge \frac{1}{2}||v||^2 - C_1||v||^{2^*}, \quad \text{for all } w \in E.$$

Therefore, by choosing ρ small, we get $\Phi_\kappa(v) \ge \alpha > 0$ when $||v|| = \rho$.

In order to prove the existence of $e \in E$ such that $\Phi_\kappa(e) < 0$, consider $\varphi \in C_0^\infty(\mathbb{R}^N,[0,1])$ satisfying $\operatorname{supp}(\varphi) = \overline{B_1}$. We will prove that $\Phi_\kappa(s\varphi) \to -\infty$ as $s \to +\infty$, which suffices to prove the result if we take $e = s\varphi$ with s large enough. Note, by property (c) of Lemma 2.1, that we get

$$\Phi_\kappa(s\varphi) \le 3s^2\bigg(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N}V(x)\varphi^2\,dx\bigg) - \int_{B_1}H(x,G^{-1}(s\varphi))\,dx.$$

By (3.3), it follows that $H(x,s) = F(s)$ in B_1. By hypothesis (f$_3$), there exist positive constants C_1 and C_2 such that

$$F(s) \ge C_1|s|^\theta - C_2, \quad \text{for all } s \in \mathbb{R}.$$

Therefore, it follows that

$$\Phi_\kappa(s\varphi) \le \frac{1}{2}s^2\bigg(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N}V(x)\varphi^2\,dx\bigg) - C_1\int_{B_1}|G^{-1}(s\varphi)|^\theta\,dx + C_3.$$

Using again property (c) of Lemma 2.1, we have

$$\Phi_\kappa(s\varphi) \le \frac{1}{2}s^2\bigg(\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} V(x)\varphi^2\,dx\bigg) - C_1 s^\theta \int_{B_1} |\varphi|^\theta\,dx + C_3.$$

Since $\theta > 2$, it follows that $\Phi_\kappa(s\varphi) \to -\infty$ as $s \to +\infty$ □

Consequently, using a version of the mountain pass theorem found in [27], there is a Palais–Smale sequence $(v_n) \subset E$ ($(\mathrm{PS})_{c_\kappa}$ sequence) such that

$$\Phi_\kappa(v_n) \to c_\kappa \quad \text{and} \quad \Phi'_\kappa(v_n) \to 0 \quad \text{as } n \to +\infty,$$

where

$$c_\kappa = \inf_{\gamma\in\Gamma_\kappa} \sup_{t\in[0,1]} \Phi_\kappa(\gamma(t)) \ge \alpha > 0, \tag{3.9}$$

with

$$\Gamma_\kappa = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) \ne 0 \ \text{ and } \Phi_\kappa(\gamma(1)) < 0\}. \tag{3.10}$$

LEMMA 3.3. *The Palais–Smale sequence (v_n) for Φ_κ is bounded in E.*

PROOF. The sequence (v_n) satisfies

$$\begin{aligned}\Phi_\kappa(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2)\,dx \\ - \int_{\mathbb{R}^N} H(x, G^{-1}(v_n))\,dx = c_\kappa + o_n(1),\end{aligned} \tag{3.11}$$

and, for every $\varphi \in E$, $\Phi'_\kappa(v)\varphi = o_n(1)||\varphi||$, that is

$$\int_{\mathbb{R}^N} \bigg[\nabla v \nabla\varphi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\,\varphi - \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}\,\varphi\bigg]\,dx = o_n(1)||\varphi||. \tag{3.12}$$

Choosing $\varphi = \varphi_n = G^{-1}(v_n)g(G^{-1}v_n)$, it follows from proprieties (c)–(d) of the Lemma 2.1 that $|\varphi| \le \sqrt{6}|v_n|$ and $|\nabla\varphi| \le |\nabla v_n|$. So,

$$\varphi \in E \quad \text{and} \quad ||\varphi|| \le \sqrt{6}||v_n||.$$

Using $\varphi_n = G^{-1}(v_n)g(G^{-1}v_n)$ as a test function in (3.12), we derive that

$$\begin{aligned}o(1)||v_n|| = \Phi'_\kappa(v_n)\varphi_n = \int_{\mathbb{R}^N} \bigg(1 + \frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\bigg)|\nabla v_n|^2\,dx \\ + \int_{\mathbb{R}^N} [V(x)|G^{-1}(v_n)|^2 - h(x, G^{-1}(v_n))G^{-1}(v_n)]\,dx.\end{aligned} \tag{3.13}$$

From property (d) of the Lemma 2.1, it follows that

$$o(1)||v_n|| \le \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 - h(x, G^{-1}(v_n))G^{-1}(v_n)]\,dx. \tag{3.14}$$

Combining (3.11) and (3.14), we have

$$\begin{aligned}(3.15)\quad \theta c_\kappa + o(1) + o(1)||v_n|| &= \theta\Phi_\kappa(v_n) - \Phi'_\kappa(v_n)\varphi_n \\ &\geq \left(\frac{\theta-2}{2}\right)\int_{\mathbb{R}^N}[|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2]\,dx \\ &\quad + \int_{\mathbb{R}^N}[h(x,G^{-1}(v_n))G^{-1}(v_n) - \theta H(x,G^{-1}(v_n))]\,dx.\end{aligned}$$

Using (3.5) and the property (c) of the Lemma 2.1, it follows that

$$\left(\frac{\mu-1}{\mu^2}\right)||v_n||^2 \leq \theta c_\kappa + o(1) + o(1)||v_n||, \tag{3.16}$$

showing that (v_n) is bounded. □

Since (v_n) is a bounded sequence in E, there exists $v_\kappa \in E$ and a subsequence of v_n, still denoted by itself, such that $v_n \rightharpoonup v_\kappa$ in E, $v_n \to v_\kappa$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [1, 2^*)$, $v_n(x) \to v_\kappa(x)$ almost everywhere on $\mathbb{R}^N$.

Lemma 3.4. *Suppose (v_n) is a $(\mathrm{PS})_{c_\kappa}$ sequence. The following statements hold*:

(a) *For each $\varepsilon > 0$ there exists $r > R$ such that*

$$\limsup_{n\to+\infty}\int_{|x|\geq 2r}[|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2]\,dx < \varepsilon.$$

(b) $\displaystyle\lim_{n\to+\infty}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2\,dx = \int_{\mathbb{R}^N} V(x)|G^{-1}(v_\kappa)|^2\,dx.$

(c) $\displaystyle\lim_{n\to+\infty}\int_{\mathbb{R}^N} h(x,G^{-1}(v_n))G^{-1}(v_n)\,dx = \int_{\mathbb{R}^N} h(x,G^{-1}(v_\kappa))G^{-1}(v_\kappa)\,dx.$

(d) $\displaystyle\lim_{n\to+\infty}\int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\,dx = \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}v_\kappa\,dx.$

(e) $\displaystyle\lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\,dx = \int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v_\kappa))}{g(G^{-1}(v_\kappa))}v_\kappa\,dx.$

(f) $\displaystyle\lim_{n\to+\infty}\int_{\mathbb{R}^N} H(x,G^{-1}(v_n))\,dx = \int_{\mathbb{R}^N} H(x,G^{-1}(v_\kappa))\,dx.$

Proof. (a) Consider $r > R$ and a function $\eta = \eta_r \in C_0^\infty(B_r^c)$ such that $\eta \equiv 1$ in B_{2r}^c, *eta* $\equiv 0$ in B_r, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq 2/r$, for all $x \in \mathbb{R}^N$. As (v_n) is bounded in E, the sequence $(\eta\varphi_n)$, where $\varphi_n = G^{-1}(v_n)g(G^{-1}v_n)$, is also bounded. Hence, $\Phi'(v_n)\eta\varphi_n = o_n(1)$, that is

$$\int_{\mathbb{R}^N}\left(1+\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right)|\nabla v_n|^2\eta\,dx+\int_{\mathbb{R}^N}V(x)|G^{-1}(v_n)|^2\eta\,dx$$
$$=-\int_{\mathbb{R}^N}\nabla v_n\nabla\eta(G^{-1}(v_n)g(G^{-1}(v_n)))\,dx$$
$$+\int_{\mathbb{R}^N}h(x,G^{-1}(v_n))G^{-1}(v_n)\eta\,dx+o_n(1).$$

From properties (c) and (d) of the Lemma 2.1 it follows that

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\eta\,dx+\int_{\mathbb{R}^N}V(x)|G^{-1}(v_n)|^2\eta\,dx$$
$$\le\sqrt{6}\int_{\mathbb{R}^N}|\nabla v_n||\nabla\eta||v_n|\,dx+\int_{\mathbb{R}^N}h(x,G^{-1}(v_n))G^{-1}(v_n)\eta\,dx+o_n(1).$$

Once that $\eta\equiv 0$ in B_r, the last inequality combined with (3.2) yields

$$\left(1-\frac{1}{\mu}\right)\int_{[|x|\ge r]}[|\nabla v_n|^2+V(x)G^{-1}(v_n)]\eta\,dx$$
$$\le\sqrt{6}\int_{[|x|\ge r]}|w_n||\nabla w_n||\nabla\eta|\,dx+o_n(1),$$

that is,

$$\left(1-\frac{1}{\mu}\right)\int_{[|x|\ge r]}[|\nabla v_n|^2+V(x)G^{-1}(v_n)]\eta\,dx \tag{3.17}$$
$$\le\frac{2\sqrt{6}}{r}\int_{[r\le|x|\le 2r]}|v_n||\nabla v_n|\,dx+o_n(1).$$

By Hölder inequality,

$$\int_{[r\le|x|\le 2r]}|v_n||\nabla v_n|\,dx\le\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\right)^{1/2}\left(\int_{[r\le|x|\le 2r]}v_n^2\,dx\right)^{1/2}.$$

Since $v_n\to v_\kappa$ in $L^2(B_{2r}\setminus B_r)$ and (v_n) is bounded in E, it follows that

$$\limsup_{n\to+\infty}\int_{[r\le|x|\le 2r]}|v_n||\nabla v_n|\,dx\le C\left(\int_{[r\le|x|\le 2r]}v_\kappa^2\,dx\right)^{1/2}, \tag{3.18}$$

for some constant $C>0$. On the other hand, using again Hölder inequality,

$$\left(\int_{[r\le|x|\le 2r]}v_\kappa^2\,dx\right)^{1/2}\le\left(\int_{[r\le|x|\le 2r]}|v_\kappa|^{2^*}\,dx\right)^{1/2^*}|B_{2r}\setminus B_r|^{1/N}. \tag{3.19}$$

Noting that $|B_{2r}\setminus B_r|\le|B_{2r}|=\omega_N(2r)^N$, from (3.18) and (3.19), we have

$$\limsup_{n\to+\infty}\int_{[r\le|x|\le 2r]}|v_n||\nabla v_n|\,dx\le 2rC\omega_N^{1/N}\left(\int_{[r\le|x|\le 2r]}|v_\kappa|^{2^*}\,dx\right)^{1/2^*}, \tag{3.20}$$

and from (3.17) and (3.20), it follows that

$$\tag{3.21} \limsup_{n\to+\infty} \int_{[|x|\geq 2r]} [|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2]\,dx \\ \leq 4\sqrt{6}C\omega_N^{1/N}\bigg(1-\frac{1}{\mu}\bigg)^{-1}\bigg(\int_{[r\leq|x|\leq 2r]}|v_\kappa|^{2^*}\,dx\bigg)^{1/2^*}.$$

Thus, for every $\varepsilon > 0$, we choose $r > R$ such that

$$4\sqrt{6}C\omega_N^{1/N}\bigg(1-\frac{1}{\mu}\bigg)^{-1}\bigg(\int_{[r\leq|x|\leq 2r]}|v_\kappa|^{2^*}\,dx\bigg)^{1/2^*} < \varepsilon,$$

and this concludes part (a) of the proof.

(b) Note first that from part (a), for each $\varepsilon > 0$, there exists $r > R$ such that

$$\limsup_{n\to+\infty} \int_{[|x|\geq 2r]} V(x)|G^{-1}(v_n)|^2\,dx < \frac{\varepsilon}{4}$$

and consequently,

$$\int_{[|x|\geq 2r]} V(x)|G^{-1}(v_\kappa)|^2\,dx \leq \frac{\varepsilon}{4}.$$

Hence,

$$\tag{3.22} \bigg|\int_{\mathbb{R}^N} V(x)[|G^{-1}(v_n)|^2 - |G^{-1}(v_\kappa)|^2]\,dx\bigg| \\ \leq \frac{\varepsilon}{2} + \bigg|\int_{[|x|\leq 2r]} V(x)[|G^{-1}(v_n)|^2 - |G^{-1}(v_\kappa)|^2]\,dx\bigg|.$$

Since $v_n \to v_\kappa$ in $L^2(B_{2r})$, using the Lebesgue Dominated Convergence Theorem, it follows that

$$\tag{3.23} \lim_{n\to+\infty} \int_{[|x|\leq 2r]} V(x)|G^{-1}(v_n)|^2\,dx = \int_{[|x|\leq 2r]} V(x)|G^{-1}(v_\kappa)|^2\,dx.$$

From (3.22) and (3.23), we have

$$\limsup_{n\to+\infty} \bigg|\int_{\mathbb{R}^N} V(x)[|G^{-1}(v_n)|^2 - |G^{-1}(v_\kappa)|^2]\,dx\bigg| \leq \frac{\varepsilon}{2},$$

for every $\varepsilon > 0$. Therefore,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2\,dx = \int_{\mathbb{R}^N} V(x)|G^{-1}(v_\kappa)|^2\,dx.$$

(c) It follows from (3.2) and part (a) that, for each $\varepsilon > 0$, there exists $r > R$ such that

$$\limsup_{n\to+\infty} \int_{[|x|\geq 2r]} h(x, G^{-1}(v_n))G^{-1}(v_n)\,dx < \frac{\varepsilon}{4}$$

and

$$\int_{[|x|\geq 2r]} h(x, G^{-1}(v_\kappa))G^{-1}(v_\kappa)\,dx \leq \frac{\varepsilon}{4}.$$

Therefore,

$$\left|\int_{\mathbb{R}^N}[h(x,G^{-1}(v_n))G^{-1}(v_n)-h(x,G^{-1}(v_\kappa))G^{-1}(v_\kappa)]\,dx\right| \\ \leq \frac{\varepsilon}{2}+\left|\int_{[|x|<2r]}[h(x,G^{-1}(v_n))G^{-1}(v_n)-h(x,G^{-1}(v_\kappa))G^{-1}(v_\kappa)]\,dx\right|. \tag{3.24}$$

Since

$$v_n(x)\to v_\kappa(x)\quad\text{a.e. on }\mathbb{R}^N,\qquad \frac{h(\,\cdot\,,G^{-1}(s))G^{-1}(s)}{|G^{-1}(s)|^{2^*}}\to 0\quad\text{as } s\to+\infty$$

and

$$\sup_n\int_{\mathbb{R}^N}|G^{-1}(v_n)|^{2^*}<+\infty,$$

it follows from the Compactness Lemma of Strauss [12] that

$$\lim_{n\to+\infty}\int_{[|x|<2r]}h(x,G^{-1}(v_n))G^{-1}(v_n)\,dx \\ =\int_{[|x|<2r]}h(x,G^{-1}(v_\kappa))G^{-1}(v_\kappa)\,dx. \tag{3.25}$$

From (3.24) and (3.25), the result follows. This completes the proof of part (c).

Using similar arguments we prove (d), (e) and (f). □

As a consequence of Lemma 3.4, we conclude that

COROLLARY 3.5. *We have that v_κ is non-trivial critical point of Φ_κ and $\Phi_\kappa(v_\kappa)=c_\kappa$. Moreover, the functional Φ_κ satisfies the $(\mathrm{PS})_{c_\kappa}$ condition.*

PROOF. Our first goal is proving that v_κ is critical point of Φ_κ. To this end, it suffices to show that

$$\Phi_\kappa'(v_\kappa)\phi=0,\quad\text{for all }\phi\in C_0^\infty(\mathbb{R}^N).$$

As in the proof of previous lemma, it is easy to deduct that

$$\int_{\mathbb{R}^N}V(x)\bigg[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\bigg]\phi\,dx\to 0\quad\text{as } n\to+\infty, \tag{3.26}$$

and

$$\int_{\mathbb{R}^N}\bigg[\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{h(x,G^{-1}(v_\kappa))}{g(G^{-1}(v_\kappa))}\bigg]\phi\,dx\to 0\quad\text{as } n\to+\infty, \tag{3.27}$$

for all $\phi\in C_0^\infty(\mathbb{R}^N)$. Moreover, since $v_n\rightharpoonup v_\kappa$ we have

$$\int_{\mathbb{R}^N}\nabla(v_n-v_\kappa)\nabla\phi\,dx\to 0\quad\text{as } n\to+\infty. \tag{3.28}$$

Combining (3.26)–(3.28) it is proved that

$$\lim_{n\to+\infty}\Phi_\kappa'(v_n)\phi=\Phi_\kappa'(v_\kappa)\phi,\quad\text{for all }\phi\in C_0^\infty(\mathbb{R}^N).$$

Since $\Phi'_\kappa(v_n)\phi = o_n(1)$, the last limit yields $\Phi'_\kappa(v_\kappa)\phi = 0$, for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

Let us show that $v_\kappa \neq 0$. To prove this, we argue by contradiction supposing that $v_\kappa = 0$. From Lemma 3.4(b), it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2\,dx = 0, \tag{3.29}$$

which implies in

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|v_n|^2\,dx = 0, \tag{3.30}$$

and consequently,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\,dx = 0. \tag{3.31}$$

Since $g(0) \neq 0$ and $H(x,0) = 0$, using conditions (e) and (f) of Lemma 3.4, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v_n))v_n}{g(G^{-1}(v_n))}\,dx = 0. \tag{3.32}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} H(x,G^{-1}(v_n))\,dx = 0. \tag{3.33}$$

Using (3.31) and (3.32) we have, from $\Phi'_\kappa(v_n).v_n = 0$, that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \to 0, \tag{3.34}$$

and thus we obtain

$$\Phi_\kappa(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\,dx - \int_{\mathbb{R}^N} H(x,G^{-1}(v_n))\,dx \to 0,$$

but this is a contradiction with $\Phi_\kappa(v_n) \to c_\kappa > 0$. Hence, $v_\kappa \neq 0$.

Now, we will show that $\Phi_\kappa(v_\kappa) = c_\kappa$. Once $\Phi'(v_n)v_n = o(1)$ and using the limits (d)–(e) of Lemma 3.4, together with $\Phi'(v_\kappa)v_\kappa = 0$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx = \int_{\mathbb{R}^N} |\nabla v_\kappa|^2\,dx. \tag{3.35}$$

The last limits combined with (b) and (f) of the Lemma 3.4, imply

$$\Phi_\kappa(v_n) = \int_{\mathbb{R}^N} \left[\frac{1}{2}(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) - H(x,G^{-1}(v_n))\right]dx \to \Phi_\kappa(v_\kappa).$$

Hence, $\Phi_\kappa(v_\kappa) = c_\kappa$.

To show that the functional Φ_κ satisfies $(\mathrm{PS})_{c_\kappa}$ condition, it remains to show that $||v_n - v_\kappa|| \to 0$. Proceeding as in the proof of Lemma 3.4(b), it follows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V(x)v_n^2\,dx = \int_{\mathbb{R}^N} V(x)v_\kappa^2\,dx.$$

Using this limit and (3.35), we conclude that

$$||v_n - v_\kappa||^2 = \int_{\mathbb{R}^N} [|\nabla v_n - \nabla v_\kappa|^2 + V(x)(v_n^2 - v_\kappa^2)]\, dx \to 0.$$

Consequently, Φ_κ satisfies the Palais–Smale condition. □

4. L^∞ estimate of the solution of the modified equation

In this section, we will establish an L^∞ estimate for solution v_κ obtained in Corollary 3.5.

LEMMA 4.1. *For $R > 1$, any solution v_κ of the equation* (3.6)

$$\|v_\kappa\|^2 \le \frac{\theta\mu^2 c_\kappa}{\mu - 1}.$$

PROOF. We know that $\Phi_\kappa(v_\kappa) = c_\kappa$. Then

$$\begin{aligned}\theta c_\kappa &= \theta\Phi_\kappa(v_\kappa) - \Phi_\kappa'(v_\kappa)G^{-1}(v_\kappa)g(G^{-1}(v_\kappa)) \\ &= \frac{\theta}{2}\int_{\mathbb{R}^N} (|\nabla v_\kappa|^2 + V(x)|G^{-1}(v_\kappa)|^2\, dx - \theta \int_{\mathbb{R}^N} H(x, G^{-1}(v_\kappa))\, dx \\ &\quad - \int_{\mathbb{R}^N} \bigg(1 + \frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\bigg)|\nabla v_n|^2\, dx \\ &\quad + \int_{\mathbb{R}^N} [V(x)|G^{-1}(v_n)|^2 - h(x, G^{-1}(v_n))G^{-1}(v_n)]\, dx\end{aligned}$$

From property (d) of the Lemma 2.1, we have

$$\begin{aligned}\theta c_\kappa \ge \bigg(\frac{\theta - 2}{2}\bigg) &\int_{\mathbb{R}^N} (|\nabla v_\kappa|^2 + V(x)|G^{-1}(v_\kappa)|^2)\, dx \\ &+ \int_{\mathbb{R}^N} [h(x, G^{-1}(v_n))G^{-1}(v_n) - \theta H(x, G^{-1}(v_\kappa))\, dx]\, dx.\end{aligned}$$

Due to (3.5), it follows that

$$\begin{aligned}\theta c_\kappa \ge \bigg(\frac{\theta - 2}{2}\bigg) &\int_{\mathbb{R}^N} (|\nabla v_\kappa|^2 + V(x)|G^{-1}(v_\kappa)|^2)\, dx \\ &+ \bigg(\frac{2-\theta}{2}\bigg)\frac{1}{\mu}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_\kappa)|^2\, dx.\end{aligned} \tag{4.1}$$

Picking $\mu > \theta/(\theta - 2)$, we obtain

$$\bigg(\frac{\mu - 1}{\mu^2}\bigg)\int_{\mathbb{R}^N} (|\nabla v_\kappa|^2 + V(x)|G^{-1}(v_\kappa)|^2)\, dx \le \theta c_\kappa, \tag{4.2}$$

that is,

$$\|v_\kappa\|^2 \le \frac{\theta\mu^2 c_\kappa}{\mu - 1}. \tag{4.3}$$

□

REMARK 4.2. In the previous lemma, $\|v_\kappa\|$ is bounded by a constant that does not depend on $R > 1$. However, this constant depends on $\kappa > 0$.

To obtain, for v_κ, an uniform boundedness of the Sobolev norm independent on $\kappa > 0$, we denote by B the unitary ball in $\mathbb{R}^N$, that is, $B = B_1(0)$ and we consider the functional $\Phi_0\colon H_0^1(B) \to \mathbb{R}$ given by

$$\Phi_0(v) = 3\int_B (|\nabla v|^2\,dx + V(x)v^2)\,dx - \int_B F(v)\,dx \tag{4.4}$$

and the set

$$\Gamma_0 = \{\gamma \in C([0,1], H_0^1(B)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } \Phi_0(\gamma(1)) < 0\}, \tag{4.5}$$

Since the function F is non-decreasing, using the Lemma 2.1(c) we have $\Phi_\kappa(v) \leq \Phi_0(v)$ and thereby $\Gamma_0 \subset \Gamma_\kappa$. Hence,

$$c_\kappa = \inf_{\gamma\in\Gamma_\kappa}\sup_{t\in[0,1]}\Phi_\kappa(\gamma(t)) \leq \inf_{\gamma\in\Gamma_0}\sup_{t\in[0,1]}\Phi_\kappa(\gamma(t)) \leq \inf_{\gamma\in\Gamma_0}\sup_{t\in[0,1]}\Phi_0(\gamma(t)) := d,$$

where d is a constant independent on κ. Consequently, by Lemma 4.1, the solution v_κ must satisfy the estimate

$$\|v_\kappa\|^2 \leq \frac{\theta\mu^2 d}{\mu - 1}. \tag{4.6}$$

Now, following the same ideas present in Aires and Souto [1], we will establish an important estimate involving $L^\infty(\mathbb{R}^N)$ norm for a solution v_κ of the equation (3.6). We will use the following estimate result which proof follows from Proposition 5.3 and Corollary 5.4 in [1] (see also Proposition 2.6 in Alves and Souto [3]).

Proposition 4.3. *Let $N > 2$, $r > 2^*$ and $v \in E\cap L^r(\mathbb{R}^N)$ be a weak solution of the problem*

$$-\triangle v + b(x)v = L(x,v) \quad \textit{in } \mathbb{R}^N, \tag{4.7}$$

where $L\colon \mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ is a continuous function verifying

$$|L(x,s)| \leq C_0|s|^{2^*-1}, \quad \textit{for all } s\in\mathbb{R},\ x\in\mathbb{R}^N$$

and b is a nonnegative function in $\mathbb{R}^N$. Then there exists a constant $C = C(C_0, ||v||_{L^r(\mathbb{R}^N)}) > 0$ such that $\|v\|_{L^\infty(\mathbb{R}^N)} \leq C\|v\|$.

In order to obtain the boundedness in the L^∞ norm, we consider for a solution v_κ of the equation (3.6), the following function

$$L(x,t) = \begin{cases} \dfrac{f(G^{-1}(t))}{g(G^{-1}(t))} & \text{if } |x| < R \text{ or } f(G^{-1}(t)) \leq \dfrac{V(x)}{\mu}G^{-1}(t), \\ 0 & \text{if } |x| \geq R \text{ and } f(G^{-1}(t)) > \dfrac{V(x)}{\mu}G^{-1}(t), \end{cases} \tag{4.8}$$

and the following non-negative mensurable function

$$b(x) = \begin{cases} \dfrac{1}{v_\kappa}V(x)\dfrac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}, & \text{if } |x| < R \text{ or } f(G^{-1}(v_\kappa)) \leq \dfrac{V(x)}{\mu}G^{-1}(v_\kappa), \\ \left(1 - \dfrac{1}{\mu}\right)\dfrac{1}{v_\kappa}\dfrac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))} & \text{if } |x| \geq R \text{ and } f(G^{-1}(v_\kappa)) > \dfrac{V(x)}{\mu}G^{-1}(v_\kappa). \end{cases}$$

Note that v_κ satisfies an equation such as (4.7). From Lemma 2.1 and (2.2), we derive that $|L(x,t)| \le C_1|t|^{2^*-1}$, for some constant $C_1 > 0$.

To apply Proposition 4.3, it remains to show the boundedness $L^r(\mathbb{R}^N)$ norm, for some $r > 2^*$.

LEMMA 4.4. *Let $N > 2$ and $\beta = N/(N-2)$. There exists a constant $C = C_\varepsilon > 0$, such that*

$$\|v_\kappa\|_{L^{2^*\beta}(\mathbb{R}^N)} \le C\|v_\kappa\|_{L^{2^*}(\mathbb{R}^N)}.$$

PROOF. In proof of this Lemma, we denote v_κ by v. Proceeding as in the proof of the Lemma 5.5 (see Aires and Souto [1]); let v a positive solution of (4.7), and for each $m \in \mathbb{N}$, consider the sets $A_m = \{x \in \mathbb{R}^N : |v|^{\beta-1} \le m\}$ and $B_m = \mathbb{R}^N \setminus A_m$. Let us define

$$v_m = \begin{cases} v|v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m, \end{cases} \quad \text{and} \quad z_m = \begin{cases} v|v|^{\beta-1} & \text{in } A_m, \\ mv & \text{in } B_m. \end{cases}$$

Using v_m as a test function and since $0 \le b(x)z_m^2 = b(x)vv_m$ in $\mathbb{R}^N$ and $\beta > 1$, we deduce that

$$\int_{\mathbb{R}^N} (|\nabla z_m|^2 + b(x)z_m^2)\,dx \le \beta^2 \int_{\mathbb{R}^N} L(x,v)v_m\,dx. \tag{4.9}$$

Note that the function L defined in (4.8) verifies the following conditions:

(L_1) $|L(x,t)| \le c_0|t|^{2^*-1}$, for t sufficiently small,

(L_2) $\lim\limits_{s\to+\infty} L(x,t)/|t|^{2^*-1} = 0$.

Observe that the conditions (L_1) and (L_2) imply that, for each $\varepsilon > 0$, there is $C = C_\varepsilon(\varepsilon, c_0) > 0$ such that

$$|L(x,t)| \le \varepsilon|t|^{2^*-1} + C_\varepsilon|t|, \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Using this inequality in (4.9), we have

$$\int_{\mathbb{R}^N} (|\nabla z_m|^2 + b(x)z_m^2)\,dx \le \beta^2\varepsilon \int_{\mathbb{R}^N} |v|^{2^*-1}|v_m|\,dx + C\beta^2 \int_{\mathbb{R}^N} z_m^2\,dx. \tag{4.10}$$

Observe that

$$\int_{\mathbb{R}^N} |v|^{2^*-1}|v_m|\,dx \le \int_{\mathbb{R}^N} |v|^{2^*-2}z_n^2\,dx \le ||z_m||^2_{L^{2^*}(\mathbb{R}^N)}\bigg(\int_{\mathbb{R}^N} |v|^{2^*}\,dx\bigg)^{2^*-2},$$

that is,

$$\int_{\mathbb{R}^N} |v|^{2^*-1}|v_n|\,dx \le S||v||^{2^*-2}_{L^{2^*}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla z_m|^2\,dx,$$

which combined with (4.10) results in

$$\int_{\mathbb{R}^N} (|\nabla z_m|^2 + b(x)z_m^2)\,dx \le \beta^2\varepsilon S||v||^{2^*-2}_{L^{2^*}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla z_m|^2\,dx + C\beta^2 \int_{\mathbb{R}^N} z_m^2\,dx. \tag{4.11}$$

By estimate (4.6), we can choose $\varepsilon > 0$ such that $\varepsilon\beta^2||v||^{2^*-2}_{L^{2^*}(\mathbb{R}^N)}S < 1/2$, from which it follows that,

$$\int_{\mathbb{R}^N} (|\nabla z_m|^2 + b(x)z_m^2)\,dx \le 2C\beta^2 \int_{\mathbb{R}^N} z_m^2\,dx.$$

Using Sobolev embedding, we have

$$\left(\int_{A_m} |z_m|^{2^*}\,dx\right)^{2/2^*} \le S\int_{\mathbb{R}^N} |\nabla z_m|^2\,dx \le 2SC\beta^2 \int_{\mathbb{R}^N} z_m^2\,dx.$$

Since $|z_m| = |v|^\beta$ in A_m and $|z_m| \le |v|^\beta$ in $\mathbb{R}^N$, it follows that

$$\left[\int_{A_m} |v|^{2^*\beta}\,dx\right]^{1/2^*\beta} \le (2SC\beta^2)^{1/2\beta}\left[\int_{\mathbb{R}^N} |v|^{2\beta}\,dx\right]^{1/2\beta}.$$

By the Monotone Convergence Theorem, letting $m \to +\infty$, we have

$$||v||_{L^{2^*\beta}(\mathbb{R}^N)} \le (2SC\beta^2)^{1/2\beta}||v||_{L^{2^*}(\mathbb{R}^N)}. \qquad \square$$

It follows from Lemma 4.4 that v_κ is bounded in $L^r(\mathbb{R}^N)$, with $r = 2^*\beta > 2^*$. Applying the Proposition 4.3, we conclude that there exists a constant $C = C(C_\varepsilon, ||v_\kappa||_{L^r(\mathbb{R}^N)}) > 0$ such that $\|v_\kappa\|_{L^\infty(\mathbb{R}^N)} \le C\|v_\kappa\|$, for any $v_\kappa \in E \cap L^r(\mathbb{R}^N)$ weak solution of the problem (4.7). Hence, any weak solution v_κ of the equation (3.6) satisfies the estimate

$$\|v_\kappa\|_{L^\infty(\mathbb{R}^N)} \le M, \tag{4.12}$$

where $M = C(\theta\mu^2 d/(\mu-1))^{1/2} > 0$ is independent of $\kappa > 0$.

Lemma 4.5. *For $R > 1$, any positive solution v_κ of the equation* (3.6) *satisfies*

$$v_\kappa(x) \le \frac{R^{N-2}\|v_\kappa\|_{L^\infty(\mathbb{R}^N)}}{|x|^{N-2}} \le \frac{R^{N-2}M}{|x|^{N-2}}, \quad \text{for all } |x| \ge R.$$

Proof. Let u be the $C^\infty(\mathbb{R}^N \setminus \{0\})$ harmonic function given by

$$u(x) = R^{N-2}M/|x|^{N-2}.$$

By estimate (4.12), we have $v_\kappa(x) \le u(x)$ for $|x| = R$. It follows that $(v_\kappa - u)^+ = 0$ for $|x| = R$, and the function given by

$$\phi = \begin{cases} (v_\kappa - u)^+ & \text{if } |x| \ge R, \\ 0 & \text{if } |x| < R, \end{cases}$$

belongs to $D^{1,2}(\mathbb{R}^N)$. Moreover, $\phi \in E$. Employing ϕ as a test function and using the fact that v_κ is a solution of (3.6), we have

$$\int_{\mathbb{R}^N} \nabla v_\kappa \nabla\phi\,dx + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\phi\,dx = \int_{\mathbb{R}^N} \frac{h(x, G^{-1}(v_\kappa))}{g(G^{-1}(v_\kappa))}\phi\,dx. \tag{4.13}$$

On the other hand, by definition of ϕ it follows that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx = \int_A \nabla v_\kappa\nabla\phi\,dx - \int_A \nabla u\nabla\phi\,dx, \tag{4.14}$$

where $A = \{x \in \mathbb{R}^N : |x| \ge R \text{ and } v_\kappa(x) > u(x)\}$.

Since $\Delta u = 0$ in $\mathbb{R}^N \setminus B_R(0)$, $\phi = 0$ for $|x| = R$ and $\phi \ge 0$, we have

$$\int_A \nabla u\nabla\phi\,dx = 0.$$

Thus using (4.13) and (4.14) it follows that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx = \int_A \frac{h(x, G^{-1}(v_\kappa))}{g(G^{-1}(v_\kappa))}\phi\,dx - \int_A V(x)\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\phi\,dx,$$

and from (3.2), we conclude that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx \leq \left(\frac{1}{\mu}-1\right)\int_A V(x)\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))}\phi\,dx \leq 0.$$

Hence, we have $\phi = 0$, in $\mathbb{R}^N$, which implies that $(v_\kappa - u)^+ = 0$, in $|x| \geq R$. From this we conclude that $v_\kappa \leq u$ in $|x| \geq R$ and the lemma is proved. □

5. Proof of the main result

Proof of Theorem 1.1. By Remark 3.1, to show that v_κ is also solution of the equation (2.6), it is sufficient to show that

$$f(G^{-1}(v_\kappa)) \leq \frac{V(x)}{\mu}G^{-1}(v_\kappa) \quad \text{in } |x| \geq R.$$

By (2.2) and Lemma 2.1(c), we have

$$\frac{f(G^{-1}(v_\kappa))}{G^{-1}(v_\kappa)} \leq c_0|v_\kappa|^{4/(N-2)}, \quad \text{for all } x \in \mathbb{R}^N.$$

Using Lemma 4.5, it follows that,

$$\frac{f(G^{-1}(v_\kappa))}{G^{-1}(v_\kappa)} \leq c_0\frac{R^4M^{4/(N-2)}}{|x|^4}, \quad \text{in } |x| \geq R.$$

Fixing $\Lambda^* = \mu c_0 M^{4/(N-2)}$ and $\Lambda \geq \Lambda^*$, it implies that

$$\frac{f(G^{-1}(v_\kappa))}{G^{-1}(v_\kappa)} \leq \frac{1}{\mu}\Lambda^*\frac{R^4}{|x|^4} \leq \frac{1}{\mu}\Lambda\frac{R^4}{|x|^4}.$$

It follows from hypothesis (V_Λ) that

$$\frac{f(G^{-1}(v_\kappa))}{G^{-1}(v_\kappa)} \leq \frac{V(x)}{\mu} \quad \text{in } |x| \geq R,$$

which implies that v_κ is a solution for the equation (2.6), that is,

$$-\Delta v_\kappa + V(x)\frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))} = \frac{f(G^{-1}(v_\kappa))}{g(G^{-1}(v_\kappa))}, \quad x \in \mathbb{R}^N.$$

On the other hand, v_κ satisfies $\|v_\kappa\|_{L^\infty(\mathbb{R}^N)} \leq C(\theta\mu^2 d/(\mu-1))^{1/2}$. Thus,

$$\|G^{-1}(v_\kappa)\|_{L^\infty(\mathbb{R}^N)} \leq \sqrt{6}\|v_\kappa\|_{L^\infty(\mathbb{R}^N)} \leq \sqrt{6}C\left(\frac{\theta\mu^2 d}{\mu-1}\right)^{1/2}.$$

Choosing $\kappa_0 \leq (\mu-1)/(18C^2\theta\mu^2 d)$, it follows that

$$\|G^{-1}(v_\kappa)\|_{L^\infty(\mathbb{R}^N)} < \sqrt{\frac{1}{3\kappa}}, \quad \text{for all } \kappa \in [0,\kappa_0).$$

From Remark 2.2 it implies that $u = g(G^{-1}(v_\kappa))$ is a classical solution of (1.1). □

References

[1] J.F.L. Aires and M.A.S. Souto, it Existence of solutions for a quasilinear Schrödinger equation with vanishing potentials, J. Math. Anal. Appl. **416** (2014), 924–946.

[2] C.O. Alves, Y. Wang and Y. Shen, *Soliton solutions for for class of quasilinear Schrödinger equations with a parameter*, J. Differential Equations **259** (2015), 318–343.

[3] C.O. Alves and M.A.S. Souto, *Existence of solutions for a class of elliptic equations in R^N with vanishing potentials*, J. Differential Equations **252** (2012), 5555–5568.

[4] ______, *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity*, J. Differential Equations **254** (2013), 1977–1991.

[5] A. Ambrosetti, V. Felli and A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. **7** (2005), 117–144.

[6] A. Ambrosetti, A. Malchiodi and D. Ruiz, *Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Anal. Math. **98** (2006), 317–348.

[7] A. Ambrosetti and Z.-Q. Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials*, Differential Integral Equations **18** (2005), 1321–1332.

[8] W.D. Bastos, O.H. Miyagaki and R.S. Vieira, *Existence of solutions for a class of degenerate quasilinear elliptic equation in RN with vanishing potentials*, Boundary Value Problems **92** (2013), doi:10.1186/1687-2770-2013-92.

[9] V. Benci and G. Cerami, *Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$*, J. Funct. Anal. **88** (1) (1990), 90–117.

[10] V. Benci, C.R. Grisanti and A.M. Micheletti, *Existence and non existence of the ground state solution for the nonlinear Schr̆odinger equations with $V(\infty) = 0$*, Topol. Methods in Nonlinear Anal. **26** (2005), 203–219.

[11] bysame, *Existence of solutions for the nonlinear Schr̆odinger equation with $V(\infty) = 0$*, Progr. Nonlinear Differential Equations Appl. **66** (2005), 53–65.

[12] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations* I*, Existence of a ground state*, Arch. Ration. Mech. Anal. **82** (1983), 313–346.

[13] C. Borovskiĭ and A. Galkin, *Dynamical modulation of an ultrashort high-intensity laser pulse in matter*, J. Exp. Theor. Phys. **77** (1983), 562–573.

[14] D. Bonheure and J. Van Schaftingen, *Groundstates for nonlinear Schrödinger equation with potential vanishing at infinity*, Ann. Mat. Pura Appl. *189* (2010), 273–301.

[15] L. Brüll, H. Lange and Köln, *Stationary, oscillatory and solitary waves type solutions of singular nonlinear Schrödinger equations*, Math. Mech. Appl. sci. **8** (1986), 559–575.

[16] X.L. Chen and R.N. Sudan, *Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse*, Phys. Rev. Lett. **70** (1993), 2082–2085.

[17] M. Colin and L. Jeanjean, *Solutions for a quasilinear Schrödinger equation: a dual approach*, Nonlinear Anal. **56** (2004), 213–226.

[18] A. De Bouard and J. Saut, *Global existence of small solutions to a relativistic nonlinear Schrödinger equation*, Comm. Math. Phys. **189** (1997), 73–105.

[19] M. Del Pino and P. Felmer, *Local mountain pass for semilinear elliptic problems in unbounded domains*, Calc. Var. **4** (1996), 121–137.

[20] M. Ghimenti and A.M. Micheletti, *Existence of minimal nodal solutions for the nonlinear Schrödinger equations with $V(\infty) = 0$*, Adv. Differential Equations **11** (2006), no. 12, 1375–1396.

[21] S. Kurihara, *Large-amplitude quasi-solitons in superfluids films*, J. Phys. Soc. Japan **50** (1981), 3262–3267.

[22] H. Lange, M. Poppenberg and H. Teisniann, *Nash–More methods for the solution of quasilinear Schrödinger equations*, Commun. Partial Differential Equations **24** (7–8) (1999), 1399–1418.

[23] P.L. LIONS, *The concentration compactness principle in the calculus of variations, The locally compact case*, Parts I and II, Ann. Inst. H. Poincaré Anal. Non. Lineairé (1984), 109–145, 223–283.

[24] J. LIU, Y. WANG AND Z. WANG, *Soliton solutions for quasilinear Schrödinger equations* II, J. Differential Equations **187** (2003), 473–493.

[25] Y. SHEN AND Y. WANG, *Soliton solutions for generalized quasilinear Schrödinger equations*, Nonlinear Anal. **80** (2013), 194–201.

[26] J. YANG, Y. WANG AND A.A. ABDELGADIR, *Soliton solutions for quasilinear Schrödinger equations*, J. Math. Phys. **54** (2013), doi: 10.1063/1.4811394.

[27] M. WILLEM, *Minimax Theorems*, Birkhäuser, (1986).

Manuscript received April 23, 2015

accepted June 10, 2014

JOSÉ F.L. AIRES AND MARCO A. S. SOUTO
Universidade Federal de Campina Grande
Unidade Acadêmica de Matemática
CEP:58429-900, Campina Grande – PB, BRAZIL

E-mail address: fernando@dme.ufcg.edu.br, e-mail:marco@dme.ufcg.edu.br