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EQUIVARIANT NIELSEN FIXED POINT THEORY

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Abstract. We provide an alternative approach to the equivariant Nielsen fixed point theory developed by P. Wong in [24] by associating an abstract simplicial complex to any G-map and defining two G-homotopy invariants that are lower bounds for the number of fixed points and orbits in the G-homotopy class of a given G-map in terms of this complex. We develop a relative equivariant Nielsen fixed point theory along the lines above and prove a minimality result for the Nielsen-type numbers introduced in this setting.

1. Introduction

Classical Nielsen fixed point theory is devoted to the study of the following basic question: Given a self-map of a compact polyhedron, $f: X \to X$, what is the least number of fixed points of φ, as φ ranges over all maps homotopic to f.

The theory provides:

(1) A lower bound (denoted by $N(f)$, and referred to as the Nielsen number of f) for the number of fixed points of any map homotopic to f.
(2) Conditions (be they on the map f, or on the polyhedron X, or on both) under which $N(f)$ is actually a *sharp* lower bound, i.e. there is a self-map $\varphi: X \to X$ homotopic to f with exactly $N(f)$ many fixed points.
(3) In certain situations a method for computing $N(f)$ using algebraic information at the fundamental group and (singular) homology level.

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Since the pioneering work done by J. Nielsen in the twenties and F. Wecken in the early forties (Nielsen) fixed point theorists have done a considerable amount of work refining the theory for use in certain special settings.

In this paper, we combine a relative theory as developed by H. Schirmer in the mid-eighties with a reworked version of the equivariant theory developed by P. Wong in [24]. While we were inspired by [24] there are substantial differences between our approach and his; we define our invariants in terms of a simplicial complex that can be associated to a G-map. One of the invariants we thus obtain is a correct lower bound for the number of fixed points in the G-homotopy class of a given map.

We only suppose some familiarity with Nielsen Fixed Point Theory. Excellent references for this are [4] and [14].

2. Preliminaries

Let us recall some definitions and basic facts of transformation groups that we will need later. Throughout G will denote a compact (not necessarily connected) Lie group.

Let X be a G-space. For any subgroup H of G, NH is the normalizer of H in G and $WH := NH/H$ is the Weyl group of H in G. We will write (H) for the set of conjugates of H in G and refer to it as the orbit type of H (if $H \lhd G$, this set contains the single element H).

If $x \in X$, then $G_x := \{g \in G : g \cdot x = x\}$ is the isotropy subgroup of x. For each subgroup H of G, the isotropy subspace of H is $X^H := \{x \in X : H \leq G_x\}$, $X_H := \{x \in X : H = G_x\}$ and likewise $X^{(H)} := \{x \in X : gHg^{-1} \leq G_x$ for some $g \in G\}$, $X_{(H)} := \{x \in X : G_x \in (H)\}$. An orbit type (H) is called an isotropy type of X if H (hence all its conjugates) appears as an isotropy subgroup of some x in X.

The set of all isotropy types of X will be denoted by $\mathrm{Iso}(X)$. If, for a given G-action, $Iso(X) = \{(\{e\})\}$ we say the action is free, if $\mathrm{Iso}(X) = \{(\{e\}), (G)\}$ we say the action is semi-free. $\mathrm{Iso}(X) = \{(H_i)\}_{i \in \Lambda}$ can be partially ordered by declaring $(H_i) \leq (H_j)$ if and only if there is some $g \in G$ such that $gH_ig^{-1} \leq H_j$. One may (clearly) arrange for the indexing to be such that $(H_j) \leq (H_i)$ implies $i \leq j$. With such an indexing, and by an abuse of language, $\mathrm{Iso}(X)$ is said to have an admissible ordering. Given an admissible ordering on $\mathrm{Iso}(X)$, one obtains the associated filtration of X by G-invariant subspaces $X_1 \subset \ldots \subset X_k = X$ (this filtration is finite for the types of G-spaces we shall consider – see Theorem 2.1 below) where $X_i := \{x \in X : (G_x) = (H_j)$ for some $j \leq i\}$ or equivalently, $X_i := \bigcup_{j \leq i} X^{(H_j)}$.

Given a G-space X, an equivariant map $f\colon X \to X$ is a map that satisfies $f(g\cdot x) = g\cdot f(x)$. Such a map must preserve the filtration of X mentioned above, i.e. $f(X_i) \subseteq X_i$ for $1 \leq i \leq k$.

A G-space X is a G-Euclidean Neighbourhood Retract (G-ENR) if and only if there exists a G-embedding, i, of X as an invariant subspace of an Euclidean G-space together with an open invariant neighbourhood that G-retracts onto $i(X)$. If there is no group acting, we simply talk about ENR's. We remark that an ENR endowed with a G-action *need not* be a G-ENR. Not by a longshot! There is a $\mathbb{Z}_2$-action on S^4 with a non-ENR (non-locally contractible) stationary set. (1) We will make use of the following characterization of G-ENRs:

Theorem 2.1 ([11, Theorem 2.1]). *Let X be a separable, finite dimensional metric G-space. Then X is a G-*ENR *if and only if X is locally compact, has a finite number of isotropy types, and for every isotropy subgroup $H \leq G$, X^H is an* ENR.

Jawarowski's theorem singles out the G-ENR's among the finite dimensional separable metric G-spaces as the most appropriate subclass for us to work with for two reasons. Firstly, we will require the existence of a fixed point index on each of the isotropy subspaces X^H; such an index exists if these subspaces are ENR's. Secondly, we will require that X have only finitely many isotropy types; had we chosen to work with G-ANR's (2) (for which the isotropy subspaces are ANR's and we would have had a fixed point index available) we would not have been able to conclude, in general, that X had finitely many isotropy types.

In order to address so-called "minimality" issues we will need to work with the following particularly pleasant types of G-spaces.

Suppose G is finite. A simplicial G-complex is a simplicial complex endowed with an action of G by simplicial homeomorphisms such that if $g \in G$ leaves a simplex invariant, then g fixes this simplex pointwise. A G-space (G is still finite) that admits an equivariant triangulation will also be called a simplicial G-complex.

A compact smooth (C^∞) G-manifold X is a compact smooth (C^∞) manifold endowed with an action of G by (C^∞) diffeomorphisms. It is well known that a (compact) smooth G-manifold X has a triangulable orbit space, X/G (see [10, pp. 488–489]). More importantly for us will be the fact that smooth G-manifolds have "fixed sets" X^H ($(H) \in \mathrm{Iso}(X)$) such that each connected component is a (smooth) submanifold.

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(1) We thank R. Edwards for pointing this out to us.
(2) See [1] for a definition.

3. Equivariant Nielsen fixed point theory

Definition 3.1. Let X be a connected ENR and $f\colon X \to X$ a self-map. Then $\widetilde{f}$ is said to be a lift of f to $\widetilde{X}$ (the universal cover of X) if the following diagram commutes

$$\begin{tikzcd} \widetilde{X} \arrow[r, "f"] \arrow[d, "P_X"'] & \widetilde{X} \arrow[d, "P_X"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

where P_X is the (universal) covering projection. Denote the set of all such lifts of f by LIFT(f). With $\widetilde{f}$ any fixed lift of f, one has: $\mathrm{LIFT}(f) = \{\alpha\widetilde{f} :$ α is a lift to the universal cover of $\mathrm{id}_X\}$. So, by standard covering space theory, LIFT(f) is in bijective correspondence with $\pi_1(X)$.This correspondence is not canonical.

Defintion 3.2. Let X be a connected ENR. One can define the following equivalence relation on LIFT(f): $\widetilde{f_1} \sim \widetilde{f_2}$ if $\widetilde{f_1} = \alpha\widetilde{f_2}\alpha^{-1}$ for some lift α of id_X. Denote the set of equivalence classes by $\mathrm{LIFT}_\sim(f)$.

In what follows we will make use of the following:

Lemma 3.3. *Let X, Y be connected* ENR*s. Suppose f, g, h are maps such that the diagram below commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Then, for a fixed lift $\widetilde{h}\colon \widetilde{X} \to \widetilde{Y}$ (of h), and a lift $\widetilde{f}$ (of f) there is a unique lift $\widetilde{g}$ (of g) such that

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[d, "\widetilde{h}"] \\ X \arrow[r, "\widetilde{g}"'] & X \end{tikzcd}$$

commutes. Thus one obtains a function $\psi\colon \mathrm{LIFT}(f) \to \mathrm{LIFT}(g)$, by setting $\psi(\widetilde{f}) = \widetilde{g}$.

Proof. Standard, see [26]. $\square$

Lemma 3.4. *The function ψ above is equivariant with respect to the actions of* $\mathrm{Cov}(\widetilde{X})$, $\mathrm{Cov}(\widetilde{Y})$ *on* LIFT(f), LIFT(g) *by conjugation. Thus it induces* *a function* $\overline{\psi}\colon \mathrm{LIFT}_{\sim}(f) \to \mathrm{LIFT}_{\sim}(g)$, *which is independent of the choice of lift* $\widetilde{h}$ *originally involved in its definition.*

Proof. Standard, see [26]. □

If h is a homeomorphism, then ψ and $\overline{\psi}$ are bijections.

Let X be a compact G-ENR, $(G_j) \in \mathrm{Iso}(X)$, $G'_j = gG_jg^{-1}$ and denote by $X_c^{G_j}$ a connected component of X^{G_j} that is mapped into itself by f. If $f\colon X \to X$ is a G-map, then the following diagram commutes:

$$\begin{array}{ccc} X_c^{G_j} & \xrightarrow{f|X_c^{G_j}} & X_c^{G_j} \\ {\scriptstyle g\cdot}\downarrow & & \downarrow{\scriptstyle g\cdot} \\ X_c^{G'_j} & \xrightarrow[f|X_c^{G'_j}]{} & X_c^{G'_j} \end{array}$$

($g\cdot$ is the homeomorphism from X^{G_j} to $X^{G'_j}$ given by $g\cdot(x) = g\cdot x$). By Lemma 3.4, there is a bijection $\overline{\phi_{g\cdot}}\colon \mathrm{LIFT}_{\sim}(f|X_c^{G_j}) \to \mathrm{LIFT}_{\sim}(f|X_c^{G'_j})$.

Now define:

$$\mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)}) := \bigcup_{X_c, G'_j} \mathrm{LIFT}_{\sim}^{G'_j}(f|X_c^{G'_j})$$

The functions $\{\overline{\phi_g}\}_{g\in G}$ give an action of G on $\mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)})$.

Remark 3.5. If Y is a connected ENR, $f\colon Y \to Y$ is a map, then

$$P_Y^*\colon \mathrm{LIFT}_{\sim}(f) \to Y$$

is given by $P_Y^*([\widetilde{f}]) = P_Y(\mathrm{Fix}(\widetilde{f}))$, where $P_Y\colon \widetilde{Y} \to Y$ is the universal covering projection. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{LIFT}_{\sim}(f|X_c^{G_j}) & \xrightarrow{\overline{\phi_{g\cdot}}} & \mathrm{LIFT}_{\sim}(f|X_c^{G'_j}) \\ {\scriptstyle P^*_{X_c^{G_j}}}\downarrow & & \downarrow{\scriptstyle P^*_{X_c^{G'_j}}} \\ X_c^{G_j} & \xrightarrow[g\cdot]{} & X_c^{G'_j} \end{array}$$

We are ready to define the notion of a (G_j)-fpc (*f*ixed *p*oint *c*lass) of $f^{(G_j)}$.

Definition 3.6. Let X be a compact G-ENR, $f\colon X \to X$ a G-map and suppose $(G_j) \in \mathrm{Iso}(X)$. We define the set of all (G_j)-fpcs of $f^{(G_j)}$, as follows:

$$\mathrm{FPC}_{(G_j)}(f^{(G_j)}) := \frac{\mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)})}{\sim}$$

the quotient space under the orbit equivalence relation induced by the G-action described above. If the (local) index $I(f|X_c^{G'_j}, F) \neq 0$ for any representative of an element of $\mathrm{FPC}_{(G_j)}(f^{(G_j)})$ (F is the projection onto $X_c^{G'_j}$ of the fixed point set of this representative as in Remark 3.5) we say the (G_j)-fpc is essential, otherwise it is inessential.

As usual in Nielsen Theory, there is an equivalent defintion in terms of paths: x, y are in the same (G_j)-fpc (or are "(G_j) equivalent" if one wants to think of the underlying equivalence relation on the fixed points of f on $X^{(G_j)}$) if and only if there is a path $\gamma\colon I \to X_c^{gG_jg^{-1}}$ (some $g \in G$) from x to $h \cdot y$ (some $h \in G$) with γ homotopic relative endpoints to $f \circ \gamma$ in $X_c^{gG_jg^{-1}}$.

Next we define what it means for a (G_k)-fpc of $f^{(G_k)}$ to "contain" a (G_j)-fpc of $f^{(G_j)}$, where $(G_j), (G_k) \in \mathrm{Iso}(X)$ and $(G_j) \geq (G_k)$. Suppose $G'_j \in (G_j)$, $G'_k \in (G_k)$ are such that $G'_j \geq G'_k$, and $X_c^{G'_j} \subset X_c^{G'_k}$. One has the following commutative diagram:

$$\begin{tikzcd}
X_c^{G'_j} \arrow[r, "f|X_c^{G'_j}"] \arrow[d, "i"'] & X_c^{G'_j} \arrow[d, "i"] \\
X_c^{G'_k} \arrow[r, "f|X_c^{G'_k}"'] & X_c^{G'_k}
\end{tikzcd}$$

where $i\colon X_c^{G'_j} \hookrightarrow X_c^{G'_k}$ is the inclusion. By Lemma 3.4, one can define a function

$$\tau^{(G_k)\leq(G_j)}\colon \mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)}) \to \mathrm{LIFT}_{\sim}^{(G_k)}(f^{(G_k)}).$$

Furthermore, $\tau^{(G_k)\leq(G_j)}$ is equivariant with respect to the actions of G on $\mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)})$ and $\mathrm{LIFT}_{\sim}^{(G_k)}(f^{(G_k)})$ described above. That is:

$$\begin{aligned}
\tau^{(G_k)\leq(G_j)}(g \cdot [\widetilde{f|X_c^{G'_j}}]) &= \tau^{(G_k)\leq(G_j)}([\widetilde{g}\cdot \widetilde{f|X_c^{G'_j}}\, \widetilde{g}\cdot^{-1}]) \\
&= [\widetilde{g}\cdot \widetilde{f|X_c^{G'_k}}\, \widetilde{g}\cdot^{-1}] = g \cdot \tau^{(G_k)\leq(G_j)}([\widetilde{f|X_c^{G'_j}}]).
\end{aligned}$$

With $\overline{\tau}^{(G_k)\leq(G_j)}$ denoting the map induced by $\tau^{(G_k)\leq(G_j)}$ on the orbit spaces, we arrive at the following important:

Definition 3.7. Let X be a compact G-ENR, $f\colon X \to X$ a G-map and suppose $(G_j), (G_k) \in \mathrm{Iso}(X)$ with $(G_k) \leq (G_j)$. Then define

$$\tau_{(G_k)\leq(G_j)}\colon \mathrm{FPC}_{(G_j)}(f^{(G_j)}) \to \mathrm{FPC}_{(G_k)}(f^{(G_k)})$$

by setting

$$\tau_{(G_k)\leq(G_j)}((G \cdot F, [\widetilde{f|X_c^{G_j}}]_G)) = (G \cdot F', \overline{\tau}^{(G_k)\leq(G_j)}([\widetilde{f|X_c^{G_j}}]_G)).$$

It follows that $G \cdot F \subseteq G \cdot F'$.

Let X be a compact G-ENR, and $f\colon X \to X$ a G-map. We associate an abstract simplicial complex, K_f to f. The vertices of K_f are the essential (G_i)-fpc's of $f^{(G_i)}$ where $(G_i) \in \mathrm{Iso}(X)$; we denote these vertices by $v_{\ell,f}$, $\ell \in \Gamma$ (some indexing set) for short. The simplexes of K_f are subsets (of $\mathrm{Vert}(K_f)$), $\sigma = \{v_{i,f}\}_{i\in\Lambda}$ where the $v_{i,f}$, $i \in \Lambda$ contain a *common* (G_j)-fpc of $f^{(G_j)}$. That is, there exists a *single* (G_j)-fpc, $(G\cdot F, [\widetilde{f|X_c^{G_j}}]_G)$, such that for any $v_{i,f} \in \sigma$ denoting an essential (G_{k_i})-fpc $(G\cdot F', [\widetilde{f|X_c^{G_{k_i}}}]_G)$, one has:

$$\tau_{(G_{k_i})\le(G_j)}((G\cdot F, [\widetilde{f|X_c^{G_j}}]_G)) = (G\cdot F', [\widetilde{f|X_c^{G_{k_i}}}]_G).$$

Remark 3.8. It may seem at first glance that one only needs to figure out what the essential (G_j)-fpcs are, as (G_j) runs over all the isotropy types of X, to determine K_f, since these are the vertices. This is far from the truth. To determine which subsets of vertices are simplices, one needs to consider also the *inessential* (G_j)-fpcs. For example, three vertices might form a simplex if they contain a *common inessential* (G_j)-fpc.

Definition 3.9. Let X be a compact G-ENR, $f\colon X \to X$ a G-map, and K_f its associated abstract simplicial complex. A set of simplexes of K_f is said to span K_f if every vertex of K_f is contained in at least one simplex of this set. We define

$$\mathrm{NO}_G(f) := \min\{|\varrho| : \varrho \text{ is a set of simplexes that spans } K_f\}.$$

To simplify notation in the proof below, we let $H^{G_j} := H|X_c^{G_j} \times I \to X_c^{G_j}$, where $H\colon X\times I \to X$ is a G-homotopy.

Theorem 3.10. *Let X be a compact G-ENR, $f\colon X \to X$ a G-map and suppose that $\varphi\colon X \to X$ is equivariantly homotopic to f. One has:*

(a) (G-Homotopy Invariance) $\mathrm{NO}_G(f) = \mathrm{NO}_G(\varphi)$.

(b) (Lower Bound) $\mathrm{NO}_G(f) \le |\{\theta \mid \theta$ *is an orbit of fixed points of* $\varphi\}|$.

Proof. For (a) we show that the abstract simplicial complexes K_φ and K_f are the same. To show that $|\mathrm{Vert}(K_f)| = |\mathrm{Vert}(K_\varphi)|$ we verify that a G-homotopy, $H\colon X\times I \to X$ between f and φ establishes a 1–1 correspondence between the essential (G_j)-fpc's of $f^{(G_j)}$ and those of $\varphi^{(G_j)}$. To this end, given a lift, $\widetilde{f|X_c^{G'_j}}$, there is a unique lift, $\widetilde{H^{G'_j}}$ with $\widetilde{H^{G_j}}(\,\cdot\,,0) = \widetilde{f|X_c^{G'_j}}$. If we associate $\widetilde{H^{G_j}}(\,\cdot\,,1)$ (a lift of $\varphi|X_c^{G'_j}$) to $\widetilde{f|X_c^{G'_j}}$ we obtain a function from the lifts of $f|X_c^{G'_j}$ to the lifts of $\varphi|X_c^{G'_j}$ which in turn yields an index-preserving bijection

$$\Phi^{(G_j)}\colon \mathrm{LIFT}_\sim^{(G_j)}(f^{(G_j)}) \to \mathrm{LIFT}_\sim^{(G_j)}(\varphi^{(G_j)})$$

(the index of $[\widetilde{f|X_c^{G'_j}}] \in \mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)})$ is understood to be $I(f|X_c^{G'_j}, F)$, where $F = P_{X_c^{G'_j}}(\mathrm{Fix}(\widetilde{f|X_c^{G'_j}})))$.

One can check the above bijection is equivariant with respect to the actions of G on $\mathrm{LIFT}_{\sim}^{(G_j)}(f^{(G_j)})$ and $\mathrm{LIFT}_{\sim}^{(G_j)}(\varphi^{(G_j)})$ and thus induces an index-preserving bijection.

$$\widetilde{\Phi}^{(G_i)}\colon \mathrm{FPC}_{(G_i)}(f^{(G_i)}) \to \mathrm{FPC}_{(G_i)}(\varphi^{(G_i)}).$$

Thus $|\mathrm{Vert}(K_f)| = |\mathrm{Vert}(K_\varphi)|$.

Moreover, for any $(G_j), (G_i) \in \mathrm{Iso}(X)$ with $(G_j) \le (G_i)$ one has, by the equivariance of H that Φ commutes with $\tau_{(G_j)\le(G_i)}$, so if $v_{\ell,f} \in \mathrm{FPC}_{(G_j)}(f^{(G_j)})$ contains a class $\alpha \in \mathrm{FPC}_{(G_i)}(f^{(G_i)})$, then $\Phi^{(G_j)}(v_{\ell,f}) \in \mathrm{FPC}_{(G_j)}(\varphi^{(G_j)})$ contains $\Phi^{(G_i)}(\alpha)$, so K_f is a subcomplex of K_φ. Reversing the direction of the argument we obtain that K_φ is a subcomplex of K_f and so $K_f = K_\varphi$

For (b), consider the function $\psi\colon \{\theta \mid \theta \text{ is an orbit of fixed points of } \varphi\} \to K_\varphi$ given by $\psi(\theta) = \{v_{i,\varphi}\}_{i\in\Lambda}$ the set of *all* essential fpcs that have (the (G_j)-fpc containing) θ in common. Then, since (for any $(G_j) \in \mathrm{Iso}(X)$) each essential (G_j)-fpc of φ must consist of at least one orbit of fixed points, $\mathrm{Im}(\psi)$ is a spanning set for $K_\varphi(= K_f)$ (if not, say $v_{\ell,\varphi} \notin \sigma$ for any $\sigma \in \mathrm{Im}(\psi)$, then $v_{\ell,\varphi}$ would be an essential (G_j)-fpc of $\varphi^{(G_j)}$ without any orbits of fixed points). Thus

$$|\{\theta \mid \theta \text{ is an orbit of fixed points of } \varphi\}| \ge |\mathrm{Im}(\psi)| \ge \mathrm{NO}_G(f)$$

as required. □

Example 3.11 (see [24, Example 3.9]). Let $X = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^2$ and $G = \mathbb{Z}_6 = \langle\alpha\rangle \times \langle\beta\rangle$ where $\mathbb{Z}_2 = \langle\alpha\rangle$, $\mathbb{Z}_3 = \langle\beta\rangle$. Suppose the G-action on X is given by:

$$\begin{aligned}\alpha \cdot (e^{i\theta_1}, \dots, e^{i\theta_5}, (x,y,z)) &= (e^{i\theta_2}, e^{i\theta_1}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, (x,y,-z)),\\ \beta \cdot (e^{i\theta_1}, \dots, e^{i\theta_5}, (x,y,z)) &= (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_5}, e^{i\theta_3}, e^{i\theta_4}, (x,y,z)).\end{aligned}$$

Let $f\colon X \to X$ be the G-map defined by

$$f((e^{i\theta_1}, \dots, e^{i\theta_5}, (x,y,z))) = (e^{i2\theta_2}, e^{i2\theta_1}, e^{i2\theta_3}, e^{i2\theta_4}, e^{i2\theta_5}, (x,-y,-z)).$$

Now, let Y denote the product of the first two factors and the last S^2, and let Z denote the product of the third, fourth and fifth factors. The G-action factors into a $\langle\beta\rangle$-action on Z and an $\langle\alpha\rangle$-action on Y, we also have that f factors i.e. $f = f_Y \times f_Z$. Clearly f_Z has a single fixed point of non-zero index $(1,1,1)$, so to obtain $\mathrm{NO}_G(f)$, it suffices to consider $\mathrm{NO}_{\langle\alpha\rangle}(f_Y)$. One has:

$$\mathrm{Fix}(f_Y) = \{(m, m^2, (n,0,0)) \mid m^3 = 1, n = 1, -1\}.$$

Now, $(1,1,(1,0,0))$ and $(1,1,(-1,0,0)$ constitute two distinct essential $(\langle\alpha\rangle)$-fpcs (which we denote by v_1 and v_2),

$$\{(e^{2\pi i/3}, e^{4\pi i/3}, (1,0,0)), (e^{2\pi i/3}, e^{4\pi i/3}, (-1,0,0)),\\ (e^{4\pi i/3}, e^{2\pi i/3}, (1,0,0)), (e^{4\pi i/3}, e^{2\pi i/3}, (-1,0,0))\}$$

constitutes a single essential $(\{e\})$-fpc (denoted by v_3), and

$$\{(1,1,(1,0,0)), (1,1,(-1,0,0)\}$$

denotes the other essential $(\{e\})$-fpc (denoted by v_4). So in this case, K_{f_Y} consists of an isolated vertex, v_3 and the two edges $\{v_1, v_4\}$ and $\{v_2, v_4\}$. So a minimal spanning set would have to consist of three simplices and hence $\mathrm{NO}_{\langle\alpha\rangle}(f_Y) = 3$. Thus also $\mathrm{NO}_G(f) = 3$.

Remark 3.12. Note that in [24], P. Wong states that there are 8 fixed points, while in fact there are only six, but this is only a minor error, since his description of the fixed point set is correct, and one can clearly see from this description that there are only six fixed points. We work with the defintion of (G_j)-fpcs in terms of paths given in Defintion 3.6, to compute K_{f_Y}.

Let us now turn our attention to the study of the least number of fixed *points* of maps in the G-homotopy class of a given G-map. First we define $N_G(f)$ in terms of K_f and a certain "weight function".

Let X be a compact G-ENR with G finite and $f\colon X \to X$ a G-map. Our definition of $N_G(f)$ is in terms of K_f. Given a simplex, σ, of K_f the set of (G_j)-fpc's ($(G_j) \in \mathrm{Iso}(X)$) that are "common to σ" (i.e. that are contained in all the $(\,\cdot\,)$-fpc's in σ) is given an ordering, $\preceq$, by declaring:

$$(G\cdot F, [\widetilde{f|X_c^{G_i}}]_G) \preceq (G\cdot F', [\widetilde{f|X_c^{G_j}}]_G) \text{ if and only if } [G:G_i] \le [G:G_j].$$

Define the following "weight" function on K_f: $\omega_f\colon K_f \to \mathbb{N}$ is given by $\omega_f(\sigma) = [G:G_i]$ if G_i is such that there is a minimal (G_i)-fpc common to σ.

Definition 3.13. Let X be a compact G-ENR, with G finite and $f\colon X \to X$ a G-map. Suppose the "weight" function ω_f, is as above. Then

$$N_G(f) := \min\left\{ \sum_{\sigma\in S_{K_f}} \omega_f(\sigma) \;\middle|\; S_{K_f} \text{ is a spanning set of simplexes of } K_f \right\}.$$

Theorem 3.14. *Let X be a compact G-ENR with G finite, $f\colon X \to X$ a G-map and suppose that φ is G-homotopic to f. Then:*

(a) (G-Homotopy Invariance) $N_G(f) = N_G(\varphi)$.
(b) (Lower Bound) $N_G(f) \le |\mathrm{Fix}(\varphi)|$.

Proof. For (a) we have already seen that $K_f = K_\varphi$, and by the proof of Theorem 3.9, if $\sigma_f \in K_f$ corresponds to $\sigma_\varphi \in K_\varphi$ a minimal (G_i)-fpc common to σ_f corresponds to a minimal (G_i)-fpc common to σ_φ, so $\omega_f = \omega_\varphi$.

For (b) we may assume φ is fix-finite. With ψ as in the proof of Theorem 3.9, and noting that $|\theta| \geq \omega_\varphi(\sigma)$ if $\psi(\theta) = \sigma$, one has (3):

$$|\mathrm{Fix}(\varphi)| = \sum_i |\theta_i| \geq \sum_{\sigma \in \mathrm{Im}(\psi)} \omega_\varphi(\sigma) \geq N_G(f). \qquad \square$$

Example 3.15. Let $f\colon X \to X$ be as in Example 3.11.

We compute $\sum_{\sigma \in S_{K_f}} \omega_f(\sigma)$ for the following spanning set of K_f:

$$S = \{\{v_{1,f}, v_{2,f}, v_{3,f}, v_{6,f}\}, \{v_{3,f}, v_{4,f}, v_{5,f}\}, \{v_{7,f}, v_{8,f}\}\}.$$

One has

$$\begin{aligned}\omega_2 = \sum_{\sigma \in S} \omega_f(\sigma) &= \omega_f(\{v_{1,f}, v_{2,f}, v_{3,f}, v_{6,f}\}) \\ &\quad + \omega_f(\{v_{3,f}, v_{4,f}, v_{5,f}\}) + \omega_f(\{v_{7,f}, v_{8,f}\}) = 1 + 1 + 2 = 4.\end{aligned}$$

As it turns out, $N_G(f) = 4$.

Remark 3.16. In some cases, there may be many different spanning sets for K_f, and it would seem, from the above example, that one really need only consider sums of the form $\sum_{\sigma \in S_{K_f}} \omega(\sigma)$ for S_{K_f} a *minimal* spanning set of K_f in determining $N_G(f)$; this would lead to errors.

4. Relative equivariant Nielsen fixed point theory

Definition 4.1. Let X be a G-ENR and $A \subset X$ a closed G-invariant subspace that is a G-ENR when endowed with the restriction of the G-action. We call (X, A) a G-ENR pair. If X (hence A) is compact, we call (X, A) a compact G-ENR pair. Given topological pairs (X, A) and (Y, B) a map of pairs $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with $f(A) \subset B$. Given two maps of pairs $f_i\colon (X, A) \to (Y, B)$, i=0,1, we say they are relative homotopic if there is a map of pairs $H\colon (X \times I, A \times I) \to (Y, B)$ with $h_0 = f_0$ and $h_1 = f_1$.

Suppose that (X, A) is a compact G-ENR pair, and $f\colon (X, A) \to (X, A)$ is a G-map of pairs. Let $(G_i) \in \mathrm{Iso}(A)$ and $\widetilde{A_c^{G_i}} \subseteq A \cap \widetilde{X_c^{G_i}}$. As usual, given a lift, $\widetilde{f|A_c^{G_i}}$ (of $f|A_c^{G_i}$), there is a unique lift $\widetilde{f|X_c^{G_i}}$ (of $f|X_c^{G_i}$) such that the diagram below commutes (or, so that $P_{A_c^{G_i}}(\mathrm{Fix}(\widetilde{f|A_c^{G_i}})) \subset P_{X_c^{G_i}}(\mathrm{Fix}(\widetilde{f|X_c^{G_i}})))$. $\widetilde{i}$ is a lift

(3) The summation on the left is over an indexing of the orbits of fixed points of φ.

of the inclusion $A_c^{G_i} \hookrightarrow X_c^{G_i}$.

$$\begin{tikzcd}
\widetilde{A_c^{G_i}} \arrow[r, "\widetilde{f|A_c^{G_i}}"] \arrow[d, "\tilde{i}"'] & \widetilde{A_c^{G_i}} \arrow[d, "\tilde{i}"] \\
\widetilde{X_c^{G_i}} \arrow[r, "\widetilde{f|X_c^{G_i}}"'] & \widetilde{X_c^{G_i}}
\end{tikzcd}$$

One defines

$$\tau^{A^{(G_i)} \subseteq X^{(G_i)}} \colon \mathrm{LIFT}_{\sim}^{(G_i)}((f|A)^{(G_i)}) \to \mathrm{LIFT}_{\sim}^{(G_i)}(f^{(G_i)})$$

by setting:

$$\tau^{A^{(G_i)} \subseteq X^{G_i)}}([\widetilde{f|A_c^{G_i}}]) = [\widetilde{f|X_c^{G_i}}]$$

if the above diagram commutes.

The function $\tau^{A^{(G_i)} \subseteq X^{(G_i)}}$ is well-defined and equivariant with respect to the actions of G on $\mathrm{LIFT}_{\sim}^{(G_i)}((f|A)^{(G_i)})$ and $\mathrm{LIFT}_{\sim}^{(G_i)}(f^{(G_i)})$, and so induces a function $\overline{\tau}^{A^{(G_i)} \subseteq X^{(G_i)}}$ on the orbit spaces.

We arrive at the following important:

Definition 4.2. Let (X, A) be a compact G-ENR pair, $f \colon (X, A) \to (X, A)$ a G-map of pairs and $(G_i) \in \mathrm{Iso}(A)$. Then define

$$\tau_{A^{(G_i)} \subseteq X^{(G_i)}} \colon \mathrm{FPC}_{(G_i)}((f|A)^{(G_i)}) \to \mathrm{FPC}_{(G_i)}(f^{(G_i)})$$

by setting

$$\tau_{A^{(G_i)} \subseteq X^{(G_i)}}((G \cdot F, [\widetilde{f|A_c^{G_i}}]_G)) := (G \cdot F', \overline{\tau}^{A^{(G_i)} \subseteq X^{(G_i)}}([\widetilde{f|A_c^{G_i}}]_G)).$$

It follows that $G \cdot F \subset G \cdot F'$.

We are ready to define the abstract simplicial complex $K_{f,f|A}$ associated to a G-map of pairs $f \colon (X, A) \to (X, A)$.

The vertices of $K_{f,f|A}$ are the essential (G_i)-fpc's of *either* $f|A^{(G_i)}$ or $f^{(G_i)}$, where $(G_i) \in \mathrm{Iso}(X)$. As for K_f, we denote these vertices by $v_{l,f}$, $l \in \Gamma$ for short. The simplexes of $K_{f,f|A}$ are subsets $\sigma = \{v_{i,f}\}_{i \in \Lambda}$ where the $v_{i,f}$, $i \in \Lambda$ contain a common (G_j)-fpc of *either* $f^{(G_j)}$ or $f|A^{(G_j)}$. That is, there is *either* a single (G_j)-fpc, $(G \cdot F,\ [\widetilde{f|X_c^{G_j}}]_G)$ of $f^{(G_j)}$ such that for any vertex $(G \cdot F, [\widetilde{f|X_c^{G_k}}]_G)$, in σ, $\tau_{(G_k) \leq (G_j)}((G \cdot F,\ [\widetilde{f|X_c^{G_j}}]_G)) = (G \cdot F', [\widetilde{f|X_c^{G_k}}]_G)$ *or* there is a single (G_j)-fpc $(G \cdot F'', [\widetilde{f|A_c^{G_j}}]_G)$ such that for any vertex in σ of the form $(G \cdot F', [\widetilde{f|X_c^{G_k}}]_G)$ or $(G \cdot F''', [\widetilde{f|A_c^{G_k}}]_G)$ one has either

$$\tau_{(G_k) \leq (G_j)}((G \cdot F, [\widetilde{f|A_c^{G_j}}]_G)) = (G \cdot F''', [\widetilde{f|A_c^{G_k}}]_G)$$

or

$$\tau_{(G_k)\leq(G_j)}(\tau_{A^{(G_j)}\subseteq X^{(G_j)}}((G\cdot F'',\widetilde{[f|A_c^{G_j}]_G}))) = (G\cdot F',\widetilde{[f|X_c^{G_k}]_G}).$$

Observe that $K_{f|A}$ and K_f are full subcomplexes of $K_{f,f|A}$.

Definition 4.3. Let (X,A) be a compact G-ENR pair and $f\colon(X,A)\to(X,A)$ a G-map of pairs, with associated simplicial complex $K_{f,f|A}$. Define:

$$\mathrm{NO}_G(f;X,A) := \min\{|\varrho| : \varrho \text{is a spanning set for } K_{f,f|A}\}.$$

Theorem 4.4. *Let (X,A) be a compact G-ENR pair, $f\colon(X,A)\to(X,A)$ a G-map of pairs and φ an arbitrary map that is relative equivariantly homotopic to f. Then:*

(a) (Relative Equivariant Homotopy Invariance)

$$\mathrm{NO}_G(f;X,A) = \mathrm{NO}_G(\varphi;X,A).$$

(b) (Lower Bound)

$$\mathrm{NO}_G(f;X,A) \leq |\{\theta \mid \theta \text{ is an orbit of fixed points of } \varphi\}|.$$

Proof. A relative equivariant homotopy establishes a 1–1 correspondence between the essential (G_j)-fpc's of $f|A^{(G_j)}$, resp. $f|X^{(G_j)}$, and those of $\varphi|A^{(G_j)}$, resp. $\varphi|X^{(G_j)}$. Thus $|\mathrm{Vert}(K_{f,f|A})| = |\mathrm{Vert}(K_{\varphi,\varphi|A})|$.

Next, we verify that with $v_{i,f}$ corresponding to $v_{i,\varphi}$, $i\in\Lambda$ under a relative equivariant homotopy (between f and φ) if $\sigma_f=\{v_{i,f}\}_{i\in\Lambda}$ is a simplex of $K_{f,f|A}$, then $\sigma_\varphi=\{v_{i,\varphi}\}_{i\in\Lambda}$ is a simplex of $K_{\varphi,\varphi|A}$. If either $v_{i,f}$, $i\in\Lambda$ are *all* $((G_j))$-fpc's of $f|A^{(G_j)}$ (different $(G_j)\in\mathrm{Iso}(X)$) or of $f^{(G_j)}$ then the preceding statement is true by the proof of Theorem 3.10. If some $v_{i,f}$'s are (essential) (G_j)-fpc's of $f|A^{(G_j)}$ and others are (essential) (G_j)-fpc's of $f^{(G_j)}$ then we need to verify commutativity of the following diagram.

$$\begin{array}{ccc}
\mathrm{FPC}_{(G_j)}(f|A^{(G_j)}) & \xrightarrow{\widetilde{\Phi}^{(G_j)}} & \mathrm{FPC}_{(G_j)}(\varphi|A^{(G_j)}) \\
{\scriptstyle \tau_{A^{(G_j)}\subseteq X^{(G_j)}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau_{A^{(G_j)}\subseteq X^{(G_j)}}} \\
\mathrm{FPC}_{(G_j)}(f^{(G_j)}) & \xrightarrow{\widetilde{\Phi}^{(G_j)}} & \mathrm{FPC}_{(G_j)}(\varphi^{(G_j)})
\end{array}$$

We leave the proof to the reader.

(b) This verification is entirely analogous to that of the corresponding inequality in Theorem 3.10. □

DEFINITION 4.5. Let (X,A) be a compact G-ENR pair with G finite, a map $f\colon (X,A) \to (X,A)$ be a G-map with associated abstract simplicial complex $K_{f,f|A}$ and "weight" function, $\omega_{f,f|A}$. Then

$$N_G(f;X,A) := \min\left\{ \sum_{\sigma\in S_{K_{f,f|A}}} \omega_{f,f|A}(\sigma) \;\middle|\; S_{K_{f,f|A}} \text{ is a spanning set of simplexes of } K_{f,f|A} \right\}.$$

THEOREM 4.6. *Let (X,A) be a compact G-ENR pair with G finite, $f\colon (X,A) \to (X,A)$ a G-map and suppose φ is relative equivariantly homotopic to f. Then:*

(a) (Relative Equivariant Homotopy Invariance)

$$N_G(f;X,A) = N_G(\varphi;X,A).$$

(b) (Lower Bound) $N_G(f;X,A) \le |\mathrm{Fix}(\varphi)|$.

PROOF. The proof is essentially the same as that of Theorem 3.13. □

5. Minimality

In order to prove our minimality theorem, we will need a "geometric characterization" of what it means for several essential (G_k)-fpc's (of a G-map $f\colon X \to X$) to contain a common (G_j)-fpc.

LEMMA 5.1. *Let X be a compact smooth G-manifold with G finite and $f\colon X \to X$ a G-map with associated abstract simplicial complex, K_f. Suppose $\sigma = \{v_{i,f}\}_{i\in\Lambda}$ is a simplex of K_f with orbits of fixed points θ_i in $v_{i,f}$ $(i \in \Lambda)$. Let us denote by $((G\cdot F, [\widetilde{f|X_c^{G_j}}]_G))$ the (G_j)-fpc that is common to σ. Then, there is an orbit $\theta \subseteq X^{(G_j)}$, and $x_i \in \theta_i$ such that there exist paths $\alpha_{x_i}\colon I \to X_c^{G_{k_i}}$ from x_i to some $y \in \theta$, together with homotopies $H_{x_i}\colon I\times I \to X_c^{G_{k_i}}$ with $(h_{x_i})_t(0) = x_i$, $(h_{x_i})_0(t) = \alpha_{x_i}(t)$, $(h_{x_i})_1(t) = f\circ\alpha_{x_i}(t)$ and $(h_{x_i})_t(1) \subseteq X_c^{G_{j'}} \subseteq X_c^{G_{k_i}}$, where $G_{j'} \in (G_j)$. If $F \ne \emptyset$ (where $F = p(\mathrm{Fix}(\widetilde{f|X_c^{G_j}}))$, then one can "take" $\theta \subseteq G\cdot F$, and $(h_{x_i})_t(1) = y$ for all $t \in I$ (i.e. $\alpha_{x_i} \sim f\circ\alpha_{x_i}$ relative endpoints).*

PROOF. This follows from the definitions and from the arguments in ([26, Theorem 2.3]). □

We are now ready to prove our minimality result. From here on, we assume X is a compact, connected smooth G-manifold, where G is finite, the action is semi-free, the dimension of each connected component of X^G is at least three and the codimension in X of each such connected component is at least two. This last requirement ensures that X^G can be bypassed in X as defined below.

Definition 5.2. Let (X, A) be a topological pair, with both X and $X - A$ path-connected. We say that A can be by-passed in X if the homomorphism (induced by the inclusion map $i: X - A \hookrightarrow X$)

$$i_*: \pi_1(X - A, \cdot\,) \to \pi_1(X, \cdot\,)$$

is surjective. If A can be bypassed in X, then any path in X with endpoints in $X - A$ is homotopic (relative endpoints) to a path in $X - A$.

Lemma 5.3. *Let X be a compact, connected smooth G-manifold as described above. Suppose $f: X \to X$ is a fix-finite map with $\theta \subseteq X_{\{e\}}$ an orbit of fixed points belonging to an essential (e)-fpc that contains only empty (G)-fpcs of f. Then there is an equivariant homotopy between f and φ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(f) - \theta \cup \{z\}$ where $z \in X^G - \mathrm{Fix}(f)$.*

Proof. Let $z \in X^G - \mathrm{Fix}(f)$ be, without loss of generality, a vertex, let $\alpha: I \to X^G$ be a PL arc homotopic (in X^G) to $\gamma(t) = H(1,t)$ (the homotopy given by Lemma 4.7) with $\alpha(0) = z$, $\alpha(1) = f(z)$, and let V_1 be a small open invariant neighbourhood of z such that $f(\mathrm{Cl}(V_1)) \cap \mathrm{Cl}(V_1) = \emptyset$ and $V_1 \cap \alpha(I)$ is a line segment. Next let $\varepsilon > 0$ be such that, if $V_{2,t} := \{y \in X^G : d(y,z) \le \varepsilon\}$, then $V_{2,t}$ $(0 \le t \le 1)$ is a conic neighbourhood of z in X^G with $V_{2,1} \subset V_1 \cap X^G$.

Subdivide (if necessary) to obtain a subcomplex $K_1 \subset V_1 \cap X^G$ with $V_{2,1} \subset \mathrm{int}_{X^G} K_1$ and define $H': K_1 \times I \to X^G$ as follows

$$H'(x,t) := \begin{cases} f(x) & \text{if } x \notin V_{2,t}, \\ f\Big(\Big(\dfrac{2}{\varepsilon t} d(x,z) - 1\Big)x + \Big(2 - \dfrac{2}{\varepsilon t} d(x,z)\Big)z\Big) & \\ & \text{if } 0 < \dfrac{\varepsilon t}{2} < d(x,z) \le \varepsilon t, \\ \alpha_g\Big(1 - t + \dfrac{2}{\varepsilon} d(x,z)\Big) & \text{if } 0 \le d(x,z) \le \dfrac{\varepsilon t}{2}. \end{cases}$$

Now, choose any $x_1 \in \theta$ and let $\beta: I \to X$ be a normal(except for the fact that $\beta(1)$ is a vertex) PL arc homotopic to $H(t,0)$ satisfying properties (α) and (β) just like γ in [2, Lemma 4.10] with $\beta(0) = x_1$, $\beta(1) = z$, $\beta([0,1)) \subset X_{\{e\}}$, and such that for any $g \ne e$, $g \cdot \beta(I) \cap \beta(I) = \{z\}$.

We next carefully equivariantly extend the homotopy, H', to all of X so that for the extension, $\overline{H'}$, one has $\mathrm{Fix}(\overline{h_1'}) = \mathrm{Fix}(\overline{h_0'}) \cup \{z\}$ and $\overline{h_1'} \circ \beta \sim \beta$ (relative endpoints).

Now, there is a retraction $r_1: V_3 \to (K_1 \times I) \cup (X^G \times \{0\})$ with

$$r_1(V_3 - (\mathrm{int}_{X^G} K_1 \times I)) \subset (\mathrm{Fr}_{X^G} K_1 \times I) \cup ((X^G - \mathrm{int}_{X^G} K_1) \times \{0\})$$

where V_3 is an open neighbourhood of $(K_1 \times I) \cup (X^G \times \{0\})$ in $X^G \times I$. Let U_1 be a neighbourhood of K_1 in X^G with

$$U_1 \times I \subset V_3, \quad r_1((U_1 - \mathrm{int}_{X^G} K_1) \times I) \subset (\mathrm{Fr}_{X^G} K_1 \times I) \cup (V_1 \times \{0\}).$$

Subdivide, if necessary, to obtain a subcomplex $K_2 \subset U_1$ with $K_1 \subset \mathrm{int}_{X^G} K_2$. There is also an equivariant retraction:

$$r_2 \colon V_4 \to \Big(\Big(K_2 \cup \bigcup_{g \in G} g \cdot \beta(I) \Big) \times I \Big) \cup (X \times \{0\})$$

where V_4 is an open invariant neighbourhood of of $((K_2 \cup \bigcup_{g \in G} g \cdot \beta(I)) \times I) \cup (X \times \{0\})$ in $X \times I$.

Next, let $1 \geq \varepsilon_1 > 0$ be the largest number such that $\beta((1-\varepsilon_1, 1]) \subset V_1$ and set $L = \beta((1-\varepsilon_1, 1])$.

Let U_2 be an open invariant neighbourhood of $K_2 \cup \bigcup_{g \in G} g \cdot L$ with $U_2 \subset V_1$, $U_2 \times I \subset V_4$ and

$$r_2(U_2 \times I) \subset \Big(\Big(K_2 \cup \bigcup_{g \in G} g \cdot L \Big) \times I \Big) \cup (V_1 \times \{0\}).$$

Next let $r_{3g} \colon (g \cdot L) \times I \to (\{z\} \times I) \cup ((g \cdot L) \times \{0\})$ be the obvious retractions.

Now let $U_3 \subset U_2 (\subset V_1)$ be an open invariant neighbourhood of $K_1 \cup \bigcup_{g \in G} g \cdot L$ with $U_3 \cap X^G \subset \mathrm{int}_{X^G} K_2$. Finally, we are able to carefully extend H' to $X \times I$ by means of a painstakingly constructed retraction. Let $R \colon (U_3 \times I) \cup (X \times \{0\}) \to (K_1 \times I) \cup (X \times \{0\})$ be given by:

$$R(x,t) := \begin{cases} r_{3g} \circ r_2(x,t) & \text{if } (x,t) \in r_2^{-1}(g \cdot L \times I), \\ r_1 \circ r_2(x,t) & \text{if } (x,t) \in r_2^{-1}(K_2 \times I), \\ r_2(x,t) & \text{if } (x,t) \in r_2^{-1}(X \times \{0\}). \end{cases}$$

Finally, let $u \colon X \to I$ be an equivariant map with

$$u(x) := \begin{cases} 1 & \text{if } x \in K_1, \\ 0 & \text{if } x \in X - U_3. \end{cases}$$

Then $\overline{H'}(x,t) = H' \circ R(x, u(x)t)$ is the desired extension.

One now proceeds to equivariant "move" $g \cdot x_1$ along $g \cdot \beta(I)$ until they are in a maximal simplex containing z as a vertex. Finally one equivariantly "coalesces" these fixed points with z as in [2, Theorem 4.13]. $\square$

Theorem 5.4. *Let X be a compact, connected, smooth G-manifold where G is finite, the action is semi-free, the dimension of each connected component of X^G is at least three and the codimension in X of each such component is at least two, and let $f \colon X \to X$ be a G-map. Then, there is an equivariant homotopy* *between f and φ where* $|\mathrm{Fix}(\varphi)| = N_G(f)$ *and* $|\{\theta \mid \theta$ *is an orbit of fixed points of* $\varphi\}| = \mathrm{NO}_G(f)$.

Proof. By [21] we obtain a partial homotopy $H\colon X \times \{0\} \cup X^G \times I \to X$ between f^G and a map that has a single fixed point in each essential (G)-fpc and has empty inessential (G)-fpcs. We extend this homotopy to an equivariant homotopy H defined on all of $X \times I$. Then, by the proofs of [2, Proposition 4.3, Lemma 4.10, Theorem 4.13], we obtain an equivariant homotopy (rel. X_G) between h_1 and a G-map that has a single orbit of fixed points in each essential (G)-fpc or $(\{e\})$-fpc, and, as usual, has empty inessential (G)-fpc and $(\{e\})$-fpcs. Finally, we apply Lemma 5.3 repeatedly (if necessary) to replace orbits of fixed points of this map in $X_{\{e\}}$ belonging to essential $(\{e\})$-fpcs with single fixed points in X_G belonging to fpcs they contain. The G-map φ, we thus obtain, clearly has the properties stated in the theorem. □

As a concluding remark, the proof above can be used to establish a similar result for equivariant self-maps of suitable finite simplicial complexes endowed with semi-free (finite) simplicial group actions. It is also tedious to generalize to the case of not necessarily semi-free group actions, but one would have to impose a series of conditions on the fixed point sets (of the group action) similar to the by-passing condition detailed above.

References

[1] K. Borsuk, *Theory of Retracts*, Warszawa, 1967.

[2] J. Better, *Nielsen fixed point theory for G-compactly fixed maps of pairs*, J. Geometry Topology **4** (2004), 269–293.

[3] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[4] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott–Foresman, 1971.

[5] ———, *Wecken properties for manifolds*, Contemp. Math. **152** (1993), 9–21.

[6] A. Dold, *Lectures on Algebraic Topology*, Springer–Verlag, Heidelberg, 1972.

[7] D. L. Ferrario, *A fixed point index for equivariant maps*, Topol. Methods Nonlinear Anal. **13** (1999), 313–340.

[8] E. Fadell and P. Wong, *On deforming G-maps to be fixed point free*, Pacific J. Math. **132** (1988), 272–281.

[9] J. Guo and P. Heath, *Equivariant coincidence Nielsen numbers* (to appear).

[10] S. Illman, *The equivariant triangulation theorem for actions of compact Lie groups*, Math. Ann. **262** (1983), 487–501.

[11] J. W. Jawarowski, *Extensions of G-maps and Euclidean G-retracts*, Math. Z. **146** (1976), 143–148.

[12] B. Jiang, *On the least number of fixed points*, Amer. J. Math. **102** (1980), 749–763.

[13] ———, *Fixed point classes from a differential viewpoint*, Lecture Notes, vol. 886, Springer–Verlag, 1981, pp. 163–170.

[14] ———, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math., vol. 14, Amer. Math. Soc., Providence, 1983.

[15] ———, *Fixed points and braids*, Invent. Math. **75** (1984), 69–74.

[16] H. SCHIRMER, *A relative Nielsen number*, Pacific J. Math. **122** (1986), 459–473.

[17] ———, *A survey of relative Nielsen fixed point theory*, Contemp. Math. **152** (1993), 291–309.

[18] E. SPANIER, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[19] N. STEENROD, *The Topology of Fibre Bundles*, Princeton Univ. Press, 1951.

[20] T. TOM DIECK, *Transformation Groups*, Walter de Gruyter, Berlin, New York, 1987.

[21] F. WECKEN, *Fixpunktclassen* I, Math. Ann. **117** (1941), 659–671; II **118** (1942), 216–234; III **118** (1942), 544–577.

[22] D. WILCZYŃSKI, *Fixed point free equivariant homotopy classes*, Fund. Math. **123** (1984), 47–60.

[23] P. WONG, *Equivariant Nielsen fixed point theory for G-maps*, Pacific J. Math. **150** (1991), 179–200.

[24] ———, *Equivariant Nielsen numbers*, Pacific J. Math. **159** (1993), 153–175.

[25] ———, *On the location of fixed points of G-deformations*, Topology Appl. **39** (199), 159–165.

[26] X. Z. ZHAO, *A relative Nielsen number for the complement*, Topological Fixed Point Theory and Applications (Proceedings, Tianjin 1988), Lecture Notes in Math., vol. 1411 Springer–Verlag, 1989, pp. 189–199.

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