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INFINITELY MANY HOMOCLINIC ORBITS FOR SUPERLINEAR HAMILTONIAN SYSTEMS

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Abstract. In this paper we study the first order nonautonomous Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(t, z),$$

where $H(t, z)$ depends periodically on t. By using a generalized linking theorem for strongly indefinite functionals, we prove that the system has infinitely many homoclinic orbits for weak superlinear cases.

1. Introduction and main results

In this paper we are interested in the existence of homoclinic orbits of the Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(t, z), \tag{HS}$$

where $z = (p, q) \in \mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$, $\mathcal{J} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ and $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is of the form

$$H(t, z) = \frac{1}{2} B(t) z \cdot z + R(t, z), \tag{1.1}$$

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with $B(t) \in C(\mathbb{R}, \mathbb{R}^{4N^2})$ being a $2N \times 2N$ symmetric matrix valued function, and $R \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is superlinear in z. Here by a homoclinic orbit of (HS) we mean a solution of the equation satisfying $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$.

Establishing the existence of homoclinic orbits for system like (HS) is a classical problem. Up to the year of 1990, there are few isolated results. In very recent years, many authors considered the existence of homoclinic orbits for Hamiltonian systems via critical point theory. For example, see [1], [8], [9], [12], [21], [22] for the second order systems, and [3], [7], [10], [13]–[16], [18], [23]–[27], [29] for the first order systems. Usually, for superlinear problem, one needs the following condition due to Ambrosetti–Rabinowitz [2];

$$\exists\, \nu > 2,\ 0 < \nu R(t, z) \le R_z(t, z) z, \quad \forall z \neq 0. \tag{1.2}$$

Generally speaking, the role of (1.2) is to ensure the boundedness of all $(\mathrm{PS})_c$-sequences for the corresponding functional. Without (1.2), it is very difficulty to get the boundedness of $(\mathrm{PS})_c$-sequences. However, it is easy to see that (1.2) does not include some superlinear nonlinearities like

$$R(t, z) = a(t)\bigg(|z|^\nu + (\nu - 2)|z|^{\nu-\varepsilon} \sin^2\bigg(\frac{|z|^\varepsilon}{\varepsilon}\bigg)\bigg), \qquad \nu > 2,\ 0 < \varepsilon < \min\{\nu - 2, \nu - \nu^*\}, \tag{1.3}$$

where $a(t) > 0$ is 1-periodic in t and $\nu^* := \nu(\nu - 2)/(\nu - 1)$. In this paper, we shall study the existence of infinitely many homoclinic orbits for the system (HS) under some superlinear conditions which cover the cases like (1.3).

Let $A := -(\mathcal{J}(d/dt) + B(t))$ be the selfadjoint operator acting in $L^2(\mathbb{R}, \mathbb{R}^{2N})$ and $\sigma(A)$ denote the spectrum of A. As we all know, the information of $\sigma(A)$ are very important in finding the homoclinic orbits for the system (HS). For example, if 0 is in the essential spectrum of the operator A, then the operator A can not lead the behavior at 0 of the equation, which brings difficulty in the usual variational arguments. So in the early results [3], [7], [18], [24], [25], [27], they assume

($\mathcal{R}$) $B(t) \equiv \widetilde{B}$ is independent of t such that $\mathrm{sp}(\mathcal{J}\widetilde{B}) \cap i\mathbb{R} = \emptyset$,

where $\mathrm{sp}(\mathcal{J}\widetilde{B})$ denotes the set of all eigenvalues of $\mathcal{J}\widetilde{B}$. Clearly, the condition ($\mathcal{R}$) means that there exists $\zeta > 0$ such that $\sigma(A) \cap (-\zeta, \zeta) = \emptyset$. That is, 0 is not in the spectrum of A, which is important for variational arguments. Recently, the above condition ($\mathcal{R}$) was relaxed by Ding and Willem [3], they handled the case when 0 may be in the essential spectrum of A, and assumed that

($\mathcal{R}_0$) $B(t)$ depends periodically on t with period 1, and there is $\alpha > 0$ such that $\sigma(A) \cap (0, \alpha) = \emptyset$.

Under the superlinear condition (1.2) and some additional conditions, [16] showed that the system (HS) has at least one homoclinic orbit. Here we point out that in the present case, 0 may be in the essential spectrum of A which brings difficulty in handing such case by variational methods. To overcome this difficulty, the authors proved an embedding theorem as a compensation (see the following Lemma 2.1). Later, under the superlinear condition (1.2), [13] also considered the case when 0 may be in the essential spectrum of A. If $H(t,z)$ is even in z, the authors proved that the system (HS) has infinitely many homoclinic orbits. In [16], [13], the condition (1.2) is important for them to get the boundedness of the $(\mathrm{PS})_c$-sequences. We emphasize that under the condition $(\mathcal{R}_0)$, the superlinear case without (1.2) is quite different and tough to be dealt with. Nearly, under the condition $(\mathcal{R}_0)$, [29] considered the superlinear case without (1.2). The authors obtained that (HS) has at least one homoclinic orbit. Motivated by present works of [16], [13], [29], in the present, we continue our work of [29] to prove that the system (HS) has infinitely many homoclinic orbits for the superlinear case. In order to state our main results, we assume that $R(t,z)$ satisfies the following conditions.

$(\mathcal{R}_1)$ $R(t,z)\in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is 1-periodic in t, and there exist positive constants c_1, c_2 and $\nu>2$ such that

$$c_1|z|^\nu \le R_z(t,z)z \le |R_z(t,z)||z| \le c_2|z|^\nu, \quad \text{for all } (t,z)\in\mathbb{R}\times\mathbb{R}^{2N}.$$

$(\mathcal{R}_2)$ $R_z(t,z)z-2R(t,z)>0$ for all $t\in\mathbb{R}$ and $z\in\mathbb{R}^{2N}\setminus\{0\}$.

$(\mathcal{R}_3)$ There exists $\mu_0>2$ such that

$$\liminf_{z\to 0}\frac{R_z(t,z)z}{R(t,z)}\ge\mu_0$$

uniformly for t$\in\mathbb{R}$.

$(\mathcal{R}_4)$ There exists $c_0>0$ such that

$$\liminf_{|z|\to\infty}\frac{R_z(t,z)z-2R(t,z)}{|z|^\beta}\ge c_0$$

uniformly for $t\in\mathbb{R}$, where $\nu>\beta>\nu^*=\nu(\nu-2)/(\nu-1)$.

$(\mathcal{R}_5)$ There exist c_3 and $\delta>0$ such that for all (t,z) and $v\in\mathbb{R}^{2N}$ with $|v|\le\delta$

$$|R_z(t,z+v)-R_z(t,z)|\le c_3(|v|^{\nu-1}+|z|^{\nu-2}|v|+|v|^{\nu-2}|z|).$$

$(\mathcal{R}_6)$ $R(t,-z)=R(t,z)$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}^{2N}$.

Remark 1.1. In [31], [32], the conditions $(\mathcal{R}_2)$–$(\mathcal{R}_5)$ have been used to weaken the Ambrosetti–Rabinowitz superlinear growth condition (1.2) for Schrödinger equations.

Remark 1.2. Let $\mu_0=\nu$ and $\beta=\nu-\varepsilon$. It is easy to see that the nonlinearity (1.3) satisfies $(\mathcal{R}_1)$–$(\mathcal{R}_6)$. However, similar to [31]. Let $z_n =$

$(\varepsilon(n\pi+3\pi/4))^{1/\varepsilon}L_{2N}$, where $L_{2N}=(1,0,\ldots,0)$. For any $\gamma>2$, one has

$$\begin{aligned}R_z(t,z_n)z_n\ &-\gamma R(t,z_n)\\&=a(t)\bigg[(\nu-\gamma)|z_n|^\nu+(\nu-2)(\nu-\varepsilon-\gamma)|z_n|^{\nu-\varepsilon}\sin^2\bigg(\frac{|z_n|^\varepsilon}{\varepsilon}\bigg)\\&\quad+(\nu-2)|z_n|^\nu\sin 2\bigg(\frac{|z_n|^\varepsilon}{\varepsilon}\bigg)\bigg]\\&=a(t)|z_n|^\nu\bigg[2-\gamma+\frac{(\nu-2)(\nu-\varepsilon-\gamma)\sin^2(|z_n|^\varepsilon/\varepsilon)}{|z_n|^\varepsilon}\bigg]\to-\infty\end{aligned}$$

as $n\to\infty$. Thus, we know that the nonlinearity (1.3) can not satisfy the Ambrostti–Rabinowitz condition (1.2) for $\gamma>2$.

Recall that, based on the periodicity condition, if z is a homoclinic orbit then for any $\iota\in\mathbb{Z}$, $\iota*z:=z(\,\cdot+\iota)$ is also a homoclinic orbit. Let $\mathcal{O}(z):=\{\iota*z;\ \iota\in\mathbb{Z}\}$ denote the orbit of z with respect to the $\mathbb{Z}$-action $*$, two homoclinic orbits z_1 and z_2 of (HS) are said to be geometrically distinct if $\mathcal{O}(z_1)\neq\mathcal{O}(z_1)$.

Now we have the following result.

THEOREM 1.3. *Let* $(\mathcal{R}_0)$–$(\mathcal{R}_6)$ *be satisfied. Then* (HS) *has infinitely many geometrically distinct homoclinic orbits.*

REMARK 1.4. If there exists $\alpha>0$ such that $(-\alpha,0)\cap\sigma(A)=\emptyset$ and $\overline{R}(t,z):=-R(t,z)$ satisfies the the assumptions $(\mathcal{R}_1)$–$(\mathcal{R}_6)$, then the same conclusion of Theorem 1.1 remains valid.

Throughout the paper we shall denote by $c>0$ various positive constants which may vary from lines to lines and are not essential to the problem.

2. The embedding theorem

In order to establish a variational setting for the system (HS), in this section we shall study the spectrum of a Hamiltonian operator.

Recall that $A:=-(\mathcal{J}(d/dt)+B(t))$ is a self-adjoint operator in $L^2(\mathbb{R},\mathbb{R}^{2N})$ with domain $\mathcal{D}(A)=H^1(\mathbb{R},\mathbb{R}^{2N})$. Let $\sigma_d(A)$ and $\sigma_{\rm ess}(A)$ be, respectively, the discrete spectrum of A and the essential spectrum of A. By Proposition 2.2 of [16], at most 0 is in the continuous spectrum of A, so we only need to consider the case $0\in\sigma_{\rm ess}(A)$. Let $|\cdot|_q$ denote the usual L^q-norm, and $(\,\cdot\,,\,\cdot\,)_2$ be the usual L^2-inner product. Set $\mathcal{H}:=L^2$.

Let $\{E(\lambda):\lambda\in\mathbb{R}\}$ be the spectral family of A. We have $A=U|A|$, called the polar decomposition, where $U=I-E(0)-E(-0)$. Clearly, $\mathcal{H}$ has orthogonal decomposition

$$\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-,$$

where $\mathcal{H}^\pm=\{z\in\mathcal{H};\ Uz=\pm z\}$. For each $z\in\mathcal{H}$, we will write $z=z^-+z^+$, where $z^\pm\in\mathcal{H}^\pm$.

Let E be the completion space of $\mathcal{D}(|A|^{1/2})$ under the norm

$$\|z\|_E = ||A|^{1/2} z|_2.$$

E is a Hilbert space with the inner product

$$(z_1, z_2)_E := (|A|^{1/2} z_1, |A|^{1/2} z_2)_2.$$

By Lemma 6.3 in Appendix, we have that for all $z \in \mathcal{D}(|A|^{1/2})$,

$$c_1 \|z\|_{H^{1/2}} \leq \|z\|_E + a|z|_2 \leq c_2 \|z\|_{H^{1/2}} + 2a|z|_2, \tag{2.1}$$

where $c_1, c_2 > 0$ and $a > 4 \sup_{t \in \mathbb{R}} |B(t)|$.

Let $E^+ := \mathcal{H}^+ \cap \mathcal{D}(|A|^{1/2})$. Since the spectrum of A on E^+ is bounded away from 0, thus we have

$$\|u\|_E^2 = (Au, u)_2 = \int_\alpha^\infty \lambda d(E(\lambda)u, u)_2 \geq \alpha |u|_2^2, \quad \text{for all } u \in E^+.$$

Thus, it follows from (2.1) that E^+ is a closed set and

$$\| \cdot \|_E \sim \| \cdot \|_{H^{1/2}} \quad \text{on } E^+, \tag{2.2}$$

where the notation "$\sim$" denotes the equivalence. Then E has an orthogonal decomposition

$$E = E^+ \oplus E^-,$$

with

$$E^- \supseteq \mathcal{H}^- \cap \mathcal{D}(|A|^{1/2}). \tag{2.3}$$

However, since 0 may belong to a spectrum of A, then $\|\cdot\|_E$ may not be equivalent to $H^{1/2}$-norm on E^-. Therefore, in the following we use the spectrum family of A to sperate $\sigma(A) \cap (-\infty, 0]$ into two segments. That is, for any $\varepsilon > 0$, set

$$\mathcal{H}_\varepsilon^- := E(-\varepsilon)\mathcal{H},$$

and $E_\varepsilon^- = \mathcal{H}_\varepsilon^- \cap \mathcal{D}(|A|^{1/2}) = \mathcal{H}_\varepsilon^- \cap E^-$. Let $\widehat{\mathcal{H}}_\varepsilon^- := \mathcal{H}^- \cap (\mathrm{cl}_{\mathcal{H}}(\bigcup_{\lambda < -\varepsilon} E(\lambda)\mathcal{H}))^\perp$, where $\mathrm{cl}_{\mathcal{H}}(B)$ denotes the closure of the set B in $\mathcal{H}$. Similarly to E^+, since the spectrum of A restrict to E_ε^- is bounded away from 0. Thus,

$$\| \cdot \|_E \sim \| \cdot \|_{H^{1/2}} \quad \text{on } E_\varepsilon^-. \tag{2.4}$$

However, $\widehat{\mathcal{H}}_\varepsilon^-$ is not complete with respect to the norm $\|\cdot\|_E$, thus it is reasonable to introduce a new norm. Define

$$\|z\|_\nu = (||A|^{1/2} z|_2^2 + |z|_\nu^2)^{1/2}. \tag{2.5}$$

Let $E_{\varepsilon,\nu}^-$ be the completion of $\widehat{\mathcal{H}}_\varepsilon^-$ under the norm $\| \cdot \|_\nu$.

Now let E_ν^- denote the completion of $\mathcal{D}(A) \cap \mathcal{H}^-$ with respect to the norm $\|\cdot\|_\nu$. Since $H^{1/2}$ is continuously embedded in L^p for any $p \in [2, \infty)$, by (2.4), E_ε^-

is a closed subspace of E_ν^-. Moreover, noting that $E_{\varepsilon,\nu}^- \subset E^-$, it is orthogonal to E_ε^- with respect to $(\,\cdot\,,\,\cdot\,)_E$, we have

$$E_\nu^- = E_\varepsilon^- \oplus E_{\varepsilon,\nu}^-. \tag{2.6}$$

Lemma 2.1. *$E_{\varepsilon,\nu}^- \subset H^1_{\mathrm{loc}}(\mathbb{R})$ and is embedded compactly in L^∞_{loc}, and continuously in L^p for all $\nu \le p \le \infty$.*

Proof. The proof was actually given in [16], we state it here for reader's convenience. By the spectral theory of self-adjoint operators, $\widehat{\mathcal{H}}_\varepsilon^- \subset \mathcal{D}(A) = H^1$. Let $\{z_n\} \subset \widehat{\mathcal{H}}_\varepsilon^-$ be Cauchy sequence with respect to $\|\cdot\|_\nu$. Then

$$\begin{aligned} |A(z_n - z_m)|_2^2 &= \int_{-\varepsilon}^0 \lambda^2 d|E(\lambda)(z_n - z_m)|_2^2 \\ &\le -\varepsilon \int_{-\varepsilon}^0 \lambda d|E(\lambda)(z_n - z_m)|_2^2 = \varepsilon||A|^{1/2}(z_n - z_m)|_2^2 \to 0, \end{aligned} \tag{2.7}$$

as $n, m \to \infty$. For any finite interval $I \subset \mathbb{R}$, one has

$$\int_I |z_n - z_m|^2\,dt \le |I|^{1-2/\nu}|z_n - z_m|_\nu^2 \to 0.$$

Together with (2.7), we have

$$\begin{aligned} \int_I |\dot z_n - \dot z_m|^2\,dt &= \int_I |A(z_n - z_m) + B(t)(z_n - z_m)|^2\,dt \\ &\le 2|A(z_n - z_m)|_2^2 + 2\int_I |B(t)(z_n - z_m)|^2\,dt \to 0, \end{aligned}$$

as $n, m \to \infty$. Therefore the limit z of $\{z_n\}$ with respect to $\|\cdot\|_\nu$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, since $H^1(I)$ is compactly embedded in $L^\infty(I)$ for any finite interval I, one sees that $E_{\varepsilon,\nu}^-$ is compactly embedded in $L^\infty(I)$.

By (2.7), $\{Az_n\}$ is a Cauchy sequence in L^2. Hence $Az_n \to w$ in L^2. Since $Az_n \to Az$ in L^2_{loc}, $w = Az$, i.e. $Az \in L^2$. Note that for any finite interval $I \subset \mathbb{R}$

$$\begin{aligned} \int_I |\dot z|^2\,dt = \int_I |Az + Bz|^2\,dt &\le 2\int_I (|Az|^2 + |Bz|^2)\,dt \\ &\le c\bigg(\int_I |Az|^2 + |I|^{1-2/\nu}\bigg(\int_I |z|^\nu\bigg)^{2/\nu}\bigg). \end{aligned} \tag{2.8}$$

Obviously, we have

$$z(\tau) = z(t) + \int_t^\tau \dot z(s)\,ds, \quad \text{for } \tau \in \mathbb{R}.$$

Integrating from $\tau - 1/2$ to $\tau + 1/2$ in the above equality, one has

$$|z(\tau)| \le \bigg(\int_{\tau-1/2}^{\tau+1/2} |z|^\nu\,dt\bigg)^{1/\nu} + \bigg(\int_{\tau-1/2}^{\tau+1/2} |\dot z|^2\,dt\bigg)^{1/2}. \tag{2.9}$$

Since $z \in \mathcal{H}$ and $Az \in \mathcal{H}$, (2.8) and (2.9) show that

$$|z(\tau)| \to 0 \quad \text{as } |\tau| \to \infty.$$

That is, $z \in L^\infty$. Therefore $z \in L^\nu \cap L^\infty$ and so $z \in L^p$ for any $p \geq \nu$. Replacing z by $z_n - z$ in (2.8) and (2.9) one sees that $E^-_{\varepsilon,\nu}$ is continuously embedded in L^∞ and so is in L^p for any $p \geq \nu$. □

Let E_ν denote the completion of the set $\mathcal{D}(A)$ under the norm $\|\cdot\|_\nu$. It follows from (2.2), (2.4), (2.6) and Lemma 2.1 that E^-_ν and E^+ are closed sets. Moreover, since $E_\nu \subset E$, and using the decomposition of E, it is easy to check that $E^-_\nu \cap E^+ = \{0\}$, and so

$$E_\nu = E^-_\nu \oplus E^+. \tag{2.10}$$

We now come to the following embedding theorem.

Theorem 2.2. *Suppose* $(\mathcal{J}_1)$ *is satisfied, and* E_ν *is defined in* (2.10). *Then* E_ν *is embedded continuously in* L^p *for all* $p \geq \nu$ *and compactly in* L^q_{loc} *for any* $q \geq 2$.

Proof. By (2.2), (2.4), (2.10) and Lemma 2.1, one can easily get the desired conclusion. □

3. The abstract critical point theorems

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space and $\Phi(z) \in C^1(\mathbb{E}, \mathbb{R})$. In order to study the critical points of $\Phi(z)$, we now recall some abstract critical point theory developed recently in [5], see also [19], [4], [31] for earlier results on that direction.

Assume that $\mathbb{E}$ has direct sum decomposition $\mathbb{E} = X \oplus Y$, let $\mathcal{P}_X$ and $\mathcal{P}_Y$ denote projections from $\mathbb{E}$ onto X and Y, respectively. For a functional $\Phi(z)$, we write $\Phi_a := \{z \in \mathbb{E} : \Phi(z) \geq a\}$, $\Phi^b := \{z \in \mathbb{E} : \Phi(z) \leq b\}$ and $\Phi^b_a = \Phi_a \cap \Phi^b$. Next, let's us recall some definitions:

(i) Φ is said to be weakly sequentially upper semi-continuous if $z_n \rightharpoonup z$ in $\mathbb{E}$ implies $\Phi(z) \geq \liminf_{n\to\infty} \Phi(z_n)$;

(ii) Φ' is said to be weakly sequentially continuous if $z_n \rightharpoonup z$ in $\mathbb{E}$ implies $\lim_{n\to\infty} \Phi'(z_n)w = \Phi'(z)w$ for each $w \in \mathbb{E}$;

(iii) A sequence $\{z_n\} \subset \mathbb{E}$ is said to be a $(\mathrm{C})_c$-sequence if $\Phi(z_n) \to c$ and $(1 + \|z_n\|)\Phi'(z_n) \to 0$. Φ is said to satisfy the $(\mathrm{C})_c$-condition if any $(\mathrm{C})_c$-sequence has a convergent subsequence.

In what follows, a set $\mathcal{B} \subset \mathbb{E}$ is said to be a $(\mathrm{C})_c$-attractor if for any $\varepsilon, \delta > 0$ and any $(\mathrm{C})_c$-sequence $\{z_n\}$ one has, along a subsequence $z_n \in \mathcal{U}_\varepsilon(\mathcal{B} \cap \Phi^{c+\delta}_{c-\delta})$. Here (and in the sequel), $\mathcal{U}_\varepsilon(\mathcal{K}) := \{z \in \mathbb{E} : \|z - \mathcal{K}\| < \varepsilon\}$ for any subset $\mathcal{K} \subset \mathbb{E}$. For any interval $I \subset \mathbb{R}$, a set $\mathcal{B}$ is called a $(\mathrm{C})_I$-attractor if it is a $(\mathrm{C})_c$-attractor for any $c \in I$ (cf. [11], [5], [4]).

From now on we assume that X is separable and reflexive subspace. For a countable dense subset $B \subset X^*$ and $b \in B$, we define a semi-norm on $\mathbb{E}$ by

$$P_b \colon \mathbb{E} = X \oplus Y \to \mathbb{R}, \quad P_b(x+y) = q_b(x) + \|y\|, \quad \text{for } x+y \in X \oplus Y,$$

where $q_b(x) = |(x,b)_{X,X^*}| = |b(x)|$. We denote by $\mathcal{T}_B$ the induced topology. Let w^* denote the weak*-topology on $\mathbb{E}^*$.

Assume:

- $(\mathcal{A}_0)$ For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_B$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_B) \to (\mathbb{E}^*, w^*)$ is continuous.
- $(\mathcal{A}_1)$ There exists $\varrho > 0$ with $\kappa := \inf \Phi(S_\varrho Y) > 0$ where $S_\varrho Y := \{z \in Y : \|z\| = \varrho\}$.
- $(\mathcal{A}_2)$ There exists an increasing sequences of finite dimensional subspace $Y_n \subset Y$ and $R_n > \varrho$ such that $\sup \Phi(X \times Y_n) < \infty$ and $\sup \Phi(X \times Y_n \setminus K_n) < \gamma := \inf \Phi(\{z \in X : \|z\| \le \varrho\})$, where $K_n := \{z \in X \times Y_n : \|z\| \le R_n\}$.
- $(\mathcal{A}_3)$ Φ has a $(\mathrm{C})_I$-attractor $\mathcal{F}$ with $\mathcal{P}_Y \mathcal{F} \subset Y \setminus \{0\}$ bounded and such that

  $$\mu := \inf\{\|\mathcal{P}_Y u - \mathcal{P}_Y v : u, v \in \mathcal{F},\ \mathcal{P}_Y u \neq \mathcal{P}_Y v\|\} > 0$$

  and there exists $\widetilde{\beta} > 0$ with

  $$\|z\| \le \widetilde{\beta} \|\mathcal{P}_Y z\|, \quad \text{for all } u \in \Phi_a^b,$$

  where $I := [a,b]$ and $a, b \in \mathbb{R}$.

Then we have the following theorem:

Theorem 3.1. *If Φ satisfies $(\mathcal{A}_0)$–$(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ for any compact interval $I \subset (0,\infty)$, then Φ has unbounded sequence of critical values.*

The proof was given in Theorem 4.8 of [5] (see also [11]).

4. Some preliminary works

4.1. Properties of the functional. Set $\mathbb{E} := E_\nu = E_\nu^- \oplus E^+$, where $Y = E^+$, $X = E_\nu^-$. Let

$$\Psi(z) = \int_{\mathbb{R}} R(t, z(t))\, dt.$$

By assumptions and Theorem 2.2, $\Psi(z) \in C^1(E_\nu, \mathbb{R})$ and

$$\Psi'(z)v = \int_{\mathbb{R}} R_z(t, z(t)) v(t)\, dt, \quad \text{for all } z, v \in E_\nu.$$

Now, let us consider the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|_E^2 - \frac{1}{2}\|z^-\|_E^2 - \Psi(z), \quad \text{for } z = z^- + z^+ \in E_\nu.$$

Then $\Phi \in C^1(E_\nu, \mathbb{R})$. Moreover, for $\psi \in C_0^\infty(\mathbb{R})$

$$\Phi'(z)\psi = \int_{\mathbb{R}} (-\mathcal{J}\dot{z} - Bz - R_z(t,z), \psi)\, dt.$$

It follows that critical points of $\Phi(z)$ are solutions of (HS). Moreover, if z is a solution of (HS), by Theorem 2.2, $R_z(t,z) \in L^s(\mathbb{R}, \mathbb{R}^{2N})$ for any $s \in [2,\infty)$. Thus $R_z(t,z) \in \mathcal{H}$. A standard argument shows that z is also a homoclinic orbit of (HS) (see [16]). So we have

Proposition 4.1.1. *Assume that the conditions* $(\mathcal{R}_0)$–$(\mathcal{R}_6)$ *hold. If* $z(t) \neq 0$ *is a solution of* (HS)*, then* z *is a homoclinic orbit of* (HS)*.*

In the following we will study the linking structure of Φ.

Lemma 4.1.2. *Let* $(\mathcal{R}_0)$–$(\mathcal{R}_1)$ *be satisfied. Then there exists* $\varrho > 0$ *such that* $\kappa := \inf \Phi(S_\varrho^+) > 0$*, where* $S_\varrho^+ := \{z \in E^+ : \|z\|_\nu = \varrho\}$*.*

Proof. For all $z \in E^+$, by the Theorem 2.2 and $(\mathcal{R}_1)$, we have

$$\Phi(z) = \frac{1}{2}\|z\|_E^2 - \int_{\mathbb{R}} R(t,z)dt \geq \frac{1}{2}\|z\|_E^2 - c|z|_\nu^\nu \geq \frac{1}{2}\|z\|_E^2 - c\|z\|_E^\nu. \qquad \square$$

Now we obtain the desired results.

Lemma 4.1.3. *Let* $(\mathcal{R}_0)$–$(\mathcal{R}_1)$ *be satisfied. Then, for any finite dimensional subspace* $\mathcal{W} \subset E^+$*, there exists* $R_\mathcal{W} > \varrho$ *such that* $\sup \Phi(E_\mathcal{W}) < \infty$ *and* $\sup \Phi(E_\mathcal{W} \setminus B_\mathcal{W}) < \gamma := \inf \Phi(\{z \in E_\nu^- : \|z\|_\nu \leq \varrho\})$*, where* $B_\mathcal{W} := \{z \in E_\mathcal{W} : \|z\|_\nu \leq R_\mathcal{W}\}$ *and* $E_\mathcal{W} := E_\nu^- \oplus \mathcal{W}$*.*

Proof. It suffices to show that $\Phi(z) \to -\infty$ as $z \in E_\mathcal{W}$ and $\|z\|_\nu \to \infty$. For $z \in E_\mathcal{W}$, let $z = z_\mathcal{W}^+ + z^-$, where $z_\mathcal{W}^+ \in \mathcal{W}$ and $z^- \in E_\nu^-$. By Theorem 6.4 in Appendix, there exists a continuous projection from the closure of $E_\mathcal{W}$ in L^ν to $\mathcal{W}$. Thus $|z_\mathcal{W}^+|_\nu \leq c|z_\mathcal{W}^+ + z^-|_\nu$. Moreover, since $\mathcal{W}$ is finite dimensional subspace, and from $(\mathcal{R}_1)$, we have

$$\begin{aligned}
\Phi(z) &= \frac{1}{2}\|z_\mathcal{W}^+\|_E^2 - \frac{1}{2}\|z^-\|_E^2 - \int_{\mathbb{R}} R(t,z)\, dt \\
&\leq c_1|z_\mathcal{W}^+|_\nu^2 - \frac{1}{2}\|z^-\|_E^2 - c_3|z^- + z_\mathcal{W}^+|_\nu^\nu \\
&\leq c_2|z^- + z_\mathcal{W}^+|_\nu^2 - \frac{1}{2}\|z^-\|_E^2 - c_3|z^- + z_\mathcal{W}^+|_\nu^\nu,
\end{aligned}$$

where $c_i > 0$ $(i = 1,2,3)$. It follows that $\Phi(z) \to -\infty$ as $\|z\|_\nu \to \infty$. $\square$

4.2. The $(\mathrm{C})_c$**-sequences.** In this section we discuss the Cerami-sequences for the functional Φ.

Lemma 4.2.1. *Let conditions* $(\mathcal{R}_0)$–$(\mathcal{R}_6)$ *be satisfied. Then any* $(\mathrm{C})_c$*-sequence is bounded.*

Proof. Let $z_n \in E_\nu$ be such that

$$\Phi(z_n) \to c \quad \text{and} \quad (1 + \|z_n\|_\nu)\Phi'(z_n) \to 0. \tag{4.1}$$

By $(\mathcal{R}_1)$ and (4.1), one sees

$$o(1) = \Phi'(z_n)z_n = \|z_n^+\|_E^2 - \|z_n^-\|_E^2 - \int_{\mathbb{R}} R_z(t, z_n)z_n\, dt.$$

Thus

$$o(1) + \|z_n^+\|_E^2 - \|z_n^-\|_E^2 = \int_{\mathbb{R}} R_z(t, z_n)z_n\, dt \geq c|z_n|_\nu^\nu. \tag{4.2}$$

Therefore, $\|z_n^-\|_E^2 \leq \|z_n^+\|_E^2 + o(1)$, $|z_n|_\nu^\nu \leq c\|z_n^+\|_E^2 + o(1)$, $|z_n|_\nu \leq c\|z_n^+\|_E^{2/\nu} + o(1)$. Clearly, it suffices to prove the boundedness of $\|z_n^+\|_E^2$.

By $(\mathcal{R}_3)$ and $(\mathcal{R}_4)$, let $\varepsilon_0 > 0$ such that $\mu_0 - \varepsilon_0 > 2$, then there exist $R_1 \geq R_0 > 0$ such that

$$R_z(t, z)z \geq (\mu_0 - \varepsilon_0)R(t, z), \quad \text{for all } t \in \mathbb{R},\ |z| \leq R_0, \tag{4.3}$$

and

$$R_z(t, z)z - 2R(t, z) \geq c_0|z|^\beta, \quad \text{for all } t \in \mathbb{R},\ |z| \geq R_1.$$

Furthermore, by $(\mathcal{R}_2)$, we can choose $\varepsilon > 0$ small enough such that

$$R_z(t, z)z - 2R(t, z) \geq \varepsilon|z|^\beta, \quad \text{for all } t \in \mathbb{R},\ |z| \geq R_0. \tag{4.4}$$

By (4.1), there exists $d > 0$ such that

$$\begin{aligned} d \geq \Phi(z_n) - \frac{1}{\mu_0 - \varepsilon_0}\Phi'(z_n)z_n &= \left(\frac{1}{2} - \frac{1}{\mu_0 - \varepsilon_0}\right)(\|z_n^+\|_E^2 - \|z_n^-\|_E^2) \\ &\quad + \int_{\mathbb{R}} \left(\frac{1}{\mu_0 - \varepsilon_0}R_z(t, z_n)z_n - R(t, z_n)\right) dt. \end{aligned}$$

Hence, by (4.3) and $(\mathcal{R}_1)$–$(\mathcal{R}_2)$, we get that

$$\begin{aligned} \|z_n^+\|_E^2 - \|z_n^-\|_E^2 &\leq c + c\int_{\mathbb{R}} \left(R(t, z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t, z_n)z_n\right) dt \\ &= c + c\left(\int_{|z_n| \geq R_0} + \int_{|z_n| \leq R_0}\right)\left(R(t, z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t, z_n)z_n\right) dt \\ &\leq c + c\int_{|z_n| \geq R_0} \left(R(t, z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t, z_n)z_n\right) dt \\ &\leq c + c\left(\frac{1}{2} - \frac{1}{\mu_0 - \varepsilon_0}\right)\int_{|z_n| \geq R_0} R_z(t, z_n)z_n\, dt \\ &\leq c + c\int_{|z_n| \geq R_0} |z_n|^\nu\, dt. \end{aligned} \tag{4.5}$$

Moreover, by (4.1), there exists $d_1 > 0$ such that $\Phi(z_n) - (1/2)\Phi'(z_n)z_n \le d_1$. $(\mathcal{R}_2)$ and (4.4) imply that

$$c \ge \int_{\mathbb{R}} \left(\frac{1}{2}R_z(t,z_n)z_n - R(t,z_n)\right) \ge \frac{\varepsilon}{2}\int_{|z_n|\ge R_0} |z_n|^\beta \, dt. \tag{4.6}$$

Choose $t \in ((\nu-2)/\beta(\nu-1), 1/\nu) \subset (0,1)$, since $\nu(\nu-2)/(\nu-1) = \nu^* < \beta < \nu$, then by (4.6), Höder inequality and Theorem 2.2, we have

$$\begin{aligned}\int_{|z_n|\ge R_0} |z_n|^\nu \, dt &= \int_{|z_n|\ge R_0} |z_n|^{\beta t\nu}|z_n|^{(1-\beta t)\nu} \, dt \\ &\le \left(\int_{|z_n|\ge R_0} |z_n|^\beta \, dt\right)^{t\nu}\left(\int_{|z_n|\ge R_0} |z_n|^{(1-t\beta)\nu/(1-t\nu)} \, dt\right)^{1-t\nu} \\ &\le c|z_n|_{p^*}^{(1-t\beta)\nu} \le c\|z_n\|_\nu^{(1-t\beta)\nu} \\ &\le c(\|z_n^+\|_E + \|z_n^-\|_E + |z_n|_\nu)^{(1-t\beta)\nu} \\ &\le c\|z_n^+\|_E^{(1-t\beta)\nu} + c\|z_n^+\|_E^{2(1-t\beta)} + o(1),\end{aligned} \tag{4.7}$$

where $p^* = (1-t\beta)\nu/(1-t\nu) > \nu$. Consequently, (4.2), (4.5) and (4.6) imply that

$$\begin{aligned}c\int_{\mathbb{R}} |z_n|^\nu \, dt &\le \|z_n^+\|_E^2 - \|z_n^-\|_E^2 + o(1) \le c + c\int_{|z_n|\ge R_0} |z_n|^\nu \, dt + o(1) \\ &\le c + c\|z_n^+\|_E^{(1-t\beta)\nu} + c\|z_n^+\|_E^{2(1-t\beta)} + o(1),\end{aligned}$$

that is, $|z_n|_\nu \le c + c\|z_n^+\|_E^{(1-t\beta)} + c\|z_n^+\|_E^{(2/\nu)(1-t\beta)} + o(1)$. On the other hand, (4.1) and $(\mathcal{R}_1)$ imply that

$$\begin{aligned}o(1) + \|z_n^+\|_E^2 &= \int_{\mathbb{R}} R_z(t,z_n)z_n^+ \, dt \le c\int_{\mathbb{R}} |z_n|^{\nu-1}|z_n^+| \, dt \\ &\le c|z_n|_\nu^{\nu-1}|z_n^+|_\nu \le c(c + c\|z_n^+\|_E^{1-t\beta} + c\|z_n^+\|_E^{(2/\nu)(1-t\beta)} + o(1))^{\nu-1}\|z_n^+\|_E \\ &\le c\|z_n^+\|_E + c\|z_n^+\|_E^{(1-t\beta)(\nu-1)+1} + c\|z_n^+\|_E^{(2(\nu-1)/\nu)(1-t\beta)+1} + o(1)\|z_n^+\|_E.\end{aligned}$$

Since $(1-t\beta)(\nu-1)+1 < 2$, we have that $\|z_n^+\|_E < \infty$. □

Let $\{z_n\}$ be an arbitrary $(C)_c$-sequence. By Lemma 4.2.1 it is bounded, hence, we may assume without loss of generality that $z_n \rightharpoonup z$ in E_ν, $z_n \to z$ in L^q_{loc} for $q \ge 2$ and $z_n(t) \to z(t)$ almost everywhere in t. Clearly, z is a critical point of Φ. Set $z_n^1 := z_n - z$.

Lemma 4.2.2. *Under the assumptions of Theorem 1.1, along a subsequence:*

(a) $\Phi(z_n^1) \to c - \Phi(z)$;
(b) $\Phi'(z_n^1) \to 0$.

Proof. Similar to the proof of Lemma 4.6 in [13], we sketch it here for reader's convenience.

(a) Observe that

$$\lim_{n\to\infty}\Phi(z_n^1)=\lim_{n\to\infty}\Phi(z_n)-\Phi(z)+\lim_{n\to\infty}\int_{\mathbb{R}}(R(t,z_n)-R(t,z_n^1)-R(t,z))\,dt.$$

It suffices to check that

$$\lim_{n\to\infty}\int_{\mathbb{R}}(R(t,z_n)-R(t,z_n^1)-R(t,z))\,dt=0.$$

Since z is a critical point of Φ, it follows from Proposition 4.1.1 that for any $\varepsilon\in(0,\delta)$, where $\delta>0$ is given in $(\mathcal{R}_5)$, choose $R>0$ such that, letting $J_R:=[-R,R]$ and $J_R^c=\mathbb{R}\setminus J_R$,

$$|z|_{L^\infty(J_R^c)}<\varepsilon,\qquad |z|_{L^\nu(J_R^c)}<\varepsilon. \tag{4.8}$$

Then by $(\mathcal{R}_1)$,

$$\int_{J_R^c}R(t,z)\,dt<c\varepsilon,$$

by mean value theorem and $(\mathcal{R}_1)$

$$\bigg|\int_{J_R^c}(R(t,z_n^1+z)-R(t,z_n^1)\,dt\bigg|\le c\int_{J_R^c}|z|(|z_n^1|^{\nu-1}+|z|^{\nu-1})\,dt\le c\varepsilon.$$

Since $z_n^1\to 0$ in $L^p(J_R)$ $(p\ge\nu)$, we have

$$\bigg|\int_{J_R}(R(t,z_n)-R(t,z_n^1)-R(t,z))\,dt\bigg|\le\varepsilon,$$

for n large. Hence

$$\int_{\mathbb{R}}(R(t,z_n)-R(t,z_n^1)-R(t,z))dt\to 0\quad\text{as } n\to\infty.$$

(b) Let $\varphi\in E_\nu$ with $\|\varphi\|_\nu\le 1$. Using the equation (4.8) and $(\mathcal{R}_1)$, we deduce that

$$\bigg|\int_{J_R^c}R_z(t,z)\varphi\,dt\bigg|\le c|z|^{\nu-1}_{L^\nu(J_R^c)}\|\varphi\|_\nu^\nu\le c\varepsilon,$$

and, by $(\mathcal{R}_5)$,

$$\bigg|\int_{J_R^c}(R_z(t,z_n^1+z)-R_z(t,z_n^1))\varphi\,dt\bigg|$$
$$\le c\int_{J_R^c}|z|(|z_n^1|^{\nu-2}+|z|^{\nu-2}+|z|^{\nu-2}|z_n^1|)|\varphi|\,dt\le c\varepsilon.$$

That is,

$$\bigg|\int_{J_R^c}(R_z(t,z_n^1+z)-R_z(t,z_n^1)-R_z(t,z))\varphi\,dt\bigg|\le c\varepsilon.$$

On the other hand, since $z_n^1\to 0$ and $z_n\to z$ in $L^p(J_R)$ $(p\ge\nu)$,

$$\bigg|\int_{J_R}(R_z(t,z_n^1+z)-R_z(t,z_n^1)-R_z(t,z))\varphi\,dt\bigg|\le c\varepsilon, \tag{4.9}$$

for n large. So

$$\sup_{\|\varphi\|_\nu \le 1} \left| \int_{\mathbb{R}} (R_z(t, z_n^1 + z) - R_z(t, z_n^1) - R_z(t, z))\varphi \, dt \right| \to 0 \quad \text{as } n \to \infty.$$

Therefore, the conclusion (b) follows from

$$\Phi'(z_n^1)\varphi = \Phi'(z_n)\varphi + \int_{\mathbb{R}} (R_z(t, z_n^1 + z) - R_z(t, z_n^1) - R_z(t, z))\varphi \, dt$$

and (4.9). □

5. Infinite number of homoclinics

In this section we are going to show that if the function Φ is even, then (HS) has infinitely many geometrically distinct homoclinic orbits. Let $\mathcal{K} := \{z \in E_\nu : \Phi'(z) = 0\}$ and $\mathcal{F} := \mathcal{K}/\mathbb{Z}$, the set $\mathcal{F}$ consisting of arbitrarily chosen representative of the orbits of $\mathcal{K}$. By $(\mathcal{R}_6)$, we may assume that $\mathcal{F} = -\mathcal{F}$. In view of the invariance of Φ under the shift $*$,

$$\mathcal{O}(z_1) \neq \mathcal{O}(z_1) \quad \text{if } z_1, z_2 \in \mathcal{K} \text{ with } \Phi(z_1) \neq \Phi(z_2).$$

By virtue of $(\mathcal{R}_2)$,

$$\Phi(z) = \Phi(z) - \frac{1}{2}\Phi'(z)z = \int_{\mathbb{R}} \left(\frac{1}{2} R_z(t, z)z - R(t, z) \right) dt > 0,$$

for all $z \in \mathcal{F} \setminus \{0\}$. Theorem 1.1 will be proved by showing that $\mathcal{F}$ is an infinite set. That is, $\mathcal{K}$ is an infinite set. To this purpose, arguing by contradiction, we suppose that

$$(\mathcal{A}^*) \qquad \mathcal{F} \setminus \{0\} \text{ is a finite set.}$$

Then there are $\widehat{\alpha}, \widehat{\beta} > 0$ such that

$$\widehat{\alpha} < \min_{\mathcal{F}\setminus\{0\}} \Phi = \min_{\mathcal{K}\setminus\{0\}} \Phi \le \max_{\mathcal{F}\setminus\{0\}} \Phi = \max_{\mathcal{K}\setminus\{0\}} \Phi < \widehat{\beta}. \tag{5.1}$$

In the following we are going to apply Theorem 3.1 to Φ.

Definition 5.1. Let $\{z_n\} \subset E_\nu$ be a bounded sequence. Then, up to a subsequence, either

(a) there exist $\gamma > 0$, $R > 0$ and $y_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{y_n - R}^{y_n + R} |z_n|^2 \, dt \ge \gamma > 0,$$

or

(b) for all $0 < R < \infty$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |z_n|^2 \, dt = 0.$$

In the first case we say that $\{z_n\}$ is non-vanishing, and in the second case that it is vanishing (see [26]).

Lemma 5.2. *Let $a > 0$ and $\{z_n\} \subset H^{1/2}$ be bounded. If*

$$\sup_{y\in\mathbb{R}} \int_{B(y,a)} |z_n|^2 \to 0, \quad n \to \infty, \tag{5.2}$$

where $B(y,a)$ is the interval $(y-a, y+a)$, then $z_n \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$. Particularly, if $\{z_n\} \subset E^+$ is bounded and satisfies (5.2)*, then $z_n \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$.*

Proof. Usually, this lemma is stated for $z_n \subset H^1$ (see [30], [20]). However, a simple modification of the argument of Lemma 1.21 in [30] shows that the conclusion remains valid in $H^{1/2}$. Since the norms $\|\cdot\|_\nu$ and $\|\cdot\|_{H^{1/2}}$ are equivalent in E^+, one sees that the second conclusion follows. □

Lemma 5.3. *Suppose that $\mathcal{F}$ is a finite set, and the conditions of Theorem* 1.1 *are satisfied. Let $\{z_n\} \subset E_\nu$ be a* $(\mathrm{C})_c$*-sequence. Then either*

(a) *$z_n \to 0$ (corresponding to $c = 0$), or*
(b) *$c \geq \widehat{\alpha}$ and there exists a positive integer $\ell \leq [c/\widehat{\alpha}]$, points $\overline{z}_1, \dots, \overline{z}_\ell \in \mathcal{F}\setminus\{0\}$ (not necessarily distinct), a subsequence of denote again by $\{z_n\}$ and sequence $\{k_n^i\} \subset \mathbb{Z}$ $(i = 1, \dots, \ell)$ such that*

$$\bigg\| z_n - \sum_{i=1}^{\ell} k_n^i * \overline{z}_i \bigg\|_\nu \to 0, \quad |k_n^i - k_n^j| \to \infty \quad (i \neq j), \ \text{as } n \to \infty,$$

and

$$\sum_{i=1}^{\ell} \Phi(\overline{z}_i) = c.$$

Proof. From Lemma 4.2.1, we know that the sequence is bounded. It follows from (4.4) that

$$\begin{aligned}\Phi(z_n) - \frac{1}{2}\Phi'(z_n)z_n &= \int_{\mathbb{R}} \left(\frac{1}{2}R_z(t,z_n)z_n - R(t,z_n)\right) dt \\ &\geq \frac{\varepsilon}{2}\int_{|z_n|\geq R_0} |z_n|^\beta \, dt \geq 0.\end{aligned} \tag{5.3}$$

Thus $c \geq 0$. Moreover, we infer from

$$\Phi(z_n) = \frac{1}{2}\|z_n^+\|_E^2 - \frac{1}{2}\|z_n^-\|_E^2 - \int_{\mathbb{R}} R(t,z_n)\, dt \leq \|z_n\|_\nu^2$$

that $c = 0$ if $z_n \to 0$. Conversely, if $c = 0$, using the arguments as in the proof Lemma 4.2.1, one can easily get that

$$
\begin{aligned}
(5.4)\quad c|z_n|_\nu^\nu &\le o(1) + \|z_n^+\|_E^2 - \|z_n^-\|_E^2 \le o(1) + c\int_{|z_n|\ge R_0} |z_n|^\nu\,dt \\
&\le \Big(\int_{|z_n|\ge R_0} |z_n|^\beta\,dt\Big)^{t\nu}\Big(\int_{|z_n|\ge R_0} |z_n|^{(1-t\beta)\nu/(1-t\nu)}\,dt\Big)^{1-t\nu} + o(1) \\
&\le c\Big(\int_{|z_n|\ge R_0} |z_n|^\beta\,dt\Big)^{t\nu} + o(1),
\end{aligned}
$$

where $(1-t\beta)\nu/(1-t\nu) > \nu$. On the other hand, by (5.3), we know that $\int_{|z_n|\ge R_0} |z_n|^\beta\,dt \to 0$ as $n \to \infty$. Thus, it follows from (5.4) that $|z_n|_\nu \to 0$ as $n \to \infty$. Since $\Phi'(z_n)(1 + \|z_n\|_\nu) \to 0$, then

$$
\begin{aligned}
\|z_n^+\|_E^2 &= \int_{\mathbb{R}} R_z(t, z_n)z_n^+\,dt + o(1) \\
&\le c|z_n|_\nu^{\nu-1}|z_n^+|_\nu + o(1) \le c|z_n|_\nu^{\nu-1} + o(1) \to 0
\end{aligned}
$$

as $n \to \infty$. Furthermore, by (4.2), one has

$$c|z_n|_\nu^\nu + \|z_n^-\|_E^2 \le \|z_n^+\|_E^2 + o(1) \to 0$$

as $n \to \infty$. That is, $\|z_n\|_\nu \to 0$ as $n \to \infty$. It follows that $z_n \to 0$ if and only if $c = 0$.

If $c > 0$ and z_n^+ is vanishing, that is,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B(y,a)} |z_n^+|^2\,dt = 0.$$

Then, by Lemma 5.2, we have $z_n^+ \to 0$ in $L^t(\mathbb{R})$ for $t > 2$. Therefore, by $(\mathcal{R}_1)$ and Hölder inequality, one has

$$\left|\int_{\mathbb{R}} R_z(t, z_n)z_n^+\,dt\right| \le c\int_{\mathbb{R}} |z_n|^{\nu-1}|z_n^+|\,dt \le c|z_n^+|_\nu|z_n|_\nu^{\nu-1} \to 0.$$

Since $\Phi'(z_n)z_n^+ \to 0$ and $\Phi'(z_n)z_n^+ = \|z_n^+\|_E^2 - \int_{\mathbb{R}} R_z(t, z_n)z_n^+\,dt$, we know that $\|z_n^+\|_E \to 0$ and

$$\Phi(z_n) \le \|z_n^+\|_E \to 0,$$

a contradiction. Thus z_n^+ is non-vanishing, that is, there exist $\gamma > 0$, $\iota > 0$ and $\widehat{y}_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{\widehat{y}_n-\iota}^{\widehat{y}_n+\iota} |z_n^+|^2\,dt \ge \gamma > 0.$$

Hence we can find $k_n \in \mathbb{Z}$ such that, setting $u_n := k_n * z_n(t) = z_n(t + k_n)$,

$$(5.5)\qquad \lim_{n\to\infty} \int_{-\iota-1}^{\iota+1} |u_n^+|^2\,dt \ge \gamma > 0,$$

where $u_n^{\pm} := z_n^{\pm}(t + k_n)$. Since $\|z_n\|_\nu = \|u_n\|_\nu$ and $\Phi(z_n) = \Phi(u_n)$, then $\{u_n\}$ is still bounded, so a subsequence of $\{u_n\}$ (still denoted by the same symbol) converges weakly to some $z^1 \in E_\nu$. That is, there exists $\{k_n^1\} \subset \mathbb{Z}$ such that $u_n = k_n^1 * z_n(t) \rightharpoonup z^1$. By (5.5), we know that $z^1 \in \mathcal{K} \setminus \{0\}$. Let $\overline{z}_1$ be the representative in which z^1 lies, and let $k^1 \in \mathbb{Z}$ be such that $k^1 * z^1 = \overline{z}_1$. Set $\overline{k}_n^1 := k^1 + k_n^1$ and $z_n^1 := \overline{k}_n^1 * z_n - \overline{z}_1$. By $\mathbb{Z}$-invariance of Φ (i.e. $\Phi(\overline{k}_n^1 * z_n) = \Phi(z_n)$) and Lemma 4.2.2, $\{z_n^1\}$ is Cerami sequence at level $c - \Phi(\overline{z}_1)$. By (5.1), (5.3), $\widehat{\alpha} < \Phi(\overline{z}_1) \leq c$. There are two possibilities: $c = \Phi(\overline{z}_1)$ or $c > \Phi(\overline{z}_1)$.

If $c = \Phi(\overline{z}_1)$, repeating the arguments for the proof of the conclusion (a), we have that $z_n^1 \to 0$ in E_ν. Consequently, the conclusions of this lemma hold with $\ell = 1$ and $k_n^1 = -\overline{k}_n^1$.

If $c > \Phi(\overline{z}_1)$, then we argue again as in above with $\{z_n\}$ and c replaced by $\{z_n^1\}$ and $c - \Phi(\overline{z}_1)$, respectively, and obtain $\overline{z}_2 \in \mathcal{F}$ with $\widehat{\alpha} < \Phi(\overline{z}_2) \leq c - \Phi(\overline{z}_1)$. So, after at most $[c/\widehat{\alpha}]$ steps, we get the desired results. □

Given $\ell \in \mathbb{N}$ and a finite set $\mathcal{N} \subset E_\nu$, let

$$[\mathcal{N}, l] := \left\{ \sum_{n=1}^{j} (k_n * z_n) : 1 \leq j \leq \ell,\ k_n \in \mathbb{Z},\ z_n \in \mathcal{N} \right\}.$$

Lemma 5.4. *For any $\ell \in \mathbb{N}$,*

$$\inf\{\|z - z'\|_\nu : z, z' \in [\mathcal{N}, l],\ z \neq z'\} > 0. \tag{5.6}$$

The proof was given in Proposition 1.55 of [8] (see also [7]).

In view of Lemma 5.3, we have:

Corollary 5.5. *If $\{z_n\}$ is a $(\mathrm{C})_c$-sequence, $c \geq \widehat{\alpha}$, then one has*

$$\|z_n - [\mathcal{F}, \ell]\|_\nu \to 0$$

provided that $\ell \geq [c/\widehat{\alpha}]$.

Lemma 5.6. *Φ satisfies $(\mathcal{A}_3)$.*

Proof. Recall that $\mathcal{F}$ is a finite set. Since Φ' is odd, then we may assume $\mathcal{F}$ is symmetric. For any compact interval $I \subset (0, \infty)$, denote $I := [a, b]$, set $\ell = [b/\widehat{\alpha}]$ and take $\mathcal{B} = [\mathcal{F}, \ell]$. Then, $\mathcal{P}^+\mathcal{B} = [\mathcal{P}^+\mathcal{F}, \ell]$, where $\mathcal{P}^+$ stands for the projector onto E^+. By $(\mathcal{A}^*)$, $\mathcal{P}^+\mathcal{F}$ is finite set and

$$\|z\|_\nu \leq \ell \max\{\|\overline{z}\|_\nu,\ \overline{z} \in \mathcal{F}\} \quad \text{for all } z \in \mathcal{B},$$

which implies that $\mathcal{B}$ is bounded. In addition, By Corollary 5.5, $\mathcal{B}$ is a $(\mathrm{C})_I$-attractor, and by (5.5),

$$\begin{aligned} &\inf\{\|z^+ - v^+\|_\nu : z,\ v \in \mathcal{B},\ z^+ \neq v^+\} \\ &\qquad = \inf\{\|z' - v'\|_\nu : z',\ v' \in \mathcal{P}^+\mathcal{B},\ z' \neq v'\} > 0. \end{aligned}$$

For each $z \in \Phi_a^b$, one has

$$0 < a \le \Phi(z) = \frac{1}{2}\|z^+\|_E^2 - \frac{1}{2}\|z^-\|_E^2 - \int_{\mathbb{R}} R(t,z)\,dt.$$

Then

$$\frac{1}{2}\|z^-\|_E^2 + |z|_\nu^\nu \le \frac{1}{2}\|z^+\|_E^2.$$

It follows that $\|z\|_\nu \le \widetilde{\beta}\|z^+\|_E$ for some $\widetilde{\beta} > 0$. Then Φ satisfies $(\mathcal{A}_3)$ for $I = [a,b]$ and $a > 0$. □

Lemma 5.7. *Φ satisfies $(\mathcal{A}_0)$.*

Proof. Let $a \in \mathbb{R}$. Assume that $z_m \in \Phi_a$ with $z_m \to z$ in τ. Then $a \le (1/2)\|z_m^+\|_E^2 - ((1/2)\|z_m^-\|_E^2 + \Psi(z))$. Since $z_m^+ \to z^+$, then $\|z_m^+\|_E$ is bounded. It follows from $\|z_m^-\|_E^2 \le \|z_m^+\|_E^2 - 2a$ that $\|z_m^-\|_E$ is bounded. By $(\mathcal{R}_1)$ one see further that $|z_m|_\nu^\nu$ is bounded and so is $\|z_m\|_\nu$. Therefore, $z_m \rightharpoonup z$ in E_ν, which implies $z_m \to z$ in L^q_{loc} $(q \ge 2)$ and along a subsequence $z_m(t) \to z(t)$ for almost every $t \in \mathbb{R}$. Consequently, by the weakly semi-continuous of norm and Fatou's lemma we get $a \le \Phi(z)$. Now let $z_m \to z$ in τ(in Φ_a). Similar to above arguments shows that $\|z_m\|_\nu$ is bounded, and so $z_m \rightharpoonup z$ in E_ν. Then $z_m \to z$ in L^p_{loc} and $R_z(t, z_m) \to R_z(t,z)$ in $L^{p/(p-1)}_{\mathrm{loc}}$ $(p \ge 2)$. Hence $\Phi'(z_m)\psi \to \Phi'(z)\psi$ for $\psi \in E_\nu$. It follows that the condition of $(\mathcal{A}_0)$ is satisfied. □

Proof of Theorem 1.1. Assume that $\mathcal{F}$ is finite set, i.e., $(\mathcal{A}^*)$ holds. According to Lemmas 4.1.2–4.1.3 and Lemmas 5.6–5.7, we know that Φ satisfies the assumptions of Theorem 3.1. Therefore Φ possesses a sequence of critical values, $c_n \to \infty$, a contradiction. The proof is completed. □

Corollary 5.8. *Let $H(t,z)$ be the form of* (1.1). *Assume that $A = -(\mathcal{J}d/dt + B(t))$ satisfies the conditions of Remark* 1.2. *Then* (HS) *has infinitely many geometrically distinct homoclinic orbits.*

It follows from the Remark 1.2 and Theorem 1.1. □

6. Appendix

Recalling that $A = -(\mathcal{J}d/dt + B(t))$ is a self-adjoint operator in $\mathcal{H}$. By $(\mathcal{J}_1)$, we have $\mathcal{D}(|A|^{1/2}) = H^{1/2}$, where $|A|^{1/2}$ denotes the square root of $|A|$. In this Appendix, we mainly refer to the paper [16]. For reader's convenience, some of the results, together with the proofs, will be provided here. Set $W^{1,s} := W^{1,s}(\mathbb{R}, \mathbb{R}^{2N})$ for $s \ge 1$, $H^1 := W^{1,2}$ and $H^{1/2} := H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$. For a self-adjoint operator A in $\mathcal{H}$, we denote by $|A|$ its absolute value.

Definition 6.1. Let $S(t) \in C(\mathbb{R}; \mathbb{R}^{4N^2})$ be a symmetric matrix valued function, and let $F(t)$ be the fundamental matrix with $F(0) = I$ for the equation

$$\dot{x}(t) = \mathcal{J}S(t)x,$$

$S(t)$ is said to have an exponential dichotomy if there is a projector P and positive constants K, ξ such that

$$\begin{cases} |F(t)PF^{-1}(s)| \le Ke^{-\xi(t-s)} & \text{if } s \le t, \\ |F(t)(I-P)F^{-1}(s)| \le Ke^{-\xi(s-t)} & \text{if } s \ge t, \end{cases} \tag{6.1}$$

see [6].

Proposition 6.2. *Suppose that $S(t)$ has an exponential dichotomy and $s \ge 1$. Then the following conclusions hold:*

(a) *The operator*

$$B_s\colon L^s \supset W^{1,s} \to L^s, \qquad u \mapsto -\bigg(\mathcal{J}\frac{d}{dt} + S(t)\bigg)u,$$

has a bounded inverse B_s^{-1} satisfying with some $d = d(s,\sigma) > 0$

$$|B_s^{-1}z|_\sigma \le d|z|_s, \quad \text{for all } z \in L^s,$$

for all $\sigma \ge s$;

(b) *$B := B_2$ is s self-adjoint, and there are $b > 0$, $b_1 > 0$, $b_2 > 0$ such that $\sigma(B) \cap [-b, b] = \emptyset$ and*

$$b_1\|z\|_{H^1} \le |Bz|_2 \le b_2\|z\|_{H^1} \quad \text{for all } z \in H^1;$$

(c) *$\mathcal{D}(|B|^{1/2}) = H^{1/2}$, and there are d_1, $d_2 > 0$ such that*

$$d_1\|z\|_{H^{1/2}} \le ||B|^{1/2}z|_2 \le d_2\|z\|_{H^{1/2}} \quad \text{for all } z \in H^{1/2}.$$

Proof. For any $z \in L^s$, $s \ge 1$, there is a unique $u \in W^{1,s}$ satisfying

$$-\bigg(\mathcal{J}\frac{d}{dt} + S\bigg)u = z$$

given by

$$u(t) = \int_{-\infty}^t F(t)PF^{-1}(s)\mathcal{J}z\,ds - \int_t^\infty F(t)(I-P)F^{-1}(s)\mathcal{J}z\,ds.$$

Set

$$\lambda^+(s) = \lambda^-(-s) = \begin{cases} 1 & \text{if } s \ge 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Then

$$\begin{aligned} u(t) &= \int_{\mathbb{R}} F(t)PF^{-1}(s)\lambda^+(t-s)\mathcal{J}z\,ds \\ &\quad - \int_{\mathbb{R}} F(t)(I-P)F^{-1}(s)\lambda^-(t-s)\mathcal{J}z\,ds := u_1(t) + u_2(t), \end{aligned}$$

and by (6.1)

$$\begin{aligned}|u_1(t)| \ &\leq K\int_{\mathbb{R}} e^{-\xi(t-s)}\lambda^+(t-s)|z|\,ds,\\ |u_2(t)| \ &\leq K\int_{\mathbb{R}} e^{-\xi(s-t)}\lambda^-(t-s)|z|\,ds.\end{aligned}$$

Setting $f^+(\tau) = e^{-\xi\tau}\lambda^+(\tau)$ and $f^-(\tau) = e^{\xi\tau}\lambda^-(\tau)$, one has

$$|u_1(t)| \leq K(f^+ * |z|)(t) \quad \text{and} \quad |u_2(t)| \leq K(f^- * |z|)(t),$$

where $*$ denotes the convolution. Observe that

$$\int_{\mathbb{R}} |f^+|^\sigma = \int_{\mathbb{R}} |f^-|^\sigma = \frac{1}{\xi\sigma} \quad \text{for all } \sigma \geq 1 \text{ and } |f^\pm|_\infty = 1.$$

By the convolution inequality, for any $\vartheta \geq 1$ satisfying $1/\vartheta = 1/s + 1/\sigma - 1$,

$$|u_j|_\vartheta \leq K(\xi\sigma)^{-1/\sigma}|z|_s, \quad j = 1, 2$$

and for $1/s + 1/s' = 1$, $s > 1$

$$|u_j|_\infty \leq K(\xi s')^{-1/s'}|z|_s, \quad j = 1, 2.$$

and also

$$|u_j|_\infty \leq K|z|_1, \quad \text{if } s = 1,\ j = 1, 2.$$

Therefore,

$$|u|_\vartheta \leq K(\xi\sigma)^{-1/\sigma}|z|_s, \quad \vartheta, s, \sigma \geq 1 \quad \text{and} \quad \frac{1}{\vartheta} = \frac{1}{s} + \frac{1}{\sigma} - 1. \tag{6.2}$$

Now the conclusion (a) follows from equation (6.2).

It is easy to verify that $B = B_2$ is self-adjoint. Note that if there is a sequence of positive numbers $b_n \to 0$ such that $\sigma(B) \cap [-b_n, b_n] = \emptyset$, then there is a sequence $\{z_n\} \subset \mathcal{D}(A)$ with $|z_n|_2 = 1$ and $|Bz_n|_2 \to 0$, contradicting (6.2). That is, $0 \notin \sigma(B)$. The inequality of (b) is clear by (6.2).

We now verify (c). Let $\Gamma := -d^2/dt^2$. Then $\mathcal{D}(\Gamma) = H^2$. By an interpolation theory (see [16, p. 764] or [23, Section 2.5.2])

$$(\mathcal{D}(\Gamma^0), \mathcal{D}(\Gamma))_{\theta,2} = (\mathcal{H}, H^2)_{\theta,2} = H^{2\theta}, \quad 0 < \theta < 1.$$

On the other hand (see [16, p. 764]) or [23, Theorem 1.18.10])

$$(\mathcal{D}(\Gamma^0), \mathcal{D}(\Gamma))_{\theta,2} = \mathcal{D}(\Gamma^\theta).$$

Consequently,

$$\mathcal{D}(\Gamma^\theta) = H^{2\theta}$$

equipped with the norm

$$\|z\|^2_{\mathcal{D}(\Gamma^\theta)} = \int_0^\infty (1 + \lambda^{2\theta})\,d|E_\lambda z|_2^2 = |z|_2^2 + |\Gamma^\theta z|_2^2,$$

where $\{E_\lambda; -\infty < \lambda < \infty\}$ is the spectral family of Γ. In particular, let $\theta = 1/4$,

$$H^{1/2} = \mathcal{D}(\Gamma^{1/4}), \quad \|u\|^2_{H^{1/2}} \le |z|_2^2 + |\Gamma^{1/4} z|_2^2.$$

Since $|\Gamma^{1/2} z|_2 = |\dot{z}|_2 \le c_1 |Bz|_2$ for $z \in H^1$ by (b), one has $(\Gamma^{1/2} z, z)_2 \le c_2(|B|z, z)_2$ (see [8, Theorem 4.12]), and so $|\Gamma^{1/4} z|_2 \le c_2 ||B|^{1/2} z|_2$. Together with equation (6.2), it follows that the first inequality of (c) holds. Similarly, considering the operator $\widetilde{\Gamma} := d^2/dt^2 + 1$, one can check the second one of (c).□

Lemma 6.3. *Under the assumption of* $(\mathcal{J}_1)$, *we have*

$$c_1 \|z\|_{H^{1/2}} \le ||A|^{1/2} z|_2 + a|z|_2 \le c_2 \|z\|_{H^{1/2}} + 2a|z|_2, \quad \text{for } z \in H^{1/2},$$

where $c_i > 0$, $(i = 1, 2)$ *and* $a > 4 \sup_{t\in\mathbb{R}} |B(t)|$.

Proof. Now we consider the matrix $B_a := B(t) + a\widetilde{B}$, where $a > 0$, $B(t)$ satisfies $(\mathcal{J}_1)$ and $\widetilde{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Clearly $a\mathcal{J}\widetilde{B}$ has the eigenvalues $\lambda_1 = \ldots = \lambda_N = a$ and $\lambda_{N+1} = \ldots = \lambda_{2N} = -a$, and its fundamental matrix is $F_a = \exp\left(at \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. Therefore $a\widetilde{B}$ has an exponential dichotomy. By the roughness of the exponential dichotomy, for any

$$a > 4 \sup_{t\in\mathbb{R}} |B(t)|, \tag{6.3}$$

B_a also have an exponential dichotomy(see [6]). In (6.2), we fix an a. Consider the self-adjoint operator

$$A_a = -\left(\mathcal{J}\frac{d}{dt} + B_a\right) = A - a\widetilde{B}.$$

Since for $z \in \mathcal{D}(A)$

$$|A_a z|_2 = |(A - a\widetilde{B})z|_2 \le |Az|_2 + a|z|_2,$$

by Proposition 6.2,

$$c_1 \|z\|^2_{H^{1/2}} \le (|A_a|z, z)_2 \le (|A|z, z)_2 + a|z|_2^2 \le c_2 \|z\|^2_{H^{1/2}} + a|z|_2^2.$$

By Proposition III 8.12 of [17], we have

$$c_1 \|z\|_{H^{1/2}} \le ||A|^{1/2} z|_2 + a|z|_2 \le c_2 \|z\|_{H^{1/2}} + 2a|z|_2,$$

for all $z \in H^{1/2} = \mathcal{D}(|A|^{1/2})$, where $c_i > 0$, $(i = 1, 2)$. □

Theorem 6.4. *Suppose that* $(X, \|\cdot\|)$ *is a Banach space with* $X = X_1 \oplus X_2$, *where* X_1 *and* X_2 *are close subset. Set* $|||x||| := \|x_1\| + \|x_2\|$. *Then for* $x = x_1 + x_2 \in X$, $x_i \in X_i$ $(i = 1, 2)$ *we have that*

(a) $|||\cdot|||$ *and* $\|\cdot\|$ *are equivalent norms;*
(b) *The projector* $P\colon X \to X_1$ *is continuous.*

PROOF. (a) Since $|||\cdot|||$ is also a complete norm on the X, and for each $x \in X$

$$\|x\| \le c_1|||x|||,$$

where $c_1 > 0$. From Functional Analysis, we know that the result of (a) holds.

(b) For each $x = x_1 + x_2 \in X$, by (a)

$$\|Px\| = \|x_1\| \le c_2|||x||| \le c_3\|x\|,$$

where c_2, c_3 are positive constants. Since P is a linear operator, we know that the conclusion (b) of this lemma follows. $\square$

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