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A PRIORI BOUNDS VIA THE RELATIVE MORSE INDEX OF SOLUTIONS OF AN ELLIPTIC SYSTEM

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Abstract. We prove a Liouville-type theorem for entire solutions of the elliptic system $-\Delta u = |v|^{q-2}v$, $-\Delta v = |u|^{p-2}u$ having finite relative Morse index in the sense of Abbondandolo. Here, $p, q > 2$ and $1/p + 1/q > (N-2)/N$. In particular, this yields a result on a priori bounds in $L^\infty \times L^\infty$ for solutions of superlinear elliptic systems obtained by means of min-max theorems, for both Dirichlet and Neumann boundary conditions.

1. Introduction

A celebrated result of A. Bahri and P. L. Lions [8] states that if $u \in C^2(\mathbb{R}^N)$ satisfies

$$-\Delta u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

with $2 < p < 2^* := 2N/(N-2)$ $(N \geq 3)$ and if u has *finite index* then $u \equiv 0$; the latter assumption means that there exists $R_0 > 0$ such that

$$\int |\nabla \varphi|^2 - (p-1) \int |u|^{p-2}\varphi^2 \geq 0, \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^N \setminus B_{R_0}(0)). \tag{1.2}$$

(Actually, in [8] it is assumed furthermore that $||u||_\infty < \infty$ but this restriction can be removed, as an inspection of its proof shows.) We observe that the

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left-hand member in (1.2) corresponds, formally, to the second derivative of the energy functional evaluated at the solution u, in the direction φ.

This type of results is known to be useful in obtaining a priori bounds for solutions of equations such as

$$-\Delta u = f(u), \quad u \in H_0^1(\Omega), \tag{1.3}$$

whenever, say, $\lim_{|s|\to\infty} f'(s)/|s|^{p-2} = \ell > 0$, since (1.1) can be seen as a limit problem of (1.3) in situations where rescalement arguments are involved; solutions of (1.3) are often constructed by means of critical point theory applied to the associated energy functional, so that the "limit property" (1.2) is expected to be a consequence of abstract results providing estimates on the Morse index of these solutions, such as the ones in e.g. [16], [22], [27], [31]. As an example, we mention that the main result in [28] strongly relies on this argument, as the authors deal with a situation where no relevant energy estimates seem to be available.

The result in [8] was later extended in several directions. In [15], [28] the authors deal with sign-changing nonlinearities of the form $f(x,s) = a(x)|s|^{p-2}s$, in [18], [19] non-homogeneous nonlinearities such as $f(s) = A(s^+)^{p-1} - B(s^-)^{q-1}$ with $2 < p$, $q < 2^*$ are considered, while the biharmonic operator Δ^2 is treated in [26]. Also, in [25], [35] it is pointed out that in fact a priori bounds for (1.3) may be obtained without relying on blow-up arguments; in [35] connexions between the Morse index and the Hausdorff measure of the nodal sets of the solutions are also displayed.

A natural extension of problem (1.1) consists in studying strongly coupled elliptic systems such as

$$-\Delta u = |v|^{q-2}v, \quad -\Delta v = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.4}$$

Here we assume $p, q > 2$ (we recall that the case of the biharmonic operator was studied in [26]) and also that p and q are subcritical in the sense of [13], [14], [20], namely that

$$\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}\cdot$$

Extending results (1.1)–(1.4) may constitute a difficult task. In connexion to our subject, we recall that a classical result [17] states that if $p < 2^*$ then (1.1) admits no positive solutions, while a corresponding statement to system (1.4) is still to be fully proved (see e.g. [24], [32] for recent developments). Also, an uniqueness result for positive solutions of $-\Delta u + u = u^{p-1}$ is known [21], whereas a corresponding one for elliptic systems seems not to have been proved.

Now, given a solution (u,v) of a system such as the one in (1.4) (satisfying some boundary conditions on, say, a bounded smooth domain), its Morse index can be defined by different methods. Let us mention here the finite dimensional reduction in [12], the relative Morse index introduced in [1] in terms of a notion of relative dimension, and also the Morse index relying on the so called spectral flow [4], [7] and the cohomological approaches in [6], [33]; we refer the reader to the books [2], [11] for an account of the theory as well as some applications.

In particular, in [7] a remarkable Liouville-type theorem extending Bahri–Lions's result [8] is proved, yielding in particular a priori bounds in $L^\infty(\Omega) \times L^\infty(\Omega)$ for superlinear and subcritical elliptic problems $-\Delta u = g(v)$, $-\Delta v = f(u)$ in Ω, $u = v = 0$ on $\partial\Omega$, for solutions having uniformly bounded Morse index in the sense of [7].

Here we aim to prove a similar conclusion with respect to the relative Morse index in [1], [5]. More precisely, our main result goes as follows.

Theorem 1.1. *Let $u, v \in C^2(\mathbb{R}^N)$ satisfy (1.4) with $0 < ||u||_\infty < \infty$, $p, q > 2$ and $1/p+1/q > (N-2)/N$. Then, for every $k \in \mathbb{N}$ there exist $\lambda = \lambda(u, v, k) \in \mathbb{R}^+$ and a subspace $X \subset \{(\lambda\phi, \phi), \phi \in \mathcal{D}(\mathbb{R}^N)\}$ with $\dim X = k$ such that*

$$I''(u, v)(\alpha + \phi, \beta - \lambda\phi)(\alpha + \phi, \beta - \lambda\phi) < 0 \tag{1.5}$$

for every $\phi \in \mathcal{D}(\mathbb{R}^N)$ and every $(\alpha, \beta) \in X$ such that $(\alpha + \phi, \beta - \lambda\phi) \neq (0, 0)$.

Here $I(u, v)$ stands (formally) for the energy functional

$$I(u, v) = \int_{\mathbb{R}^N} \left(\langle \nabla u, \nabla v \rangle - \frac{1}{p}|u|^p - \frac{1}{q}|v|^q \right),$$

and so, for $\varphi, \psi \in \mathcal{D}(\mathbb{R}^N)$, the expression in (1.5) is precisely given by

$$I''(u, v)(\varphi, \psi)(\varphi, \psi) = \int_{\mathbb{R}^N} (2\langle \nabla\varphi, \nabla\psi \rangle - (p-1)|u|^{p-2}\varphi^2 - (q-1)|v|^{q-2}\psi^2).$$

We point out that the conclusion of Theorem 1.1 may be *formally* expressed by stating that (u, v) has an infinite relative Morse index, with respect to the splitting associated to the bilinear map $\int_{\mathbb{R}^N} \langle \nabla\varphi, \nabla\psi \rangle$ (see Lemma 3.1 below). A much weaker version of Theorem 1.1 (namely, the conclusion that (1.5) holds with $\phi = 0$) is proved in [30, Lemma 1.2]. Here the point is that the full conclusion in (1.5) gives the correct information in connexion with the relative Morse index in [1], [5], so that one can combine this straightforwardly with the general abstract estimates on the Morse index of critical points constructed via minimax theorems in critical point theory (see [2], [3], [5]).

In fact, as shown in Section 3, by means of a simple Lyapunov–Schmidt type reduction it turns out that the relative Morse index can be estimated (by below) in terms of the Morse index associated to a functional J which is no longer strongly indefinite and to which we can therefore apply the well-established theory in e.g. [16], [22], [27], [31]. This, we believe, is a novel feature of our main theorem (cf. Lemma 3.1 below for details). This idea was recently proved to be successful in the study of perturbed symmetric superlinear elliptic systems, cf. [9].

We also mention that we assume for definiteness that $N \geq 3$, since an easier argument would cover the lower dimensions. This is in contrast with the main result in [7], where the authors explicitly point out their restriction on the dimension.

The proof of Theorem 1.1 is given in Section 2 (cf. Theorem 2.9). The argument is quite elementary and is much in the spirit of the original one in [8]. We use some energy estimates displayed in [7, Sections 5, 6] (cf. Lemma 2.1 below) and we fully exploit the Pohožaev's type-identity for systems stated in [23], [34], the core of this being the proper choice of the constant λ which appears in (1.5). We mention that one would hope that the assumption on the boundedness of u could be dropped, but our argument does depend on this, since the value of λ relies heavily on the fact that $||u||_\infty < \infty$. In Section 3 we are concerned with the reduction method mentioned above and we derive a priori bounds from our main result, for both Dirichlet and Neumann boundary conditions.

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2. A Liouville-type theorem

In the following we suppose $u, v \in C^2(\omega)$, $u \neq 0$, satisfy

$$-\Delta u = g(v), \quad -\Delta v = f(u) \quad \text{in } \omega \tag{2.1}$$

where either $\omega = \mathbb{R}^N$ ($N \geq 3$) or else ω is a half space which, up to rotation and translation, we may assume to be given by $\omega = \{x = (x_1, \dots, x_N) : x_N > 0\}$; in the latter case, we also impose Dirichlet ($u = 0 = v$) or Neumann ($\partial u/\partial x_N = 0 = \partial v/\partial x_N$) boundary conditions on the boundary of ω. The functions f and g are given by $f(s) = |s|^{p-2}s$, $g(s) = |s|^{q-2}s$ with

$$p, q > 2 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}. \tag{2.2}$$

In fact, for later purposes in Section 3, we keep f as above but we let $g \in C^1(\mathbb{R}; \mathbb{R})$ be such that, for some positive constants c_1, c_2 and every $s \in \mathbb{R}$,

$$g'(s)s^2 \geq g(s)s, \quad qG(s) \geq g(s)s, \quad c_1|s|^q \leq g(s)s \leq c_2|s|^q, \tag{2.3}$$

where $G(s) := \int_0^s g(\xi)\, d\xi$. We reserve the letter φ to denote a smooth cut-off function with support in an annulus $\{x : aR \leq |x| \leq bR\}$ ($0 < a < b$) or in some ball $B_R(0)$, the main feature of it being that $0 \leq \varphi(x) \leq 1$ and $|\nabla\varphi(x)| \leq C/R$

for all $x \in \mathbb{R}^N$. The radius R is taken large, as we compute limits as $R \to \infty$. Moreover, hereafter m is a large integer whose value depends only on p and q, and all integrals are taken in ω except when indicated otherwise.

For future reference, we collect in our next lemma some estimates in [7].

Lemma 2.1 ([7]). *The following holds as $R \to \infty$.*

(a) $\displaystyle\int g(v)v\,\varphi^m = (1+\mathrm{o}(1))\int |u|^p\varphi^m + \mathrm{o}(1)$.

(b) $\displaystyle\int_{\{\varphi=1\}} |\nabla u|\,|\nabla v| + \frac{1}{R}\int |u|\,|\nabla v|\varphi^{m-1} \le C\int |u|^p\varphi^m + \mathrm{o}(1)$.

Proof (sketch). The estimate $R^{-1}\int |u|\,|\nabla v|\varphi^{m-1} \le \mathrm{o}(1)\int |u|^p\varphi^m + \mathrm{o}(1)$ as well as the identity in (a) are proved in [7, Theorem 5A], using interpolation and Hölder's inequality; here assumption (2.2) plays a crucial role and m is chosen sufficiently large. As for the other estimate in (b), this follows similarly to the proof of [7, Lemma 6B] in which, however, it is furthermore assumed that $\int |u|^p < \infty$; for the reader's convenience we give a sketch of the argument: for given $\alpha, \beta > 0$ and r, s such that $1/r + 1/s = 1$, by Hölder's inequality

$$\int_{\{\varphi=1\}} |\nabla u|\,|\nabla v| \le \int |\nabla(u\varphi^\alpha)|\,|\nabla(v\varphi^\beta)| \le \left(\int |\nabla(u\varphi^\alpha)|^r\right)^{1/r}\left(\int |\nabla(v\varphi^\beta)|^s\right)^{1/s}.$$

Now, for s given by $1/s = (1/2)(1+1/q-1/p)$, the Gagliardo–Nirenberg inequality (cf. [10, p. 194]) implies that

$$||\nabla(v\varphi^\beta)||_s \le C||\Delta(v\varphi^\beta)||_{p/(p-1)}^{1/2}||v\varphi^\beta||_q^{1/2}.$$

We choose $\beta = m(p-1)/p$. Then, by (a),

$$\int |v|^q\varphi^{\beta q} \le \int |v|^q\varphi^m \le C\int |u|^p\varphi^m + \mathrm{o}(1).$$

Again by Hölder's inequality one can prove that

$$\int |\Delta(v\varphi^\beta)|^{p/(p-1)} \le C\int |u|^p\varphi^m + \mathrm{o}(1).$$

In conclusion,

$$||\nabla(v\varphi^\beta)||_s \le C\left(\int |u|^p\varphi^m\right)^{1/s} + \mathrm{o}(1).$$

By interchanging u and v (whence $1/r = (1/2)(1+1/p-1/q)$), the conclusion follows. □

Next we compare integral terms $\int \varphi^m|u|^p$ and $\int \overline{\varphi}^m|u|^p$ where φ and $\overline{\varphi}$ are both supported in some ball or annulus of radius $R > 0$.

Lemma 2.2. *If* $\operatorname{supp}\nabla\overline{\varphi} \subset \{\varphi = 1\}$ *then, for some* $C > 0$ (*independent of* R),

$$\int |u|^p \overline{\varphi}^m \le C \int |u|^p \varphi^m + \mathrm{o}(1).$$

Proof. Let $F(s) := |s|^p/p$. The following (formal) identity for solutions of (2.1)

$$(N-2)\int \langle \nabla u, \nabla v\rangle = N \int (F(u) + G(v))$$

is well-known (and, as in [7], it holds indeed in case $\int |u|^p < \infty$, thanks to Lemma 2.1. Precisely, following [23], [34] we compute $0 = \int \operatorname{div}(\overline{\varphi}^m W)$ where W is the vector field $W(x) := \langle \nabla v, x\rangle \nabla u + \langle \nabla u, x\rangle \nabla v - \langle \nabla u, \nabla v\rangle x + F(u)x + G(v)x$; by using the fact that $qG(v) \ge g(v)v$ and also the second equation in (2.1), according to which $\int \langle \nabla v, \nabla(\overline{\varphi}^m u)\rangle = \int \overline{\varphi}^m f(u)u$ we arrive at

$$\begin{aligned}\Big(\frac{1}{p} + \frac{1}{q} - \frac{N-2}{N} + \mathrm{o}(1)\Big) \int |u|^p \overline{\varphi}^m \\ \le C \int_{\operatorname{supp}\nabla\overline{\varphi}} \overline{\varphi}^{m-1} \Big(\frac{|u|}{R}\, |\nabla v| + |u|^p + g(v)v + |\nabla u|\,|\nabla v|\Big).\end{aligned}$$

The conclusion follows from our assumption that $\operatorname{supp}\nabla\overline{\varphi} \subset \{\varphi = 1\}$, together with (2.2) and Lemma 2.1. □

Remark 2.3. Since $u \ne 0$, if φ is supported in some annulus $\{x : aR < |x| < bR\}$ it follows from the preceding lemma that $\int |u|^p = \infty$ and $\int |u|^p \varphi^m \to \infty$ as $R \to \infty$ (just take $\overline{\varphi} = 1$ in $B_{aR}(0)$ in such a way that $\operatorname{supp}\nabla\overline{\varphi} \subset \{\varphi = 1\}$).

Lemma 2.4. *Let* $\lambda = \lambda(R) > 0$ *be given by* $\lambda = R^{N(1/p-1/q)}$. *Then, uniformly in* $\phi \in \mathcal{D}(\omega)$,

$$\begin{aligned}\int |uv|\, |\nabla\varphi^m|^2 + \int |v - \lambda u|\, (|\phi|\, |\Delta\varphi^m| + |\nabla\phi|\, |\nabla\varphi^m|) \\ \le \lambda \int |\nabla\phi|^2 + \mathrm{o}(1) \int |u|^p \varphi^{2m}.\end{aligned}$$

Proof. We have $|\Delta\varphi^m| + R^{-1}|\nabla\varphi^m| \le C\varphi^{m-1}R^{-2}$ and so the second integral on the left-hand side above is bounded by

$$\begin{aligned}\int_{\operatorname{supp}\nabla\varphi} \frac{1}{R}|v - \lambda u| \Big(\frac{|\phi|}{R} + |\nabla\phi|\Big)\varphi^{m-1} \\ \le \delta\lambda \int_{\operatorname{supp}\nabla\varphi} \Big(\frac{\phi^2}{R^2} + |\nabla\phi|^2\Big) + C_\delta\, \frac{1}{\lambda R^2} \int (v - \lambda u)^2 \varphi^{2m-2}\end{aligned}$$

for any small $\delta > 0$. Using Hölder's inequality (recall that φ is supported in some ball of radius CR) and the Sobolev embedding,

$$\int_{\operatorname{supp}\nabla\varphi} \frac{\phi^2}{R^2} \le C\Big(\int |\phi|^{2^*}\Big)^{2/2^*} \le C' \int |\nabla\phi|^2.$$

So, provided δ is chosen sufficiently small, the above expression is bounded by

$$\lambda \int |\nabla\phi|^2 + C\,\frac{1}{\lambda R^2}\int (v-\lambda u)^2\varphi^{2m-2} + \int |uv|\,|\nabla\varphi^m|^2. \tag{2.4}$$

On the other hand, let us denote $\alpha := N(1-2/p)-2$, $\beta := N(1-2/q)-2$, so that $\lambda^2 = R^{\beta-\alpha}$, and let us fix m large enough so that $(2m-2)p/2 \geq 2m$ and $(2m-2)q/2 \geq 2m$. Then, by Hölder's inequality and Lemma 2.1 (a) (with m replaced by $2m$),

$$\begin{aligned}\frac{1}{\lambda R^2}\int (v-\lambda u)^2\varphi^{2m-2} &\leq \frac{2}{R^2}\Big(\lambda\int u^2\varphi^{2m-2} + \frac{1}{\lambda}\int v^2\varphi^{2m-2}\Big)\\ &\leq \lambda C\Big(\int |u|^p\varphi^{2m}\Big)^{2/p} R^\alpha + \frac{C}{\lambda}\Big(\int |v|^q\varphi^{2m}\Big)^{2/q} R^\beta\\ &\leq C'\int |u|^p\varphi^{2m}\Big(\lambda R^\alpha + \frac{1}{\lambda}R^\beta\Big)\\ &= 2C'R^{(\alpha+\beta)/2}\int |u|^p\varphi^{2m} = \mathrm{o}(1)\int |u|^p\varphi^{2m},\end{aligned}$$

since, by assumption, $\alpha+\beta<0$; we have also taken into account the Remark 2.3. Similarly, by Hölder's inequality the last term in (2.4) is bounded by $CR^{(\alpha+\beta)/2}\int |u|^p\varphi^{2m}$ and the conclusion follows. □

The energy functional associated to (2.1) is formally given by

$$I(u,v) = \langle u,v\rangle - \int F(u) - \int G(v),$$

where we have denoted $\langle u,v\rangle := \int\langle\nabla u,\nabla v\rangle$. If α, β are smooth functions with compact support, the quadratic form $I''(u,v)(\alpha,\beta)(\alpha,\beta)$ is well-defined and is given by

$$I''(u,v)(\alpha,\beta)(\alpha,\beta) = 2\langle\alpha,\beta\rangle - \int f'(u)\alpha^2 - \int g'(v)\beta^2. \tag{2.5}$$

Our next result summarizes the preceding conclusions.

PROPOSITION 2.5. *Let u,v be solutions of the system* (2.1), *$m\in\mathbb{N}$ be sufficiently large and φ be supported in some ball* (*or annulus*) *of radius R. Then, provided R is large enough and $\lambda := R^{N(1/p-1/q)}$*,

$$\sup_{\phi\in\mathcal{D}(\omega)} I''(u,v)(u\varphi^m+\phi, v\varphi^m-\lambda\phi)(u\varphi^m+\phi, v\varphi^m-\lambda\phi) < -\frac{1}{2}\frac{p-2}{p-1}\int |u|^p\varphi^{2m}.$$

PROOF. We compute (2.5) with $\alpha = u\psi+\phi$, $\beta = v\psi-\lambda\phi$, $\psi := \varphi^m$. Starting from $-\Delta(u\psi) = g(v)\psi - u\Delta\psi - 2\langle\nabla u,\nabla\psi\rangle$ and similarly for $-\Delta(v\psi)$, and using integration by parts, one finds that

$$2\langle u\psi, v\psi\rangle = 2\int uv|\nabla\psi|^2 + \int f(u)u\psi^2 + \int g(v)v\psi^2.$$

Similarly, by computing $-\Delta((v-\lambda u)\psi)$ we get that

$$2\langle (v-\lambda u)\psi, \phi\rangle = 2\int f(u)\psi\phi - 2\lambda\int g(v)\psi\phi \\ + 4\int (v-\lambda u)\phi\Delta\psi + 2\int (v-\lambda u)\langle\nabla\phi,\nabla\psi\rangle.$$

Thus in our case the expression in (2.5) is given by

$$-\int (f'(u) - \frac{f(u)}{u})(u\psi+\phi)^2 - \int \frac{f(u)}{u}\phi^2 \\ -\int (g'(v) - \frac{g(v)}{v})(v\psi-\lambda\phi)^2 - \lambda^2\int \frac{g(v)}{v}\phi^2 - 2\lambda\int|\nabla\phi|^2 \\ + 4\int (v-\lambda u)\phi\Delta\psi + 2\int (v-\lambda u)\langle\nabla\phi,\nabla\psi\rangle + 2\int uv|\nabla\psi|^2.$$

According to Lemma 2.4, the last four integrals can be estimated by $o(1)\int |u|^p\psi^2$. Since $g'(v)\geq g(v)/v$, each remaining term is negative. In fact, by recalling that $f(u)=|u|^{p-2}u$, the first two integrals above can be written as

$$-\int |u|^{p-2}((p-1)\phi^2 + (p-2)u^2\psi^2 + 2(p-2)u\psi\phi) \leq -\frac{p-2}{p-1}\int |u|^p\psi^2,$$

and the conclusion follows. □

Remark 2.6. For future reference in Section 3, we mention that the conclusion of Proposition 2.5 still holds, with a much simpler proof, when we take $g=0$ in (2.1) and $0<||u||_\infty<\infty$. Indeed, in this case u is constant (by Liouville theorem) and v is bounded (by elliptic estimates). Then, by going through the computations in the proof of Proposition 2.5 with $\lambda := 1$ we see that $I''(u,v)(u\varphi^m+\phi, v\varphi^m-\phi)(u\varphi^m+\phi, v\varphi^m-\phi)$ is bounded above by

$$-\frac{p-2}{p-1}\int |u|^p\varphi^{2m} + \frac{C}{R^2}\int (v-u)^2\varphi^{2m-2} + \frac{C}{R^2}\int uv\varphi^{2m-2},$$

and the conclusion follows.

In view of extending Proposition 2.5, for a given $k\in\mathbb{N}$ we consider a family of functions $\varphi_1,\dots,\varphi_k$ supported in disjoint ordered annuli $A_1,\dots,A_k$; that is, $A_i=\{x : c_iR<|x|<d_iR\}$ with $0<c_i<d_i<1$ and $d_i<c_{i+1}$; moreover, $\varphi_i=1$ in $\{x:\alpha_iR<|x|<\beta_iR\}\subset A_i$.

Lemma 2.7. *Given $\varphi_1,\dots,\varphi_k$ we can find numbers $0<a_1<b_1<a_2<b_2$ and smooth functions ξ_1, ξ_2 in such a way that*

(a) $\xi_1=1$ *in* $B_{a_1R}(0)$, $\xi_1=0$ *in* $\mathbb{R}^N\setminus B_{b_1R}(0)$, $0\leq\xi_1\leq 1$,
$\xi_2=1$ *in* $B_{a_2R}(0)$, $\xi_2=0$ *in* $\mathbb{R}^N\setminus B_{b_2R}(0)$, $0\leq\xi_2\leq 1$,

(b) *for every* $i=1,\dots,k$ *and some* $c,c'>0$ (*independent of* R)

$$c\int |u|^p\xi_1^m \leq \int |u|^p\varphi_i^m \leq c'\int |u|^p\xi_2^m.$$

Proof. By assumption, $\varphi_1 = 1$ in $\{x : \alpha_1 R < |x| < \beta_1 R\}$ and $\operatorname{supp}\varphi_k \subset B_{d_k R}(0)$. Take $a_1 = \alpha_1$, $b_1 = \beta_1$, $a_2 = d_k$, $b_2 > a_2$ and let ξ_1, ξ_2 be defined by the conditions in (a). For every $i = 1, \dots, k$, since $\operatorname{supp}\nabla\varphi_i \subset B_{a_2 R}(0) \subset \{\xi_2 = 1\}$, it follows from Lemma 2.2 that

$$\int |u|^p \varphi_i^m \leq C \int |u|^p \xi_2^m.$$

Similarly, since $\operatorname{supp}\nabla\xi_1 \subset \{x : a_1 R < |x| < b_1 R\} \subset \{\varphi_1 = 1\}$, we have that

$$\int |u|^p \xi_1^m \leq C \int |u|^p \varphi_1^m.$$

It remains to prove the second inequality in (b) for $i = 2, \dots, k$. Now, for every such i, let us fix $\overline{\xi}_i$ such that $\overline{\xi}_i = 1$ in $B_{\alpha_i R}(0)$ and $\overline{\xi}_i = 0$ in $\mathbb{R}^N \setminus B_{\beta_i R}(0)$. Then, as above,

$$\int |u|^p \overline{\xi}_i^m \leq C \int |u|^p \varphi_i^m.$$

But since, by construction, $\operatorname{supp}\xi_1 \subset \{\overline{\xi}_i = 1\}$, we have $\xi_1^m \leq \overline{\xi}_i^m$ in $\mathbb{R}^N$ and the conclusion follows. □

Lemma2.8. *Assume $||u||_\infty < \infty$. Given $k \in \mathbb{N}$ we can find a sequence $R_n \to \infty$ and functions $\varphi_1, \dots, \varphi_k$ as in Lemma 2.7 in such a way that*

$$\max\left\{\int |u|^p \varphi_i^m : i = 1, \dots, k\right\} \leq C \min\left\{\int |u|^p \varphi_i^m : i = 1, \dots, k\right\}.$$

Proof. Let ξ_1, ξ_2 be given by Lemma 2.7. It is sufficient to find $C > 0$ and a sequence $R_n \to \infty$ such that

$$\int |u|^p \xi_2^m \leq C \int |u|^p \xi_1^m. \tag{2.6}$$

The argument is similar to the one in [28, p. 621]. Let $\theta(R) := \int_{B_{a_1 R(0)}} |u|^p$ and $\mu := b_2/a_1 > 1$, so that

$$\int |u|^p \xi_2^m \leq \theta(\mu R) \quad \text{and} \quad \theta(R) \leq \int |u|^p \xi_1^m.$$

We claim that there exists $R_n \to \infty$ such that

$$\theta(\mu R_n) \leq \mu^{N+1}\theta(R_n), \quad \text{for all } n \in \mathbb{N}.$$

Indeed, assume by contradiction that $\theta(R) \leq \theta(\mu R)/\mu^{N+1}$ for all $R \geq R_0$. By iterating this inequality and using the fact that u is bounded we get that, for every $j \in \mathbb{N}$,

$$\theta(R_0) \leq \mu^{-j(N+1)}\theta(\mu^j R_0) \leq C\mu^{-j}.$$

Taking limits we conclude that $\theta(R_0) = 0$ for every large R_0, that is $u = 0$. This is a contradiction and therefore (2.7) (whence (2.6)) holds. □

Now we can state the main result of this section.

Theorem 2.9. *Under assumptions* (2.2)–(2.3), *let u, v be solutions of the system* (2.1) *with $0 < ||u||_\infty < \infty$ and let $k \in \mathbb{N}$. Then we can find a positive constant λ and k functions $\xi_1, \dots, \xi_k \in \mathcal{D}(\mathbb{R}^N)$ with disjoint supports such that*

$$I''(u,v)(\overline{\xi}(u,v) + (\phi, -\lambda\phi))(\overline{\xi}(u,v) + (\phi, -\lambda\phi)) < 0, \tag{2.8}$$

for all $\phi \in \mathcal{D}(\omega)$ and all $\overline{\xi} = \sum_{i=1}^k \mu_i \xi_i$, $\mu_i \in \mathbb{R}$, with $\overline{\xi}(u,v) + (\phi, -\lambda\phi) \neq (0,0)$.

Proof. If $\overline{\xi} = 0$ then $\phi \neq 0$ and

$$I''(u,v)(\phi, -\lambda\phi)(\phi, -\lambda\phi) = -2\lambda \int |\nabla\phi|^2 - \int f'(u)\phi^2 - \lambda \int g'(v)\phi^2 < 0.$$

So we may assume $\overline{\xi} \neq 0$. Since ϕ is arbitrary in $\mathcal{D}(\omega)$, we may assume $\sum_{i=1}^k \mu_i^2 = 1$. We let $\xi_i := \varphi_i^m$ where m is some large integer depending on p and q, and $\varphi_1, \dots, \varphi_k$ are given by Lemma 2.8 (with m replaced by $2m$) for a sufficiently large $R > 0$; the constant $\lambda > 0$ is defined by

$$\lambda = R^{N(1/p - 1/q)}. \tag{2.9}$$

It remains to show that

$$\sup_{\phi \in \mathcal{D}(\omega), \sum \mu_i^2 = 1} I''(u,v)(\overline{\xi}(u,v) + (\phi, -\lambda\phi))(\overline{\xi}(u,v) + (\phi, -\lambda\phi)) < 0. \tag{2.10}$$

Similarly to the proof of Proposition 2.5, this expression is bounded above by

$$-\frac{p-2}{p-1} \int |u|^p \overline{\xi}^2 - 2\lambda \int |\nabla\phi|^2 + 4 \int |v - \lambda u|\, |\phi|\, |\Delta\overline{\xi}| \\ + 2 \int |v - \lambda u|\, |\nabla\phi|\, |\nabla\overline{\xi}| + 2 \int |uv|\, |\nabla\overline{\xi}|^2.$$

Since $\mu_i^2 \leq 1$ $\forall i$, we can replace $\overline{\xi}$ by $\xi := \xi_1 + \dots + \xi_k$ in the last three terms. Using the definition of λ, these can be estimated as in Proposition 2.5, leading to the conclusion that the expression in (2.10) is bounded above by

$$-\frac{p-2}{p-1} \int |u|^p \overline{\xi}^2 + \mathrm{o}(1) \int |u|^p \xi^2,$$

as $R \to \infty$. We can fix $c = c(k,m)$ such that if $\sum \mu_i^2 = 1$ then $\sum \mu_i^{2m} \geq c$ and then, since the functions φ_i have disjoint supports and by using Lemma 2.8, the above expression is dominated by

$$-c' \min\left\{ \int |u|^p \varphi_i^{2m} : i = 1, \dots, k \right\} + \mathrm{o}(1) \max\left\{ \int |u|^p \varphi_i^{2m} : i = 1, \dots, k \right\} \\ \leq -c'' \min\left\{ \int |u|^p \varphi_i^{2m} : i = 1, \dots, k \right\} \to -\infty$$

as $R \to \infty$. This implies (2.10) and completes the proof. □

Remarks 2.10. (a) An inspection of the proof of Lemma 2.4 shows that in case p and q are both less than 2^* then we can simply take $\lambda = 1$ without any reference to the special sequence $R_n \to \infty$ of Lemma 2.8. Similarly conclusion holds in case when $g = 0$.

(b) In fact, as the final estimates in the proof of Lemma 2.4 show, in the general case where $1/p + 1/q > (N-2)/N$ we could have chosen λ differently — namely, in such a way that it would better reflect the symmetries by dilation of our problem. In view of the applications in Section 3, we have chosen $\lambda = R^{N(1/p-1/q)}$ due to its simple expression.

(c) By using a density argument, we see that the conclusion in Theorem 2.9 holds in fact for every $\phi \in \mathcal{D}^{1,2}(\omega)$. Then, of course, the expression in (2.8) may take the value $-\infty$. In the case of Neumann boundary conditions, the conclusion holds for $\phi \in \mathcal{D}(\mathbb{R}^N)$, whence for $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

(d) In connexion with Theorem 1.1 as stated in the introduction, we see that

$$X := \operatorname{span}\{(\lambda^2 u + \lambda v, \lambda u + v)\xi_i,\ i = 1, \dots, k\} \subset \{(\lambda\phi, \phi), \phi \in H_0^1(\omega)\},$$

for $\xi_i = \varphi_i^m$. This follows from the observation that we can write

$$(\lambda^2 u + \lambda v, \lambda u + v)\xi_i = (1+\lambda^2)(u,v)\xi_i + (\psi, -\lambda\psi),$$

where $\psi = (\lambda v - u)\xi_i \in H_0^1(\omega)$. Moreover, indeed $\dim X = k$ if R is sufficiently large. Otherwise we would have $v = -\lambda u$, whence $-2\Delta u = g(v) - f(u)/\lambda$ over the support of some function φ_i; multiplying this identity by $\lambda u \varphi_i^{2m}$, a simple computation and Hölders's inequality would then lead to the contradiction:

$$\int |u|^p \varphi_i^{2m} \le \frac{C}{R^2} \int |u|\,|v|\,\varphi_i^{2m-2} \le \mathrm{o}(1) \int |u|^p \varphi_i^{2m}.$$

3. A priori bounds and related estimates

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 3$, and $f, g \in C^1(\mathbb{R})$. We consider the problem

$$\text{(3.1)} \qquad -\Delta u = g(v), \qquad -\Delta v = f(u) \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega,$$

where f and g satisfy the following:

(H1) $f(0) = g(0) = f'(0) = g'(0) = 0$;

(H2) $0 < (1+\delta)f(s)s \le f'(s)s^2$ and $0 < (1+\delta)g(s)s \le g'(s)s^2$, for some $\delta > 0$;

(H3) for some $p, q > 2$ with $1/p + 1/q > (N-2)/N$

$$\lim_{|s|\to\infty} \frac{f'(s)}{|s|^{p-2}} = \ell_1 > 0, \qquad \lim_{|s|\to\infty} \frac{g'(s)}{|s|^{q-2}} = \ell_2 > 0,$$

We first assume that both p and q are smaller than $2^* := 2N/(N-2)$. In this case, the energy functional

$$I(u,v) := \int_\Omega (\langle \nabla u, \nabla v\rangle - F(u) - G(v)), \quad (u,v) \in E := H_0^1(\Omega) \times H_0^1(\Omega),$$

is a well defined C^2 functional and its critical points correspond to solutions of (3.1); here, as usual, $F(s) := \int_0^s f(\xi)\,d\xi$, $G(s) := \int_0^s g(\xi)\,d\xi$. We denote $E^{\pm} := \{(\varphi, \pm\varphi) : \varphi \in H_0^1(\Omega)\}$. Following [2, Chapter 2.4] and [5, Section 1], if $I'(u,v) = 0$ we denote by $m_{E^-}(u,v)$ the relative Morse index of (u,v) with respect to E^-. This integer is given by the relative dimension

$$m_{E^-}(u,v) := \dim_{E^-} V^- := \dim(V^- \cap (E^-)^{\perp}) - \dim(E^- \cap (V^-)^{\perp}), \tag{3.2}$$

where V^- is the negative eigenspace of the quadratic form $I''(u,v)$. In particular, there is an orthogonal splitting $E = V^- \oplus V^+$, $-I''(u,v)$ is coercive on V^- and $I''(u,v)$ is non-negative on V^+; the splitting is orthogonal also with respect to the quadratic form.

Now, following [29, Section 2], for any $\lambda > 0$ we consider the functional $J_\lambda\colon H_0^1(\Omega) \to \mathbb{R}$,

$$J_\lambda(u) := I(\lambda u + \psi_u, u - \lambda\psi_u) := \max\{I(\lambda u + \psi, u - \lambda\psi) : \psi \in H_0^1(\Omega)\}. \tag{3.3}$$

Then J_λ is C^2 and

$$J'_\lambda(u)\varphi = I'(\lambda u + \psi_u, u - \lambda\psi_u)(\lambda\varphi, \varphi), \quad \text{for all } u, \varphi \in H_0^1(\Omega). \tag{3.4}$$

In particular, u is a critical point of J_λ if and only if $(\lambda u + \psi_u, u - \lambda\psi_u)$ is a critical point of I. We denote by $m_{J_\lambda}(u)$ the usual Morse index of u as a critical point of J_λ.

Lemma 3.1. *Given a critical point u of J_λ ($\lambda > 0$),*

$$m_{J_\lambda}(u) \le m_{E^-}(\lambda u + \psi_u, u - \lambda\psi_u).$$

Proof. Assume first $\lambda = 1$ and denote $J = J_1$. For any fixed $\varphi \in H_0^1(\Omega)$, the quadratic form

$$\phi \mapsto I''(u + \psi_u, u - \psi_u)(\varphi + \phi, \varphi - \phi)(\varphi + \phi, \varphi - \phi)$$

is strictly concave and admits a (unique) maximum point, call it ϕ_φ. Thus

$$I''(u + \psi_u, u - \psi_u)(\varphi + \phi_\varphi, \varphi - \phi_\varphi)(\psi, -\psi) = 0, \quad \text{for all } \psi \in H_0^1(\Omega). \tag{3.5}$$

Going back to the definition in (3.3), we have that

$$I'(u + \psi_u, u - \psi_u)(\psi, -\psi) = 0 \quad \text{for all } \psi \in H_0^1(\Omega);$$

by differentiating this and comparing with (3.5) we see that $\phi_\varphi = D_{\psi_u}\varphi$ for every $\varphi \in H_0^1(\Omega)$. As a consequence,

$$\begin{aligned} J''(u)\varphi, \varphi &= I''(u + \psi_u, u - \psi_u)(\varphi + \phi_\varphi, \varphi - \phi_\varphi)(\varphi + \phi_\varphi, \varphi - \phi_\varphi) \\ &= \max_{\phi \in H_0^1(\Omega)} I''(u + \psi_u, u - \psi_u)(\varphi + \phi, \varphi - \phi)(\varphi + \phi, \varphi - \phi). \end{aligned}$$

Now, we fix a subspace Y of $H_0^1(\Omega)$ such that $-J''(u)$ is coercive on Y and $\dim Y = m_J(u)$, and denote $X := \{(\varphi,\varphi) : \varphi \in Y\}$. It follows from the previous considerations that $-I''(u+\psi_u, u-\psi_u)$ is coercive on $X \oplus E^-$, and so $(X \oplus E^-) \cap (V^-)^\perp = \{0\}$. Thus, by definition of the relative dimension (cf. (3.2)),

$$\dim_{V^-}(X \oplus E^-) = -\dim(V^- \cap (X \oplus E^-)^\perp) \leq 0.$$

The conclusion follows then by using the following properties of the index (see [2, Chapter 2]),

$$\begin{aligned}\dim_{V^-}(X \oplus E^-) &= \dim_{E^-}(X \oplus E^-) + \dim_{V^-}(E^-)\\ &= \dim X - \dim_{E^-}(V^-) = k - m_{E^-}(u+\psi_u, u-\psi_u).\end{aligned}$$

In the general case $\lambda > 0$, by letting

$$\begin{aligned}E_\lambda^+ &:= \{(\lambda\varphi, \varphi) : \varphi \in H_0^1(\Omega)\},\\ E_\lambda^- &:= \{(\varphi, -\lambda\varphi) : \varphi \in H_0^1(\Omega)\},\\ X &:= \{(\lambda\varphi, \varphi) : \varphi \in Y)\},\end{aligned}$$

one deduces as above that $\dim X \leq \dim_{E_\lambda^-} V^-$. It suffices then to observe that

$$\dim_{E^-}(V^-) = \dim_{E_\lambda^-}(V^-) + \dim_{E^-}(E_\lambda^-) = \dim_{E_\lambda^-}(V^-),$$

where the last equality comes from the fact that $E_\lambda^- \cap (E^-)^\perp = E_\lambda^- \cap E^+ = \{0\}$ and $E^- \cap (E_\lambda^-)^\perp = E^- \cap E_\lambda^+ = \{0\}$. □

Example 3.2. Under the above conditions, let us consider the least non zero critical level of I,

$$c := \inf\{I(u,v) : I'(u,v) = 0, (u,v) \neq (0,0)\}.$$

It can be shown that c is indeed attained. Moreover, by letting $J = J_\lambda$ as in (3.3), we can rephrase the results in [29, Section 2] by stating that c can be characterized as a mountain-pass type critical level of J, namely

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma := \{\gamma\colon [0,1] \to H_0^1(\Omega)$ continuous, $\gamma(0) = 0, J(\gamma(1)) < 0\}$; moreover, if u is any non zero critical point of J then $J(tu) < J(u)$ for every $t \geq 0$, $t \neq 1$. By standard arguments, this implies that $m_J(u) = 1$ for *every* $u \in H_0^1(\Omega)$ such that $J(u) = c$ and $J'(u) = 0$. On the other hand, by combining [3, Theorem 1.1] with [29, Proposition 2.4] we can assert that $m_{E^-}(\lambda u + \psi_u, u - \lambda\psi_u) = 1$ for at least one such u.

We consider next the general case where $1/p + 1/q > (N-2)/N$ with, say, $2 < p < 2^* \leq q$. For any sequence $a_j \to \infty$, we let $g_j(s) = A_j|s|^{p-2}s + B_j$ for $s \geq a_j$, $g_j(s) = g(s)$ for $|s| \leq a_j$ and $g_j(s) = \widetilde{A}_j|s|^{p-2}s + \widetilde{B}_j$ for $s \leq -a_j$, where the coefficients are chosen in such a way that g_j is C^1. It can be checked that $g_j'(s)s^2 \geq (1+\delta)g_j(s)s > 0$ for every $s \neq 0$ if j is large enough.

Thus we have a well defined C^2 functional

$$I_j(u,v) := \int_\Omega (\langle \nabla u, \nabla v\rangle - F(u) - G_j(v)), \quad (u,v) \in E := H_0^1(\Omega) \times H_0^1(\Omega),$$

with $G_j(s) := \int_0^s g_j(\xi)\,d\xi$, whose critical points are the solutions of the system (3.6) below.

Theorem 3.3. *Under assumptions* (H1)–(H3), *let* (u_j, v_j) *be any sequence of solutions of the truncated systems*

$$(3.6) \qquad -\Delta u_j = g_j(v_j), \quad -\Delta v_j = f(u_j), \quad u_j, v_j \in H_0^1(\Omega).$$

If there exists $C > 0$ *such that* $m_{E^-}(u_j, v_j) \leq C$ *for all* j, *then* $||u_j||_\infty + ||v_j||_\infty \leq C'$ *for some constant* C' (*and so* (u_j, v_j) *solves the original problem* (3.1) *if* j *is sufficiently large*). *More generally, the conclusion holds if the reduced Morse indices* $m_{J_{\lambda_j}}$ *associated to* (u_j, v_j) *are bounded uniformly in* j.

Proof. We prove that if $||u_j||_\infty + ||v_j||_\infty \to \infty$ along a subsequence then we can find positive constants λ_j in such a way that the reduced Morse indices $m_{J_{\lambda_j}}$ are arbitrarily large (and so are the indices $m_{E^-}(u_j, v_j)$, according to Lemma 3.1). Indeed, as proved in [30, Section 1], if $||u_j||_\infty + ||v_j||_\infty \to \infty$ we can find points $x_j \in \Omega$ and constants $\alpha_j > 0$, $\beta_j > 0$, $\nu_j \to 0^+$ such that both functions

$$\widetilde{u}_j(x) := \frac{1}{\alpha_j} u_j(\nu_j x + x_j), \quad \widetilde{v}_j(x) := \frac{1}{\beta_j} v_j(\nu_j x + x_j)$$

are uniformly bounded and converge in $C^2_{\rm loc}$ to some non zero functions u, v with $||u||_\infty \leq 1$, $||v||_\infty \leq 1$; we have that

$$-\Delta \widetilde{u}_j = \frac{\nu_j^2}{\alpha_j} g_j(\beta_j \widetilde{v}_j), \quad -\Delta \widetilde{u}_j = \frac{\nu_j^2}{\beta_j} f(\alpha_j \widetilde{u}_j)$$

in $\Omega_j := (\Omega - x_j)/\nu_j$, and (u,v) satisfies some limit problem

$$-\Delta u = g_\infty(v), \quad -\Delta v = f_\infty(u) \quad \text{in } \omega,$$

where $f_\infty(s) = c|s|^{p-2}s$ $(c > 0)$ and $g_\infty(s)$ is such that

$$c_1|s|^q \leq g_\infty(s)s \leq c_2|s|^q, \quad qG_\infty(s) \geq g_\infty(s)s, \quad g'_\infty(s)s^2 \geq (p-1)g_\infty(s)s.$$

Here either $\omega = \mathbb{R}^N$ or else $\omega := \{x : \langle x, y_0\rangle < d_0\}$ for some $d_0 \geq 0$, $y_0 \in \mathbb{R}^N$, $y_0 \neq 0$, and in this case $u = 0 = v$ on $\partial\omega$. Moreover,

$$\frac{\alpha_j}{\beta_j}\nu_j^2 f'(\alpha_j \widetilde{u}_j) \to f'_\infty(u) \quad \text{and} \quad \frac{\beta_j}{\alpha_j}\nu_j^2 g_j'(\beta_j \widetilde{v}_j) \to g'_\infty(v)$$

uniformly on compact sets.

Now, for any given $k \in \mathbb{N}$ we apply the conclusion of Theorem 2.9 to the quadratic form $I''_\infty(u,v)$ associated to the limit system above, with λ given by (2.9). For $i = 1,\dots,k$ and $j \in \mathbb{N}$ we denote $\xi_{i,j}(x) = \xi_i((x - x_j)/\nu_j)$ and $\lambda_j = \lambda\beta_j/\alpha_j$.

To prove the theorem, and by taking the Remark 2.10(d) into account, it is enough to show that, provided j is large enough,

$$I''_j(u_j,v_j)\Big(\overline{\xi}_j\frac{u_j}{\alpha_j}+\phi,\overline{\xi}_j\frac{v_j}{\alpha_j}-\lambda_j\phi\Big)\Big(\overline{\xi}_j\frac{u_j}{\alpha_j}+\phi,\overline{\xi}_j\frac{v_j}{\alpha_j}-\lambda_j\phi\Big)<0 \tag{3.7}$$

for every $\phi \in H^1_0(\Omega)$, $\overline{\xi}_j = \sum_i \mu_i\xi_{i,j}$, $(\overline{\xi}_j u_j/\alpha_j + \phi, \overline{\xi}_j v_j/\alpha_j - \lambda_j\phi) \neq (0,0)$. Indeed, we may already assume that $\sum_i \mu_i^2 = 1$ and, up to a factor of $\nu_j^{N-2}\beta_j/\alpha_j$, (3.7) is given by

$$\begin{aligned}2\int\langle\nabla(\overline{\xi}\widetilde{u}_j+\phi_j),\nabla(\overline{\xi}\widetilde{v}_j-\lambda\phi_j)\rangle\\ -\frac{\nu_j^2\alpha_j}{\beta_j}\int f'(\alpha_j\widetilde{u}_j)(\overline{\xi}\widetilde{u}_j+\phi_j)^2-\frac{\nu_j^2\beta_j}{\alpha_j}\int g'_j(\beta_j\widetilde{v}_j)(\overline{\xi}\widetilde{v}_j-\lambda\phi_j)^2\end{aligned}$$

where we have denoted $\phi_j(x) = \phi(\nu_j x + x_j)$, $\overline{\xi} = \sum \mu_i\xi_i$, and we integrate over Ω_j. If we maximize this expression with respect to ϕ_j we see that $\int|\nabla\phi_j|^2 \le C = C(R)$. Thus we can take a weak limit $\phi_j \rightharpoonup \phi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Passing to the limit we get that the above expression is bounded above by

$$I''_\infty(u,v)(\overline{\xi}(u,v)+(\phi_0,-\lambda\phi_0))(\overline{\xi}(u,v)+(\phi_0,-\lambda\phi_0)). \tag{3.8}$$

The conclusion follows from the estimate in (2.10) (see also the Remark 2.10(c)).

We mention that, as proved in [30], in fact this blow-up procedure may lead to limit systems of the form

$$-\Delta u = 0,\quad -\Delta v = c|u|^{p-2}u \quad \text{in } \omega,\quad u\neq 0 \quad (2<p<2^*,\ c>0)$$

or

$$-\Delta u = c|v|^{p-2}v,\quad -\Delta v = 0 \quad \text{in } \omega,\quad v\neq 0 \quad (2<p<2^*,\ c>0).$$

However, thanks to the Remark 2.10(a), the conclusion in (3.7) still holds in this case. □

A similar conclusion holds for the Neumann boundary conditions:

THEOREM 3.4. *Under assumptions* (H1)–(H3), *let* (u_j,v_j) *be any sequence of solutions of the truncated systems*

$$-\Delta u_j + u_j = g_j(v_j),\quad -\Delta v_j + v_j = f(u_j),\quad u_j,v_j\in H^1(\Omega). \tag{3.9}$$

If there exists $C>0$ *such that* $m_{E^-}(u_j,v_j)\le C$ *for all* j *then* $||u_j||_\infty + ||v_j||_\infty \le C'$ *for some constant* C' (*and so* (u_j,v_j) *solves the original problem* (3.1) *if* j

is sufficiently large). More generally, the conclusion holds if the reduced Morse indices $m_{J_{\lambda_j}}$ associated to (u_j, v_j) are bounded uniformly in j.

Proof. The argument follows the lines of Theorem 3.3 but some care is needed in taking limits as $j \to \infty$. We must prove that (3.7) holds uniformly in $\sum_i \mu_i = 1$ and $\phi \in H^1(\Omega)$. Let us denote by ϕ^* the operator extension in $\mathbb{R}^N$, so that $||\phi^*||_{H^1(\mathbb{R}^N)} \leq c||\phi||_{H^1(\Omega)}$ for every $\phi \in H^1(\Omega)$. If we maximize (3.7) with respect to ϕ_j we see that

$$\int_{\Omega_j} |\nabla \phi_j|^2 + \nu_j^2 \int_{\Omega_j} \phi_j^2 \leq C = C(R),$$

thus also

$$\int_{\mathbb{R}^N} |\nabla \phi_j^*|^2 + \nu_j^2 \int_{\mathbb{R}^N} (\phi_j^*)^2 \leq C'.$$

Let $\phi_j^* \rightharpoonup \phi_0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. By using the differential equation satisfied by ϕ_j in Ω_j and by taking weak limits (recall that the support of $\overline{\xi}$ is fixed) we see that ϕ_0 satisfies, in ω,

$$-2\lambda\Delta\phi_0 + f'(u)\phi_0 + \lambda^2 g'(v)\phi_0 = \lambda g'(v)\overline{\xi}v - f'(u)\overline{\xi}u + \Delta(\overline{\xi}(\lambda u - v)), \tag{3.10}$$

together with Neumann boundary conditions on $\partial\omega$ (in case $\omega \neq \mathbb{R}^N$). Now, the limit as $j \to \infty$ of

$$\frac{\lambda\beta_j}{\alpha_j}\nu_j^2 \int_{\Omega_j} g_j'(\beta_j\widetilde{v}_j)\overline{\xi}\widetilde{v}_j\phi_j - \frac{\alpha_j}{\beta_j}\nu_j^2 \int_{\Omega_j} f'(\alpha_j\widetilde{u}_j)\overline{\xi}\widetilde{u}_j\phi_j + \int_{\Omega_j} \phi_j\Delta(\overline{\xi}(\lambda\widetilde{u}_j - \widetilde{v}_j))$$

is precisely

$$\lambda \int_\omega g'(v)\overline{\xi}v\phi_0 - \int_\omega f'(u)\overline{\xi}u\phi_0 + \int_\omega \phi_0\Delta(\overline{\xi}(\lambda u - v)),$$

that is, thanks to (3.10),

$$2\lambda \int_\omega |\nabla\phi_0|^2 + \int_\omega f'(u)\phi_0^2 + \lambda^2 \int_\omega g'(v)\phi_0^2.$$

Then we can pass the (rescaled) expression in (3.7) to the limit, yielding the expression in (3.8) with $\phi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Taking into account the Remark 2.10(c), the conclusion follows. $\square$

As a final remark we stress that the preceding estimates also yield compactness for special sequences of solutions of systems such as (3.1). For example, under assumptions (H1)–(H3) with, now, $2 < p, q < 2^*$, let $(u_\varepsilon, v_\varepsilon) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $\varepsilon \to 0$ be bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$ and solve the singularly perturbed system

$$-\varepsilon^2\Delta u_\varepsilon + u_\varepsilon = g(v_\varepsilon), \quad -\varepsilon^2\Delta v_\varepsilon + v_\varepsilon = f(u_\varepsilon)$$

in such a way that the rescaled sequences $\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, $\widetilde{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x)$ converge in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to a non zero solution of the limit system in $\mathbb{R}^N$,

$$-\Delta u + u = g(v), \quad -\Delta v + v = f(u).$$

In this case we have:

Proposition 3.5. *Under* (H1)–(H3) *with* $2 < p, q < 2^*$, *suppose that the relative Morse index of* $(u_\varepsilon, v_\varepsilon)$ *remains* ≤ 1 *as* $\varepsilon \to 0$. *Then* $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *and strong convergence holds* (*i.e.* $\widetilde{u}_\varepsilon \to u$ *and* $\widetilde{v}_\varepsilon \to v$ *in* $H^1(\mathbb{R}^N)$).

Proof (sketch). Let $\varphi_1 \in \mathcal{D}(B_{2R}(0))$ be such that φ_1 in $B_R(0)$ and $\varphi_2 \in \mathcal{D}(\mathbb{R}^N \setminus B_{3R}(0))$ be such that $\varphi_2 = 1$ in $\mathbb{R}^N \setminus B_{4R}(0)$. Our assumption implies that there exist $\mu_1, \mu_2 \in \mathbb{R}$, $\mu_1^2 + \mu_2^2 = 1$ and $\phi \in H^1(\mathbb{R}^N)$ such that

$$I''(\widetilde{u}_\varepsilon, \widetilde{v}_\varepsilon)\bigg(\widetilde{u}_\varepsilon \sum_{i=1}^2 \mu_i\varphi_i + \phi, \widetilde{v}_\varepsilon \sum_{i=1}^2 \mu_i\varphi_i - \phi\bigg)\bigg(\widetilde{u}_\varepsilon \sum_{i=1}^2 \mu_i\varphi_i + \phi, \widetilde{v}_\varepsilon \sum_{i=1}^2 \mu_i\varphi_i - \phi\bigg) \geq 0,$$

where I'' stands for the (rescaled) quadratic form associated to the system; we have dropped the subscript ε in order to simplify the notations. By taking the Remark 2.10(a) into account, we get that

$$\int \bigg((|\widetilde{u}_\varepsilon|^p + |\widetilde{v}_\varepsilon|^q)\bigg(\sum_{i=1}^2 \mu_i\varphi_i^2\bigg)\bigg) = \mathrm{o}(1)$$

as $R \to \infty, \varepsilon \to 0$. Since $(u, v) \neq (0, 0)$, we must have that $\mu_1 \to 0$, whence $\mu_2 \to 1$. In conclusion, given $\delta > 0$ we can find $R, \varepsilon_0 > 0$ such that

$$\int_{|x| \geq 3R} (|\widetilde{u}_\varepsilon|^p + |\widetilde{v}_\varepsilon|^q) \leq \delta, \quad \text{for all } \varepsilon < \varepsilon_0.$$

From this the conclusion follows easily. □

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