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LIMIT SETS IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

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Abstract. In this paper, we establish several fundamental properties in impulsive semidynamical systems. First, we formulate a counterpart of the continuous dependence on the initial conditions for impulsive dynamical systems, and also establish some equivalent properties. Second, we present several theorems similar to the Poincaré–Bendixson theorem for two-dimensional impulsive systems, i.e. if the omega limit set of a bounded infinite trajectory (with an infinite number of impulses) contains no rest points, then there exists an almost recurrent orbit in the limit set. Further, if the omega limit set contains an interior point, then it is a chaotic set; otherwise, if the limit set contains no interior points, then the limit set contains a periodic orbit or a Cantor-type minimal set in which each orbit is almost recurrent.

1. Introduction

The theory of impulsive differential equations now becomes an important field of investigation, for example, see books [3], [17], [22]. The researches of impulsive semidynamical systems began by Kaul [19], [20] and Rozhko [24], [25]. In particular, Kaul associated a given impulsive semidynamical system with a discrete semidynamical system defined on the range of an impulsive set under an impulsive function, thus Kaul established many important results about the limit sets and recurrent properties. He also initiated the study of stabilities for impulsive semidynamical system (see [21]). Later, Ciesielski [12], [13] presented

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several fundamental results for this theory, in fact he applied his section theory of semiflow (see [11]) to obtain the continuities of some important functions associated with impulsive semidynamical systems. Recently, Bonotto, Federson and their collaborators also published a series of important papers on this subject (for instance, see [1], [8], [9] and references therein), they developed a list of valued results in impulsive semidynamical systems, which are counterparts of basic properties in dynamical systems.

It is obvious that the dynamical behavior of an impulsive semidynamical system is much richer than that of the corresponding semidynamical system. Actually, in the next section we will present a very simple planar system in which there is highly irregular dynamical behavior (e.g. chaotic orbits in its limit sets). Our first goal of this paper is to establish the counterpart of the continuous dependence on the initial conditions for impulsive dynamical systems. In particular, we reformulate the quasi-continuous dependence property in [10], and present some equivalent formulas. Next, we investigate the structure of omega limit sets of infinite trajectories (with an infinite number of impulses). In [9], the authors stated a version of the Poincaré–Bendixson theorem, their conditions separate the omega limit set of a bounded orbit away from the impulsive set, i.e., the bounded orbit only has a finite number of impulses, thus their conclusion is a generalization of a result of Poincaré–Bendixson Theory. In this paper, we show the existence of almost recurrent orbits in the omega limit set of an infinite trajectory if the limit set contains no rest points, and also we present an example with a limit set that is a minimal set and homeomorphic to the product of a Cantor set and an interval, in which all the orbits are almost recurrent and not periodic. This shows that our result can not be strengthened to the existence of periodic orbits in a limit set. Further, if there exists an interior point in the limit set, then it is a chaotic set; otherwise, it has a periodic orbit or a Cantor-type minimal set in which each orbit is almost recurrent.

2. Basic notations and definitions

Let $X = (X, d)$ be a metric space with metric d. Throughout the paper, for $A \subset X$, $\overline{A}$ and ∂A denote the closure and boundary of A, respectively. Let $B(x, \delta) = \{y \in X : d(x, y) < \delta\}$ be the open ball with center x and radius $\delta > 0$. In addition, for $A \subset X$, the r-neighbourhood of A is denoted by $N(A, r) = \{z \in X : d(z, A) < r\}$ for $r > 0$, where $d(z, A) = \inf\{d(z, p) : p \in A\}$. Here, with no confusion, we use d for the distance between a point and a set. Similarly, we also use d for the distance between two sets A and B, i.e. $d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$. Let $\mathbb{R}^+$ and $\mathbb{Z}^+$ be the sets of nonnegative real numbers and nonnegative integers, respectively.

A *semidynamical system* (or *semiflow*) on X is a triple $(X, \pi, \mathbb{R}^+)$, where π is a continuous mapping from $X \times \mathbb{R}^+$ onto X satisfying the following axioms:

(1) $\pi(x, 0) = x$ for each $x \in X$,

(2) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for each $x \in X$ and t, $s \in \mathbb{R}^+$.

Sometimes we denote a semidynamical system $(X, \pi, \mathbb{R}^+)$ by (X, π). Replacing $\mathbb{R}^+$ by $\mathbb{R}$, we get the definition of a dynamical system. For brevity, we write $x \cdot t = \pi(x, t)$, and also let $A \cdot J = \{x \cdot t : x \in A,\ t \in J\}$ for $A \subset X$ and $J \subset \mathbb{R}^+$. If either A or J is a singleton, i.e. $A = \{x\}$ or $J = \{t\}$, then we simply write $x \cdot J$ and $A \cdot t$ for $\{x\} \cdot J$ and $A \cdot \{t\}$, respectively. For any $x \in X$, the function $\pi_x \colon \mathbb{R}^+ \to X$ defined by $\pi_x(t) = \pi(x, t)$ is clearly continuous, and we call π_x the *trajectory* of x. The set $x \cdot \mathbb{R}^+$ is said to be the (*positive*) *orbit* of x, and sometimes denoted by $C^+(x)$. The closure of $C^+(x)$ is denoted by $K^+(x)$, i.e. $K^+(x) = \overline{x \cdot \mathbb{R}^+}$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y : y \cdot t = x\}$. Similarly, for $J \subset [0, +\infty)$ and $D \subset X$, let $F(D, J) = \{F(x, t) : x \in D \text{ and } t \in J\}$. a point $x \in X$ is called a *start point*, if $F(x, t) = \emptyset$ for all $t > 0$. Note that for a semidynamical system defined on a manifold, there exist no start points (see [5]). For the elementary properties of dynamical and semidynamical systems, we refer to [5], [6], [14] and [23].

Let $\Omega \subset X$ be an open set in X, and denote the boundary of Ω in X by $M = \partial\Omega$. Assume that M is a nonempty closed set. Let $I \colon M \to \Omega$ be a continuous function and $I(M) = N$. If $x \in M$, we shall denote $I(x)$ by x^+ and say that x jumps to x^+. Meanwhile, I and M are said to be an *impulsive function* and an *impulsive set*, respectively. For each $x \in X$, by $M^+(x)$ we mean the set $(x \cdot \mathbb{R}^+ \cap M) \setminus \{x\}$. In order to avoid the uninterrupted impulses, we assume that when a trajectory π_x intersects the impulsive set M, it instantaneously exits M, i.e. the points of M are isolated in every trajectory of the system (X, π). Indeed, we suppose that for any $x \in M$, there exists an $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad x \cdot (0, \varepsilon_x) \cap M = \emptyset. \tag{2.1}$$

This condition was introduced by Kaul, see the condition (2.1) in [19]. Clearly, it implies that M contains no rest points of (X, π), here by a rest point x of (X, π) we mean that $x \cdot t = x$ for any $t \geq 0$. Thus, we can define a function $\phi \colon X \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\phi(x) = \begin{cases} s & \text{if } x \cdot s \in M \text{ and } x \cdot t \notin M \text{ for } t \in (0, s), \\ \infty & \text{if } M^+(x) = \emptyset. \end{cases}$$

In general, the function $\phi \colon X \to \mathbb{R}^+ \cup \{+\infty\}$ may not be continuous. However, its continuity is crucial for the analysis of dynamics. Fortunately, some simple conditions are given by Ciesielski in [12] to guarantee the continuity of ϕ.

ASSUMPTION I. We assume throughout the paper that ϕ is a continuous function on Ω.

Now, we define an impulsive semidynamical system $(X,\pi;\Omega,M,I)$ by portraying the orbit of each point in Ω. The *impulsive trajectory* of $x\in\Omega$ is an Ω-valued function $\widetilde{\pi}_x$ defined on the subset $[0,s)$ of $\mathbb{R}^+$ (s may be $+\infty$).

If $M^+(x)=\emptyset$, then $\phi(x)=+\infty$, and we set $\widetilde{\pi}_x(t)=x\cdot t$ for all $t\in\mathbb{R}^+$. If $M^+(x)\neq\emptyset$, it is easy to see that there is a smallest positive number t_0 such that $x\cdot t_0=x_1\in M$ and $x\cdot t\notin M$ for $0<t<t_0$. Thus, we define $\widetilde{\pi}_x$ on $[0,t_0]$ by

$$\widetilde{\pi}_x(t)=\begin{cases} x\cdot t & \text{if } 0\le t<t_0,\\ x_1^+ & \text{if } t=t_0,\end{cases}$$

where $x_1^+=I(x_1)$ and $\phi(x)=t_0$.

Since $t_0<+\infty$, we continue the process by starting with x_1^+. Similarly, if $M^+(x_1^+)=\emptyset$, i.e. $\phi(x_1^+)=+\infty$, we define $\widetilde{\pi}_x(t)=x_1^+\cdot(t-t_0)$ for $t_0\le t<+\infty$. Otherwise, let $\phi(x_1^+)=t_1$ and $x_1^+\cdot t_1=x_2\in M$, then we define $\widetilde{\pi}_x(t)$ on $[t_0,t_0+t_1]$ by

$$\widetilde{\pi}_x(t)=\begin{cases} x_1^+\cdot(t-t_0) & \text{if } t_0\le t<t_0+t_1,\\ x_2^+ & \text{if } t=t_0+t_1,\end{cases}$$

where $x_2^+=I(x_2)$.

Thus, continuing inductively, the process above either ends after a finite number of steps, whenever $M^+(x_n^+)=\emptyset$ for some n, or it continues infinitely, if $M^+(x_n^+)\neq\emptyset$ for $n=1,2,\dots$, and $\widetilde{\pi}_x$ is defined on the interval $[0,T(x))$, where $T(x)=\sum_{i=0}^{\infty}t_i$.

We call $\{t_i\}$ the *impulsive intervals* of $\widetilde{\pi}_x$, and call $\left\{\tau_n=\sum_{i=0}^{n}t_i \;\middle|\; n=0,1,\dots\right\}$ the *impulsive times* of $\widetilde{\pi}_x$. Obviously, this gives rise to either a finite or infinite number of jumps at points $\{x_n\}$ for the trajectory $\widetilde{\pi}_x$. After setting each trajectory $\widetilde{\pi}_x$ for every point x in Ω, we let $\widetilde{\pi}(x,t)=\widetilde{\pi}_x(t)$ for $x\in\Omega$ and $t\in[0,T(x))$, and then we get a discontinuous system $(\Omega,\widetilde{\pi})$ satisfying the following properties:

(i) $\widetilde{\pi}(x,0)=x$ for $x\in\Omega$,

(ii) $\widetilde{\pi}(\widetilde{\pi}(x,t),s)=\widetilde{\pi}(x,t+s)$ for $x\in\Omega$ and t, $s\in[0,T(x))$, such that $t+s\in[0,T(x))$.

We call $(\Omega,\widetilde{\pi})$, with $\widetilde{\pi}$ as defined above, an impulsive semidynamical system associated with (X,π). Also, for simplicity of exposition, in the remainder of this paper we denote the trajectory $\widetilde{\pi}(x,t)$ by $x{*}t$. Thus, (ii) reads $(x{*}t){*}s=x{*}(t+s)$. Let $x\in\Omega$ be given, if $M^+(x)=\emptyset$, the trajectory $\widetilde{\pi}_x$ is continuous; otherwise, it has discontinuities at a finite or an infinite number of its *impulsive points* $\{x_n^+\}$. At any such point, however, $\widetilde{\pi}_x$ is continuous from the right. From the point of view of an impulsive semidynamical system, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0,+\infty)$ as the interval of definition. We call them *infinite trajectories*. In this paper, we do not deal with the Zeno orbits (see [17, Chapter 2]), i.e. we are mainly interested in the case of $T(x)=+\infty$.

Remark 2.1. Under the condition of (2.1), the points of M are isolated in each orbit of the system (X,π), thus we can similarly define an impulsive semidynamical system $(X,\widetilde{\pi})$ on X, which admits $(\Omega,\widetilde{\pi})$ as a subsystem. However, we are just interested in the system $(\Omega,\widetilde{\pi})$ for the following reasons. First, each point in M of the system $(X,\widetilde{\pi})$ is a start point from the point of view of an impulsive semidynamical system. There is not much interesting dynamical behavior at the points of M, since those points jump away from M. Second, it destroys the consistency with the classical theory, e.g. the closure of an invariant set may not be invariant and a limit set may not be invariant, etc. Furthermore, ϕ is not continuous on M, see [12].

Let $x\in\Omega$, the orbit $x*\mathbb{R}^+$ of x in the system $(\Omega,\widetilde{\pi})$ will be denoted by $\widetilde{C}^+(x)$ and its closure in Ω by $\widetilde{K}^+(x)$, i.e. $\widetilde{K}^+(x)=\overline{x*\mathbb{R}^+}\cap\Omega$. Now, we introduce some concepts (see [19], [20]) that will be used in the sequel.

Definition 2.2. A subset S of Ω is said to be positively invariant if for any $x\in S$, $\widetilde{C}^+(x)\subset S$. Further, it is said to be invariant if it is positively invariant and for any $x\in S$, $t\in[0,T(x))$ there exists a $y\in S$ such that $y*t=x$.

Definition 2.3. A point x in Ω is a rest point if $x*t=x$ for every $t\in\mathbb{R}^+$. An orbit $\widetilde{C}^+(x)$ is said to be periodic of period τ and order k if $x*\mathbb{R}^+$ has k components and τ is the least positive number such that $x*\tau=x$.

Clearly, a periodic orbit of $(\Omega,\widetilde{\pi})$ is an invariant closed set in Ω, and it is not connected as long as $k\neq 1$. a point $x\in\Omega$ is a rest point of $(\Omega,\widetilde{\pi})$ if and only if it is a rest point of (X,π).

Definition 2.4. Let $x\in\Omega$ and $T(x)=+\infty$, the omega limit set of x in $(\Omega,\widetilde{\pi})$ is defined by $\widetilde{\omega}(x)=\{y\in\Omega : x*t_n\to y \text{ for some } t_n\to+\infty\}$.

Equivalently, $\widetilde{\omega}(x)=\limsup\limits_{n\to+\infty}\{x_n^+*[0,\phi(x_n^+))\}$. Thus, it is easy to see that $\widetilde{\omega}(x)$ is closed and positively invariant.

We end this section with a very simple impulsive semidynamical system defined in the plane $\mathbb{R}^2$, however, there is highly irregular dynamical behavior (e.g. chaotic trajectories) in this system.

Example 2.5. Let $X=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1\}$. We define a dynamical system on X by a differential system:

$$\dot{x}=0,\qquad \dot{y}=-1. \tag{2.2}$$

This system is very simple, each orbit goes down straightly. Now, we set $M = \{(x,0) | 0 \le x \le 1\}$ and $N = \{(x,1) | 0 \le x \le 1\}$. Then, define the impulsive function $I: M \to N$ as $I(x,0) = (\lambda x(1-x), 1)$, where $\lambda \in (0,4]$. It is easy to see that every trajectory with an initial point above M is an infinite trajectory. Note that the unimodal map $y = \lambda x(1-x)$ on the interval $[0,1]$ is well studied, we refer to [7], [15] for its dynamical behavior dependent on λ. In particular, if $\lambda = 4$, it is chaotic on $[0,1]$, see [15, Example 8.9, p. 50]. Thus, this unimodal map has dense periodic points in $[0,1]$, topological transitivity and sensitive dependence on initial conditions. Correspondingly, if $\lambda = 4$, it is easy to check that in the region $S = [0,1] \times (0,1]$, the impulsive system defined as above has dense periodic orbits and topological transitivity, which also implies sensitive dependence on initial conditions. Thus, S is a chaotic set. For the detail, see Section 4 and [7], [15].

3. Dependence on the initial conditions

In this section, our goal is to establish a kind of continuous dependence on the initial conditions for impulsive semidynamical systems. In the theory of dynamical systems, the continuous dependence on the initial conditions is a fundamental property, which is often applied to prove important dynamical behavior about close orbits. We restate it for semidynamical systems according to the famous book of Nemytskii and Stepanov [23, p. 327] as follows.

Continuous dependence on the initial conditions. *Let (X,π) be a semidynamical system. For any $x \in X$, let two positive numbers T and ε be given, there can be found a $\delta = \delta(x,\varepsilon,T) > 0$ such that if $d(x,y) < \delta$, then $d(x{\cdot}t, y{\cdot}t) < \varepsilon$ for all $t \in [0,T]$, in particular,*

$$B(x,\delta) \cdot [0,T] \subset N(x \cdot [0,T], \varepsilon). \tag{3.1}$$

Obviously, it is an immediate conclusion from the continuity of π. Sometimes, we call it CD *property* of a semiflow, for brevity. Of course, now this basic property does not hold for our impulsive semidynamical system $(\Omega, \widetilde{\pi})$. To remedy this case, for impulsive semidynamical systems in $\mathbb{R}^n$, in [10] the authors presented an assumption of quasi-continuous dependence. For every $x_0 \in \mathfrak{D} \subset \mathbb{R}^n$, there exists $J_{x_0} \subset [0,+\infty)$ such that $[0,+\infty) \setminus J_{x_0}$ is (finitely or infinitely) countable and for every $\varepsilon > 0$, $t \in J_{x_0}$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $||x_0 - y|| < \delta(\varepsilon, x_0, t)$ and $y \in \mathfrak{D}$, then $||x_0 * t - y * t|| < \varepsilon$, where $|| \cdot ||$ is the ordinary norm in $\mathbb{R}^n$. In the following, we restate this dependence on initial conditions for an impulsive semidynamical system $(\Omega, \widetilde{\pi})$ defined in the metric space (X,d).

Assumption II. For every $x \in \Omega$, let $\mathbb{T}(x) = \{t_i \mid i = 0, 1, \dots\}$ be the set of impulsive times for $\widetilde{\pi}_x$. For any $\varepsilon > 0$ and $t \in J_x = [0, T(x)) \setminus \mathbb{T}(x)$, there exists a $\delta = \delta(\varepsilon, x, t) > 0$ such that if $d(x, y) < \delta$ and $y \in \Omega$, then $d(x * t, y * t) < \varepsilon$.

Since Ω is an open set, if δ is small enough, it implies that $y \in \Omega$. We are now going to show that Assumption II is equivalent to the continuity of ϕ on Ω. First, we present a lemma that is a new version of an important result proved by Kaul in [21, Lemma 2.3]. One can see that our conclusion is consistent with the right continuity of the system $(\Omega, \widetilde{\pi})$ at impulsive points.

Lemma 3.1. *Let ϕ be continuous on Ω. Suppose that $\{x_n\}$ is a sequence in Ω, convergent to a point $y \in \Omega$. Then for any $t \in [0, T(y))$, there exists a sequence of real positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, such that $t + \varepsilon_n < T(x_n)$ and $x_n * (t + \varepsilon_n) \to y * t$.*

Note that in this lemma and its proof, $\{x_n\}$ is not the n-th discontinuous point of the trajectory $\widetilde{\pi}_x$. Actually, we will denote the i-th impulsive point of the trajectory $\widetilde{\pi}_{x_n}$ by $\{x_{n,i}^+\}$.

Proof. If $\phi(y) = +\infty$, the continuity of ϕ implies that $\phi(x_n) > t$ for any fixed $t \in [0, +\infty)$ as long as n is sufficiently large. Consequently, $x_n * t = x_n \cdot t$ for large n. Then the result follows from the continuity of π with $\varepsilon_n = 0$. If $\phi(y) < +\infty$, then we may assume that $\phi(x_n) < +\infty$, since ϕ is continuous.

Now, we first consider the case $0 \le t < \phi(y)$. Let $\varepsilon \in (0, \phi(y) - t)$. From the continuity of ϕ, we conclude that $\phi(y) - \varepsilon < \phi(x_n)$ for large n, so $t < \phi(x_n)$ and $x_n * t = x_n \cdot t$, and also the continuity of π implies the result for $\varepsilon_n = 0$.

Next, for the case $t = \phi(y)$, then $y * t = y_1^+$. If $\phi(x_n) \ge \phi(y)$, set $\varepsilon_n = \phi(x_n) - t \ge 0$. Because $x_{n,1} = x_n \cdot \phi(x_n) \to y_1$, and I is continuous, thus it follows that $x_{n,1}^+ = x_n * \phi(x_n) = x_n * (t + \varepsilon_n) \to y_1^+$. If $\phi(x_n) < \phi(y)$, we set $\varepsilon_n = 0$. Then, $d(y * t, x_n * t) = d(y_1^+, x_{n,1}^+ \cdot (t - \phi(x_n))) \to 0$, since $x_{n,1}^+ \to y_1^+$ and $t - \phi(x_n) \to 0$.

Finally, we consider the case where $t > \phi(y)$, so $t = \sum_{i=0}^{m-1} \phi(y_i^+) + t'$, $0 \le t' < \phi(y_m^+)$, where $y_0^+ = y$ and $y_{i+1}^+ = y_i^+ * \phi(y_i^+)$ $(i = 0, 1, \dots, m-1)$. Define $\{x_{n,i}^+\}$ inductively by $x_{n,i+1}^+ = x_{n,i}^+ * \phi(x_{n,i}^+)$. If $t_n = \sum_{i=0}^{m-1} \phi(x_{n,i}^+)$, then it is easy to see that $x_{n,m}^+ = x_n * t_n \to y_m^+$ as $n \to +\infty$. Indeed, for this fixed integer m, we have $x_{n,i}^+ \to y_i^+$ for each $i = 1, \dots, m$. Note that $t_n \to t - t'$ as $n \to +\infty$. Now, if $t' > 0$, the reasoning is similar to the first case, we set $\varepsilon_n = 0$. If $t' = 0$, it is similar to the second case, for $t_n \ge t$ we set $\varepsilon_n = t_n - t$, and for $t_n < t$ set $\varepsilon_n = 0$. Thus, similarly, it is easy to see that $x_n * (t + \varepsilon_n) \to y_m^+ * t' = y * t$. The proof is complete. $\square$

The conclusion in Lemma 3.1 is equivalent to the following statement. For any $\varepsilon > 0$, $\sigma > 0$ and $t \in [0, T(y))$, there exists a $\delta = \delta(y, \varepsilon, \sigma, t) > 0$ such that if $d(x, y) < \delta$, then the inequality $d(x * (t + \theta), y * t) < \varepsilon$ holds for some $\theta \in [0, \sigma)$ ($0 \le t + \theta < T(y)$). Clearly, if we adjust the definition of $\widetilde{\pi}(x, t)$ to make the trajectory $\widetilde{\pi}_x$ continuous from left at impulsive times, then we can set $\varepsilon_n \le 0$ and $\varepsilon_n \to 0^-$ in Lemma 3.1. Note that this result admits any t in $[0, T(y))$. However, in Assumption II, it requires that t is not an impulsive time. Actually, in our new definition of quasi-continuous dependence in the following, we also permit that t is an impulsive time. Now, we show the equivalence between Assumption II and the continuity of ϕ.

THEOREM 3.2. *Assumption* II *is equivalent to the continuity of* ϕ *on* Ω.

PROOF. Let ϕ be continuous on Ω. For $x \in \Omega$, if $M^+(x) = \emptyset$ (or $\phi(x) = +\infty$), then we let $J_x = [0, +\infty)$. In this case, the standard CD property of a semidynamical system works. Indeed, for each $\varepsilon > 0$ and $t \in J_x = [0, +\infty)$, there exists a $\delta_1 > 0$ such that if $d(x, y) < \delta_1$, then $\phi(y) > t + 1$, since ϕ is continuous. Thus, by (3.1) it is easy to see that there is a $\delta < \delta_1$ such that $B(x, \delta) \cdot [0, t] \subset \Omega$ and if $d(x, y) < \delta$, then $d(x * t, y * t) = d(x \cdot t, y \cdot t) < \varepsilon$.

If $M^+(x) \neq \emptyset$, by the definition of the trajectory $x * t$, we set $J_x = [0, T(x)) \setminus \mathbb{T}(x)$, where $\mathbb{T}(x)$ is the set of impulsive times of $x * t$. Let $\varepsilon > 0$ and $t \in J_x$. Since t is not an impulsive time, without loss of generality, we suppose $B(x * t, \varepsilon) \subset \Omega$. Clearly, we only need to consider the case $t > 0$. Thus, by the continuity of $\widetilde{\pi}_x$ at time t, there exists a $\sigma > 0$ satisfying $t - \sigma > 0$ and $x * [t - \sigma, t + \sigma] \subset B(x * t, \varepsilon)$.

Now, according to (3.1), one can find a positive number $\mu < \varepsilon$ so that $B(x * (t - \sigma), \mu) * [0, \sigma] \subset B(x * t, \varepsilon)$. Thus by Lemma 3.1, there exists a $\delta > 0$ such that if $d(y, x) < \delta$, then $d(y * (t - \sigma + \theta), x * (t - \sigma)) < \mu$ for some $\theta \in [0, \sigma)$. Since $y * t = (y * (t - \sigma + \theta)) * (\sigma - \theta) \in B(x * (t - \sigma), \mu) * [0, \sigma]$, it follows that $y * t \in B(x * t, \varepsilon)$, i.e. $d(y * t, x * t) < \varepsilon$. It shows that Assumption II is true.

Conversely, suppose that Assumption II is true. For $x \in \Omega$, if $t_0 = \phi(x) < +\infty$, let $d(x_1, x_1^+) = 4\rho > 0$. Given an $\varepsilon > 0$ ($\varepsilon < t_0$), without loss of generality, we suppose that ε is so small that for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, both $d(x \cdot t_0, x \cdot t) < \rho$ and $d(x_1^+, x * (t_0 + \varepsilon)) < \rho$ hold. Since the impulsive times $\mathbb{T}(x)$ of $x * t$ are isolated in $\mathbb{R}^+$, we also assume $(t_0 - \varepsilon, t_0 + \varepsilon) \cap \mathbb{T}(x) = \{t_0\}$ (if necessary, take a smaller ε).

By the CD property of a semidynamical system, there exists a $\delta_1 > 0$ such that if $d(x, y) < \delta_1$, then $d(x \cdot t, y \cdot t) < \rho$ holds for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. It implies that $y \cdot [t_0 - \varepsilon, t_0 + \varepsilon] \subset B(x_1, 2\rho)$ for $y \in B(x, \delta_1)$. Since the orbit segment $x \cdot [0, t_0 - \varepsilon]$ is compact and does not intersect M, the distance between them is positive, i.e. $d(x \cdot [0, t_0 - \varepsilon], M) = 2r > 0$. By (3.1), there exists a $\delta_2 > 0$ such that $B(x, \delta_2) \cdot [0, t_0 - \varepsilon] \subset N(x \cdot [0, t_0 - \varepsilon], r)$, i.e. for each $y \in B(x, \delta_2)$, the orbit segment $y \cdot [0, t_0 - \varepsilon]$ is disjoint from M. Hence, it means $\phi(y) > t_0 - \varepsilon$ for $y \in B(x, \delta_2)$.

Now, since $t_0+\varepsilon$ is not an impulsive time, it follows from Assumption II that there exists a $\delta_3>0$ such that if $d(x,y)<\delta_3$, then $d(x*(t_0+\varepsilon),y*(t_0+\varepsilon))<\rho$. Thus, let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, and we assert that if $d(x,y)<\delta$, then $y*(t_0+\varepsilon)\neq y\cdot(t_0+\varepsilon)$.

Actually, it is easy to see that $d(y\cdot(t_0+\varepsilon),y*(t_0+\varepsilon)\geq d(x_1,x_1^+)-d(x_1,y\cdot(t_0+\varepsilon))-d(x_1^+,y*(t_0+\varepsilon))>4\rho-2\rho-2\rho=0$. It is equivalent to that there exists at least an impulse for the trajectory $y*t$ in the time interval $[0,t_0+\varepsilon)$, then it implies $\phi(y)<t_0+\varepsilon$. So, we conclude that $\phi(y)\in(t_0-\varepsilon,t_0+\varepsilon)$ for $y\in B(x,\delta)$, i.e. ϕ is continuous at x.

Finally, we consider the case $t_0=\phi(x)=+\infty$. Let any large $\mathcal{T}>0$ be given. Since the compact set $x\cdot[0,\mathcal{T}]$ is disjoint from M, by (3.1) one can find a $\delta>0$ such that if $d(x,y)<\delta$, then $y\cdot[0,\mathcal{T}]$ does not intersect M, it implies $\phi(y)>\mathcal{T}$.□

In order to obtain a counterpart of continuous dependence on initial conditions for an impulsive semidynamical system, we introduce a concept of a time σ-reparametrization.

DEFINITION 3.3. For a $\sigma>0$, by a time σ-reparametrization we mean a homeomorphism τ from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\tau(0)=0$ such that $|\tau(t)-t|<\sigma$ for all $t\geq 0$.

Quasi-continuous dependence. *Let $x\in\Omega$. For any $\varepsilon>0$, $\sigma>0$, and a positive number $\mathcal{T}<T(x)$, there exists a $\delta>0$ such that if $d(x,y)<\delta$, then $d(x*t,y*\tau_y(t))<\varepsilon$ holds for all $t\in[0,\mathcal{T}]$, where τ_y is a σ-reparametrization.*

It is easy to see that quasi-continuous dependence property is a generalization of the standard continuous dependence property for dynamical systems to impulsive dynamical systems. Actually, by letting $\sigma=0$, the quasi-continuous dependence property turns to the classical continuous dependence on initial conditions. In the sequel, we sometimes denote the quasi-continuous dependence by QCD *property*, for brevity.

THEOREM 3.4. *The* QCD *property is equivalent to the continuity of ϕ on Ω.*

PROOF. First, let ϕ be continuous on Ω, then Assumption II holds. By definition, for $x\in\Omega$ and a fixed $\mathcal{T}\in(0,T(x))$, there exists a finite set $\{t_0,\dots,t_n\}$ satisfying $0<t_0<\dots<t_n<\mathcal{T}$, which is all impulsive times contained in $[0,\mathcal{T}]$ of the trajectory $x*t$. Suppose that $\mathcal{T}$ is not an impulsive time, otherwise we replace it by $\mathcal{T}+\nu$ $(<T(x))$, where ν is a small positive number such that $(\mathcal{T},\mathcal{T}+\nu]\cap\mathbb{T}(x)=\emptyset$.

Given an $\varepsilon>0$ and a $\sigma>0$, we assume that σ is sufficiently small so that $0<t_0-\sigma/2<t_0<t_0+\sigma/2<t_1-\sigma/2<t_1<t_1+\sigma/2<\ldots<t_n-\sigma/2<t_n<t_n+\sigma/2<\mathcal{T}$, meanwhile $x*[t_i-\sigma/2,t_i)\subset B(x_{i+1},\varepsilon/2)$ and $x*[t_i,t_i+\sigma/2]\subset B(x_{i+1}^+,\varepsilon/2)$ for each $i=0,\dots,n$, where $x_{i+1}^+=x*t_i$ and $x_{i+1}=x_i^+\cdot(t_i-t_{i-1})$.

We assert that there exists a $\delta_1 > 0$ such that if $d(x,y) < \delta_1$, then $d(x*t, y*t) < \varepsilon$ for all $t \in \mathbb{I} = [0, t_0 - \sigma/2] \cup [t_0 + \sigma/2, t_1 - \sigma/2] \cup \ldots \cup [t_n + \sigma/2, \mathcal{T}]$. Otherwise, one can find two sequences $\{y_n\}$ and $\{\tau_n\}$ such that $y_n \to x$ and $\tau_n \in \mathbb{I}$, and $d(x * \tau_n, y_n * \tau_n) \geq \varepsilon$.

Without loss of generality, suppose $\tau_n \to \tau_0 \in \mathbb{I}$. It implies that for large n, τ_n lies in a fixed subinterval $[\alpha, \beta]$ of $\mathbb{I}$, where $[\alpha, \beta]$ is a component of $\mathbb{I}$. Thus, for $y_n \to x$, by Assumption II, we have $d(x * \alpha, y_n * \alpha) \to 0$. Now, it follows from the continuity of π that

$$\begin{aligned} d(x * \tau_n, y_n * \tau_n) &= d((x * \alpha) * (\tau_n - \alpha), (y_n * \alpha) * (\tau_n - \alpha)) \\ &= d((x * \alpha) \cdot (\tau_n - \alpha), (y_n * \alpha) \cdot (\tau_n - \alpha)) \\ &\to d((x * \alpha) \cdot (\tau_0 - \alpha), (x * \alpha) \cdot (\tau_0 - \alpha)) = 0. \end{aligned}$$

It is contradictory to $d(x * \tau_n, y_n * \tau_n) \geq \varepsilon$, so we obtain the conclusion:

$$\forall \varepsilon > 0,\ \exists \delta_1 > 0, \quad d(x,y) < \delta_1 \Rightarrow d(x * t, y * t) < \varepsilon \ (t \in \mathbb{I}). \tag{3.2}$$

Consider $\mathbb{I}_i = (t_i - \sigma/2, t_i + \sigma/2)$ $(i = 0, \ldots, n)$. Since ϕ is continuous on Ω and I is continuous on M, there is a $\delta_2 \in (0, \delta_1)$ such that if $y \in B(x, \delta_2)$, the trajectory $y * t$ has just $n+1$ impulsive times $\{\tau_i : i = 0, \ldots, n\}$ in $[0, \mathcal{T}]$ and one in each $\mathbb{I}_i$, i.e. $\tau_i \in \mathbb{I}_i$.

Clearly, by (3.1), $y * [t_i - \varepsilon/2, \tau_i) \subset N(x * [t_i - \varepsilon/2, t_i), \varepsilon/2)$ holds as long as $d(y * (t_i - \varepsilon/2), x * (t_i - \varepsilon/2))$ is small, further by the continuity of I it also leads to $y * [\tau_i, t_i + \sigma/2] \subset N(x * [t_i, t_i + \sigma/2], \varepsilon/2)$. Thus, by Assumption II, we take a $\delta \in (0, \delta_2)$ such that for $i = 0, \ldots, n$, both $\sup\{d(p,q) : p \in x*[t_i - \sigma/2, t_i),\ q \in y*[t_i - \varepsilon/2, \tau_i)\} < \varepsilon$ and $\sup\{d(p,q) : p \in x*[t_i, t_i + \sigma/2],\ q \in y*[\tau_i, t_i + \varepsilon/2]\} < \varepsilon$ hold. Then, for $y \in B(x, \delta)$, define a σ-reparametrization τ_y as follows.

If $t \in \mathbb{I} \cup [\mathcal{T}, +\infty)$, let $\tau_y(t) = t$; if $t \in \mathbb{I}_i$ $(i = 0, \ldots, n)$, set

$$\begin{aligned} \tau_y(t) &= \frac{1}{\sigma}(2\tau_i - 2t_i + \sigma)\left(t - t_i + \frac{\sigma}{2}\right) + t_i - \frac{\sigma}{2} \quad && \text{for } t \in [t_i - \sigma/2, t_i), \\ \tau_y(t) &= \frac{1}{\sigma}(2t_i + \sigma - 2\tau_i)(t - t_i) + \tau_i && \text{for } t \in [t_i, t_i + \sigma/2). \end{aligned}$$

It is easy to see that τ_y is a σ-reparametrization for the trajectory $y * t$ such that if $d(x,y) < \delta$, then $d(x * t, y * \tau_y(t)) < \varepsilon$ holds for all $t \in [0, \mathcal{T}]$. So, the QCD property is true.

Next, assume that the QCD property holds. Let $x \in \Omega$ and $\phi(x) = t_0 < +\infty$. Given an $\varepsilon > 0$, we denote $d(x_1, x_1^+) = 3r > 0$ and $d(x \cdot [0, t_0 - \varepsilon], M) = \lambda > 0$. Without loss of generality, suppose that ε is so small that $x \cdot [t_0 - \varepsilon, t_0 + \varepsilon] \subset B(x_1, r)$ and $x_1^+ \cdot [0, t_0 + \varepsilon] \subset B(x_1^+, r)$. Thus, by the CD property of π, there exists a $\delta_1 > 0$ such that if $d(x,y) < \delta_1$, then $d(x \cdot t, y \cdot t) < \min\{\lambda, r\}$ for $t \in [0, t_0 + \varepsilon]$. It follows that $y \cdot [0, t_0 - \varepsilon] \subset N(x \cdot [0, t_0 - \varepsilon], \lambda)$, i.e. $y \cdot [0, t_0 - \varepsilon] \cap M = \emptyset$. Clearly, it implies that $\phi(y) > t_0 - \varepsilon$.

Meanwhile, we have $y \cdot [t_0 - \varepsilon, t_0 + \varepsilon] \subset B(x_1, 2r)$, since $d(x \cdot t, y \cdot t) < r$ for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. According to the QCD property, there exist a $\delta_2 > 0$ and a ε-reparametrization τ_y such that if $d(x, y) < \delta_2$, one has $d(x*t, y*\tau_y(t)) < r$ for all $t \in [0, t_0+\varepsilon]$. So, we obtain $d(x_1^+, y*\tau_y(t_0)) < r$. Now, we set $\delta = \min\{\delta_1, \delta_2\}$. Then, for $y \in B(x, \delta)$, both $y \cdot [t_0-\varepsilon, t_0+\varepsilon] \subset B(x_1, 2r)$ and $y*\tau_y(t_0) \in B(x_1^+, r)$ hold, it implies $y*\tau_y(t_0) \notin y \cdot [t_0-\varepsilon, t_0+\varepsilon]$. Since $|t_0 - \tau_y(t_0)| < \varepsilon$, it follows that there must be an impulsive time in $[t_0-\varepsilon, t_0+\varepsilon]$ for y, i.e. $t_0-\varepsilon < \phi(y) < t_0+\varepsilon$.□

4. Limit sets

In this section, our discussion is restricted in the impulsive system $(\Omega, \widetilde{\pi})$ defined in the Euclidean plane, i.e. Ω is an open set in $\mathbb{R}^2$. Our goal is to describe the structure of an omega limit set for two-dimensional impulsive systems and prove some theorems similar to the Poincaré–Bendixson theorem. Actually, we show that if the omega limit set of a bounded trajectory does not contain rest points, then it contains an almost recurrent orbit (see Definition 4.8). In particular, if there exists an interior point in the limit set, then it is a chaotic set (see Definition 4.5); otherwise, it has a periodic orbit or a Cantor-type minimal set in which each orbit is almost recurrent. In the following, we first recall some results of dynamics of one-dimensional system and prove that if an omega limit set of an interval map contains an interior point, then it is a chaotic set. Next, we apply those results to investigate the structure of a limit set $\widetilde{\omega}(x)$ of an infinite trajectory for two-dimensional impulsive systems.

4.1. Dynamics on an interval. In this subsection, we recall some results of dynamics in one dimension that will be used in the sequel. We show that if an omega limit set of an interval map contains an interior point, then it is a chaotic set.

Let J be a compact interval in $\mathbb{R}$ and let f denote a continuous function mapping J into J. For each natural number n, we let f^n denote the nth iterate of f. It is standard to think of f^0 as the identity function. For a fixed point $x \in J$, we denote the trajectory of f at x by $\gamma(x) = \{f^n(x) : n = 0, 1, \dots\}$. Sometimes we will view $\gamma(x)$ as a set of points (the orbit of x) and sometimes as a sequence; however, the context will always indicate which is meant. In the case where $\gamma(x)$ has a finite range, x is said to be an *eventually periodic* point. In particular, if there is a positive integer k such that $f^k(x) = x$, then f is said to be *periodic* at x, and if k is the least natural number such that $f^k(x) = x$, then x is called a *periodic point with period* k. In the special case where $k = 1$, i.e. when $f(x) = x$, x is called a *fixed point* of f. The *omega limit set* of f at x is denoted by $\omega(x, f)$, and is the set of all numbers $y \in J$ for which there exists a sequence $\{f^{n_k}(x)\}$ such that $f^{n_k}(x) \to y$ as $n_k \to \infty$, i.e. the set of accumulation points of the orbit $\gamma(x)$. a nonempty set $A \subset J$ is called *positively invariant* if $f(A) \subset A$

and *invariant* if $f(A) = A$. A nonempty invariant closed set is called *minimal* if it contains no proper invariant closed subset. The existence of a minimal set in a nonempty compact invariant set is actually valid for any dynamical systems defined in a metric space X, for $X = J$ we have the following stronger version (see [7, p. 92]).

LEMMA 4.1. *Let $f\colon J \to J$ be a self map of a compact interval J. Then any nonempty closed invariant set contains a minimal set. Further, any infinite minimal set in J is a Cantor set, i.e. a closed subset of J which has no isolated point and contains no interval.*

Topological structure of $\omega(x, f)$ was investigated in depth by Agronsky et al. [2], Block and Coppel [7]. It is well known that $\omega(x, f)$ for any $x \in J$ is non-empty, closed and invariant. The following result is a characteristic of the omega limit set, see [2], [7].

LEMMA 4.2. *A non-empty subset Λ of J is an omega limit set for some continuous self map of J if and only if either Λ is closed and nowhere dense or Λ is a finite union of pairwise disjoint, closed subintervals $\{I_i\}_{i=1}^n$ which form a cycle, that is $\Lambda = \bigcup_{i=1}^n I_i$ where $I_i \cap I_j = \emptyset$ if $i \neq j$, with $f(I_{i-1}) = I_i$ if $i = 2, \dots, n$ and $f(I_n) = I_1$.*

Clearly, if a limit set $\omega(x, f)$ has only finitely many points, then it contains a periodic orbit. Generally speaking, a limit set always has some recurrent property. In order to see some recurrent phenomena in $\omega(x, f)$, we recall the concept of almost periodicity that is termed to be almost recurrence in [20].

DEFINITION 4.3. We say that f is almost periodic at x if, for any $\varepsilon > 0$, there exists a positive integer $k = k(\varepsilon)$ such that $d(f^{i+n}(x), f^n(x)) < \varepsilon$ for any $n \in \mathbb{Z}^+$ and for some i with $0 < i \leq k$.

PROPOSITION 4.4. *For each $x \in J$, there exists an almost periodic orbit contained in the omega limit set $\omega(x, f)$ of x.*

The proof of above proposition is immediate, since $\omega(x, f)$ contains a compact minimal set and each point in a compact minimal set is almost periodic (see [7]). In Proposition 4.4, we do not have the existence of periodic orbits. Indeed, Block and Coppel in [7, p. 149] presented an example taking the classical 'middle-third' Cantor set as a minimal set, which implies the nonexistence of periodic orbits in it.

Now, we recall two important concepts in the theory of chaos. A continuous map $f\colon X \to X$, where X is a metric space, is said to be (topologically) *transitive*, if for any pair of nonempty open sets U and V in X, there exists a positive integer k such that $f^k(U) \cap V \neq \emptyset$. It is well known that, in a complete metric space with no isolated points, transitivity is equivalent (via the Baire Category Theorem) to the existence of a point with dense orbit, which in turn is equivalent to the existence of a dense G_δ set of points each of which has a dense orbit (see [16, p. 77]). The map f is said to possess *sensitive dependence on initial conditions*, if there exists an $\varepsilon > 0$ for any $x_0 \in X$ and any open set $U \subset X$ containing x_0 there exists $y_0 \in U$ and $k \in \mathbb{Z}^+$ such that $d(f^k(x_0), f^k(y_0)) > \varepsilon$. We introduce the definition of chaos according to Devaney [15] as follows.

DEFINITION 4.5. A continuous map $f: X \to X$, where X is a metric space, is said to be chaotic (or X is said to be a chaotic set) provided

(a) f is transitive;
(b) The set of periodic points of f is dense in X;
(c) f has sensitive dependence on initial conditions.

Clearly, if f^n is chaotic for some $n \in \mathbb{Z}^+$, also is f. Note that in [4], Banks et al. showed that conditions (a) and (b) imply condition (c). Furthermore, Vellekoop and Berglund [26] proved that for interval maps transitivity implies chaos, i.e. if $X = J$ and $f: J \to J$ is transitive, then it is chaotic.

THEOREM 4.6. *If $\omega(x, f)$ for $x \in J$ contains an interior point, then $\omega(x, f)$ is a chaotic set.*

PROOF. First, we consider the case $\omega(x, f) = I_0$, where I_0 is a subinterval in J. Obviously, I_0 is invariant and $f^k(x)$ lies in the interior of I_0 for some integer $k \geq 0$, so the orbit $\gamma(f^k(x))$ is a dense orbit in I_0. Thus, f is transitive in the invariant set $\omega(x, f)$, which implies $\omega(x, f)$ is a chaotic set (see [26]). Next, by Lemma 4.2, since $\omega(x, f)$ contains an interior point, we have $\omega(x, f) = \bigcup_{i=1}^{n} I_i$, where each I_i is a subinterval and $I_i \cap I_j = \emptyset$ if $i \neq j$, with $f(I_{i-1}) = I_i$ if $i = 2, \dots, n$ and $f(I_n) = I_1$. Thus, $f^n: I_1 \to I_1$ is a chaotic map on I_1. Clearly, it follows that f is chaotic on $\omega(x, f)$, or $\omega(x, f)$ is a chaotic set. $\square$

4.2. Limit sets in $(\Omega, \widetilde{\pi})$. In this subsection, we discuss the structure of the omega limit set of an infinite trajectory for the impulsive system $(\Omega, \widetilde{\pi})$ defined in the Euclidean plane. By establishing a discrete semidynamical system associated with $(\Omega, \widetilde{\pi})$, which was first proposed by Kaul [20], we apply the results of Section 4.1 to obtain our main theorems about $\widetilde{\omega}(x)$. Our investigation is focused on an infinite trajectory $\widetilde{\pi}_x$, i.e. $\widetilde{\pi}_x$ has an infinite number of impulsive times and $T(x) = +\infty$.

Let $\widehat{N} = \{x \in N : \widetilde{\pi}_x \text{ is an infinite trajectory}\}$, it is a subset of N. Define $\Pi: \widehat{N} \to M$ by letting $\Pi(y) = y \cdot \phi(y)$ for each $y \in \widehat{N}$. Since ϕ is continuous on Ω, so is Π on $\widehat{N}$. Now, given any $x \in \widehat{N}$, $x_1 = x \cdot \phi(x) \in M$, then we define $g: \widehat{N} \to \widehat{N}$ by $g(x) = I \circ \Pi(x) = x_1^+$, consequently g is continuous on $\widehat{N}$. As usual, we set g^0 =identity, $g^1 = g$ and g^n inductively for $n > 1$. Thus, we obtain a discrete semidynamical system $(\widehat{N}, g, \mathbb{Z}^+)$, where $g(x,n) = g^n(x) = x_n^+$ and $x_0^+ = x$. $(\widehat{N}, g, \mathbb{Z}^+)$ or simply $(\widehat{N}, g)$ is called the *discrete semidynamical system associated with the given impulsive system* $(\Omega, \widetilde{\pi})$ (see [20]). The orbit of a point $x \in \widehat{N}$ is denoted by $\widehat{C}^+(x)$ and its closure in $\widehat{N}$ by $\widehat{K}^+(x)$. A subset A in $\widehat{N}$ is *minimal* if for any $q \in A$, $A = \widehat{K}^+(q)$. We say that g is *almost periodic* at $p \in \widehat{N}$ if, for any $\varepsilon > 0$, there exists a positive integer $k = k(\varepsilon)$ such that for any $n \in \mathbb{Z}^+$, $d(p_{i+n}^+, p_n^+) = d(g^{i+n}(p), g^n(p)) < \varepsilon$ for some i with $0 < i \leq k$. Note that in the paper [20] by Kaul a function g fulfilling this property is said to be almost recurrent at p.

For a subset $A \subset \widehat{N}$, if A is closed in Ω and positively invariant under g, i.e. $g(A) \subset A$, then we call A a *regular set*. Clearly, if $\widehat{N}$ is closed in Ω, then $\widehat{N}$ is a regular set. In particular, for an $x \in \widehat{N}$, if the closure of $\widehat{C}^+(x)$ in Ω coincides with $\widehat{K}^+(x)$, then $\widehat{K}^+(x)$ is a regular set. If a set A in $\widehat{N}$ is a regular set, an important property of A is that the restriction $g|_A\colon A \to A$ defines a sub-system $(A, g|_A)$ of $(\widehat{N}, g)$ such that each limit set $\omega(x, g|_A)$ $(x \in A)$ in the system $(A, g|_A)$ contains all limit points of $\widehat{C}^+(x)$ in Ω.

For $x \in \Omega$, the trajectory $\widetilde{\pi}_x$ is *bounded* if its orbit $\widetilde{C}^+(x) \subset B(x,r)$ for some $r > 0$, otherwise it is said to be *unbounded*. Let $\widetilde{\pi}_x$ be a bounded trajectory, it is easy to see that the limit set $\widetilde{\omega}(x)$ is a nonempty closed set in Ω.

Before presenting the next result, we recall the concept of asymptotical stability from one side of a periodic orbit for a two-dimensional system. Let Γ be a periodic orbit of a semidynamical system $(\mathbb{R}^2, \pi)$. Clearly, Γ divides $\mathbb{R}^2$ into two disjoint connected components U_1 and U_2. Then Γ is said to be *stable from one side* in U_1 if for any neighbourhood V of Γ, there exists a neighbourhood U of Γ such that $U \subset V$ and $U \cap U_1$ is positively invariant. Further, Γ is said to be *asymptotically stable from one side* in U_1 if it is stable from one side in U_1 and there exists a neighbourhood W of Γ such that $\omega(x) \subset \Gamma$ for each $x \in W \cap U_1$. We refer the reader to [14] for details, and also refer [18, Chapter 7] for the case of differential dynamical systems.

LEMMA 4.7. *Let $x \in \widehat{N}$ and $\widetilde{\pi}_x$ be bounded. Suppose that $\widetilde{\omega}(x)$ contains no rest points. Then $\widehat{K}^+(x)$ is a regular set.*

PROOF. For $x \in \widehat{N}$, assume that $\widetilde{\pi}_x$ is bounded and $\widetilde{\omega}(x)$ contains no rest points. We prove that the limit set of $\widehat{C}^+(x)$ in Ω lies in $\widehat{N}$. Let $x_{n_k}^+ \to p$, clearly $p \in \overline{N}$ and $p \in \widetilde{\omega}(x)$. By the condition (2.1) and the continuity of ϕ, we have $p \notin M$. Since $\widetilde{\omega}(x)$ contains no rest points, p is not a rest point. We show that $\phi(p) < +\infty$.

Suppose to the contrary that $\phi(p) = +\infty$. Then $p * \mathbb{R}^+ = p \cdot \mathbb{R}^+$ lies in Ω. First, if $p \cdot \mathbb{R}^+$ is unbounded, i.e. for any $r > 0$, there exists a $t > 0$ such that $p \cdot t \notin B(p, r+1)$. From the CD property of π, it follows that for a sufficiently large n_k, $x^+_{n_k} \cdot [0,t) \subset \Omega$ and $d(p \cdot t, x^+_{n_k} \cdot t) < 1$. Thus, $x^+_{n_k} \cdot t \notin B(p,r)$, but $x^+_{n_k} \cdot t \in \widetilde{C}^+(x)$. It is a contradiction, since $\widetilde{C}^+(x)$ is a bounded set. Second, if $p \cdot \mathbb{R}^+$ is bounded, then $\omega(p)$ is a nonempty compact invariant set in $\mathbb{R}^2$. Since a limit set is an invariant set and $p \in \widetilde{\omega}(x)$, it follows that $\omega(p) \subset \widetilde{\omega}(x) \cup M$. Note that under the condition (2.1), there exist no rest points in M. Thus, $\omega(p)$ contains no rest points. By the Poincaré–Bendixson theorem for two-dimensional semiflows (see [14]), $\omega(p)$ is a periodic orbit Γ, and further the orbit $p \cdot \mathbb{R}^+$ is a spiral which tends to Γ as $t \to +\infty$. Consequently, Γ is asymptotically stable from one side (see [14]). Thus, for large n_k, we also have $\omega(x^+_{n_k}) = \omega(p) = \Gamma$, it implies $\phi(x^+_{n_k}) = +\infty$, which is contradictory to $x \in \widehat{N}$. Hence, we conclude that $\phi(p) < +\infty$.

Let $p_1 = p \cdot \phi(p) \in M$, by the continuity of ϕ and I, it is easy to see that $x^+_{n_k+1} \to p^+_1$. Similarly, we have $\phi(p^+_1) < \infty$. Thus, by induction, it follows that $\phi(p^+_n) < \infty$ for each positive integer $n \geq 1$. So, we obtain $p \in \widehat{N}$. Now, let A be the closure of $\widehat{C}^+(x)$ in Ω, then $A = \widehat{K}^+(x) \subset \widehat{N}$, i.e. $\widehat{K}^+(x)$ is a regular set. □

Definition 4.8 ([20]). We say that $\widetilde{\pi}$ is almost recurrent at $x \in \Omega$ if, for any $\varepsilon > 0$, there exists an $L \in \mathbb{R}^+$ such that for any $t \in \mathbb{R}^+$, the interval $[0, L]$ contains a real number τ with $d(x, x * (t + \tau)) < \varepsilon$.

For a point $x \in \widehat{N}$, some important relations between the dynamical behavior of $\widehat{C}^+(x)$ and $\widetilde{C}^+(x)$ were established by Kaul in [20], e.g. the following lemma was proved in [20].

Lemma 4.9. *Let $x \in \widehat{N}$, $\widehat{K}^+(x)$ be compact. If g is almost periodic at x, then $\widetilde{\pi}$ is almost recurrent at x.*

Theorem 4.10. *Let $x \in \widehat{N}$ and $\widetilde{\pi}_x$ be bounded. Suppose that $\widetilde{\omega}(x)$ contains no rest points. Then, there exists an almost recurrent orbit in $\widetilde{\omega}(x)$.*

Proof. Let $A = \widehat{K}^+(x)$, it is a bounded set, since $\widetilde{\pi}_x$ is bounded. By Lemma 4.7, we see that A is a regular set in $\widehat{N}$. Thus, A is a bounded closed set in Ω, i.e. it is compact. Define $g_1 = g|_A$ to be the restriction of g on A, then (A, g_1) is a sub-system of $(\widehat{N}, g)$. Since (A, g_1) is a system defined on a compact space, there exists a minimal set B in $\omega(x, g_1)$ (see [7, p. 92]). Thus, each point in B is almost periodic (see [7] and Corollary 5.4 in [20]). Now, the result follows from Lemma 4.9. □

Definition 4.11. For an invariant set $S \subset \Omega$, S is said to be a *chaotic set* of $(\Omega, \widetilde{\pi})$, if it has the properties of density and transitivity, i.e. periodic orbits of $(\Omega, \widetilde{\pi})$ are dense in S and there exists an $x \in S$ such that $\widetilde{C}^+(x)$ is dense in S.

Let S be an invariant set in Ω. The system $(\Omega, \widetilde{\pi})$ has *sensitive dependence on initial conditions* on S if there exists a $\delta > 0$ such that for any $x \in S$ and for any neighborhood U of x, there exist a $y \in U$ and a $t > 0$ such that $d(x*t, y*t) > \delta$. Similarly to the case of discrete systems, for an invariant set $S \subset \Omega$, if S is not a periodic orbit. Then density and transitivity imply sensitive dependence on initial conditions on S, which is at the heart of chaos.

PROPOSITION 4.12. *Let an invariant set $S \subset \Omega$ be not a periodic orbit. If it has the properties of density and transitivity, then $(\Omega, \widetilde{\pi})$ has sensitive dependence on initial conditions on S.*

PROOF. Let $\widetilde{C}^+(p_1)$ and $\widetilde{C}^+(p_2)$ be two periodic orbits in S, then

$$d(\widetilde{C}^+(p_1), \widetilde{C}^+(p_2)) = 8\delta > 0.$$

Let $x \in S$ and any neighbourhood U of x be given. Then there exists a periodic $\widetilde{C}^+(p)$ in $S \setminus B(x, 4\delta)$. Indeed, one of $\widetilde{C}^+(p_1)$ and $\widetilde{C}^+(p_2)$ is a candidate. Now, we choose two points y and q in $U \cap B(x,\delta)$ such that $\widetilde{C}^+(y)$ is a dense orbit and $\widetilde{C}^+(q)$ is a periodic orbit in S, respectively. Let $\tau > 0$ be the period of $\widetilde{C}^+(q)$. By the QCD property, take a $\mu \in (0,\delta)$ such that if $z \in B(p,\mu)$, then $d(p*t, z*h_z(t)) < \delta$ holds for $t \in [0, 2\tau]$, where $h_z(t)$ is a time τ-reparametrization. Now, there exists a $T > \tau$ such that $y * T \in B(p,\mu)$. Clearly, one can find a positive integer n such that $n\tau = T + \eta$, where $\eta \in [0,\tau)$. Let $y * T = z$, then $y*(T+\eta) = z*\eta = z*h_z(\nu)$ for some $\nu \in [0, 2\tau]$, it follows $y*(T+\eta) \in N(\widetilde{C}^+(p), \delta)$. Thus, it is easy to see that $d(q*n\tau, y*(T+\eta)) = d(q, y*(T+\eta)) > 2\delta$. Hence, either $d(x*n\tau, y*(T+\eta)) > \delta$ or $d(x*n\tau, q*n\tau) > \delta$. It means that δ is a sensitivity constant. The proof is complete. $\square$

LEMMA 4.13. *Let A be a closed invariant set in $\widehat{N}$. If A has the properties of density and transitivity in the system $(\widehat{N}, g)$, so does $A^+ = \bigcup\{x \cdot [0, \phi(x)) : x \in A\}$ in the system $(\Omega, \widetilde{\pi})$. In particular, if A is a chaotic set in $\widehat{N}$, then A^+ is also a chaotic set in Ω.*

PROOF. Let $x \in A$ be a periodic point of $(\widehat{N}, g)$, i.e.

$$\widehat{C}^+(x) = \{x, x_1^+, \dots, x_{n-1}^+\},$$
$$\text{where } g^i(x) = x_i^+ \text{ for } i = 1, \dots, n-1 \text{ and } g^n(x) = x.$$

Clearly, $\widetilde{C}^+(x) = \bigcup\{x_i^+ \cdot [0, \phi(x_i^+)) : i = 0, \dots, n-1\}$ is a periodic orbit of $(\Omega, \widetilde{\pi})$, where $x = x_0^+$. Now, by the definition of A^+, the density of periodic orbits in A^+ follows immediately from the density of periodic orbits in A. Next, the transitivity of A is equivalent to the existence of a dense orbit [16, p. 77], i.e. there exists an x in A such that $\widehat{K}^+(x) = A$. It is easy to see that $\widetilde{C}^+(x)$ is a dense orbit in A^+, then the transitivity of A^+ follows. $\square$

In the remainder of the paper, we deal with the limit set of an infinite trajectory under a suitable assumption. For an infinite trajectory $\widetilde{\pi}_x$ with $x \in \widehat{N}$, we apply the theory of one-dimensional dynamics to get the characteristic of $\widetilde{\omega}(x)$. We suppose that $\widetilde{\omega}(x)$ contains no rest points and the orbit $\widetilde{C}^+(x)$ meets with a component of $\widehat{N}$. Since the image of a connected set under a continuous map is still connected, it follows that if M is connected, then $N = I(M)$ is also connected. Thus in the case where $\widetilde{C}^+(x)$ has nonempty intersection with a finite number of components of $\widehat{N}$, it is easy to see that among those components of $\widehat{N}$, the orbit $\widetilde{C}^+(x)$ goes to a fixed component periodically, then we may consider g^k for some positive integer k such that the orbit of x under g^k lies in a component of $\widehat{N}$. Hence, for brevity, our discussion goes on under the following condition.

ASSUMPTION III. The set $\widehat{N}$ is an arc in Ω, i.e. it is a homeomorphic image of the interval $[0, 1]$.

Now, we present our main results of the paper under this assumption. Recall the definition of topological conjugacy between two discrete dynamical systems (see [7, p. 18]). Let $f_1: X \to X$ and $f_2: Y \to Y$ be continuous maps, where X and Y are metric spaces. The systems (f_1, X) and (f_2, Y) are said to be topologically conjugate if there exists a homeomorphism $h: X \to Y$ such that $h \circ f_1(x) = f_2 \circ h(x)$ for every $x \in X$. Since $\widehat{N}$ is closed in Ω, it implies that $\widehat{N}$ is a regular set. Thus, the system $(\widehat{N}, g)$ is topologically conjugate to a discrete system (J, f) on a compact interval J, and the results in Section 4.1 hold for $(\widehat{N}, g)$. Clearly, $x \in \widehat{N}$ is periodic under g if and only if $\widetilde{C}^+(x)$ is periodic under $\widetilde{\pi}$. Also, it is easy to see that $\widetilde{\omega}(x) = \omega(x, g)^+ = \bigcup\{p \cdot [0, \phi(p)) : p \in \omega(x, g)\}$ for $x \in \widehat{N}$.

THEOREM 4.14. *Let $\widehat{N}$ be an arc in Ω. For $x \in \widehat{N}$, assume that $\widetilde{\omega}(x)$ contains no rest point.*

(a) *If $\widetilde{\omega}(x)$ has an interior point, then the limit set $\widetilde{\omega}(x)$ is a chaotic set.*
(b) *If $\widetilde{\omega}(x)$ has no interior points, then $\widetilde{\omega}(x)$ has either a periodic orbit or a Cantor-type minimal set, in which every orbit is almost recurrent.*

PROOF. Since $\widehat{N}$ be an arc, ϕ is a bounded function on $\widehat{N}$, it follows that $\widetilde{C}^+(x)$ for $x \in \widehat{N}$ is bounded. Let $(\widehat{N}, g)$ be the discrete semidynamical system associated with the given impulsive system $(\Omega, \widetilde{\pi})$. For $x \in \widehat{N}$, since $\widehat{N}$ is a regular set, the limit set of $\widehat{C}^+(x)$ in Ω is equal to $\omega(x, g)$, which is an invariant closed set in $\widehat{N}$. Thus, we have $\widetilde{\omega}(x) = \omega(x, g)^+ = \bigcup\{p \cdot [0, \phi(p)) : p \in \omega(x, g)\}$. Clearly, it follows that $\widetilde{\omega}(x)$ has an interior point in Ω if and only if $\omega(x, g)$ has an interior point in $\widehat{N}$. By Theorem 4.6, $\omega(x, g)$ is a chaotic set if it has an interior point. It follows from Lemma 4.13 that $\widetilde{\omega}(x)$ has the properties of sensitivity and transitivity. Hence, $\widetilde{\omega}(x)$ is a chaotic set and it has the property of sensitive dependence on initial conditions by Proposition 4.12. Next, if $\widetilde{\omega}(x)$ has no interior points, then $\omega(x, g)$ has no interior points in $\widehat{N}$. From Lemma 4.2, it follows that

$\omega(x, g)$ is closed and nowhere dense. Clearly, $\omega(x, g)$ contains a minimal set A in which every orbit is almost periodic. If A is finite, it is a periodic orbit. Thus, $A^+ = \bigcup\{p \cdot [0, \phi(p)) : p \in A\}$ is a periodic orbit in $\widetilde{\omega}(x)$. Otherwise, if A is infinite, it is a Cantor set by Lemma 4.1. Now, the Cantor-type set A^+ is also a minimal set in $(\Omega, \widetilde{\pi})$ (see [20, Theorem 3.3]). Since A is almost periodic, it follows from Lemma 4.9 that each orbit in A^+ is almost recurrent. Finally, by the continuity of ϕ and minimality of A, it is easy to see that A^+ is homeomorphic to the product of a Cantor set and an interval. $\square$

Remark 4.15. Block and Coppel in [7, p. 149] presented an example taking the classical 'middle-third' Cantor set C as a minimal set. Thus, it is easy to get an impulsive system similar to Example 2.5 such that for an infinite trajectory $\widetilde{\pi}_x$ with $x \in C$, the limit set $\widetilde{\omega}(x)$ is a minimal set with almost recurrent orbits, which is homeomorphic to the product of a Cantor set and an interval. Hence, $\widetilde{\omega}(x)$ does not contain periodic orbits.

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