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ATTRACTORS FOR REACTION–DIFFUSION EQUATIONS ON ARBITRARY UNBOUNDED DOMAINS

Martino Prizzi — Krzysztof P. Rybakowski

Abstract. We prove existence of global attractors for parabolic equations of the form

$$\begin{aligned} u_t + \beta(x)u - \sum_{ij} \partial_i(a_{ij}(x)\partial_j u) &= f(x,u), && x \in \Omega,\ t \in [0,\infty[\,,\\ u(x,t) &= 0, && x \in \partial\Omega,\ t \in [0,\infty[\,. \end{aligned}$$

on an arbitrary unbounded domain Ω in $\mathbb{R}^3$, without smoothness assumptions on $a_{ij}(\,\cdot\,)$ and $\partial\Omega$.

1. Introduction

In this paper we study the existence of global attractors for semilinear parabolic equations of the form

$$\begin{aligned} u_t + \beta(x)u - \sum_{ij} \partial_i(a_{ij}(x)\partial_j u) &= f(x,u), \quad x \in \Omega,\ t \in [0,\infty[\,,\\ u(x,t) &= 0, \quad x \in \partial\Omega,\ t \in [0,\infty[\,. \end{aligned} \tag{1.1}$$

Here, $N = 3$ and Ω is an *arbitrary* open set in $\mathbb{R}^N$, bounded or not, $\beta\colon \Omega \to \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions and $\mathrm{L}u := \sum_{ij} \partial_i(a_{ij}(x)\partial_j u)$ is a linear second-order differential operator in divergence form. We do not make any smoothness assumption on $\partial\Omega$ and $a_{ij}(\,\cdot\,)$.

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Notice that, without smoothness assumptions on $\partial\Omega$ and $a_{ij}(\,\cdot\,)$, it is not possible to study (1.1) in the L^q setting for $q \neq 2$. The reason is that one cannot use the regularity theory of elliptic partial differential equations to characterize the fractional power spaces generated by $-\mathrm{L}+\beta(x)$. On the other hand, in order to work in the L^2 setting one must impose growth conditions on f. In particular, for $N=3$ the critical exponent is $\overline{\rho}=5$. The lack of regularity also prevents us from being able to use the ε-regular mild solutions introduced by Arrieta and Carvalho in [4] to treat the critical case. Therefore we shall assume in this paper that f has subcritical growth.

There is vast literature concerning existence of global attractors for reaction-diffusion equations on bounded domains (see e.g. [12,14,7,18,10]). In this case the asymptotic compactness property for the solutions of the equations follows from the compactness of the Sobolew embedding $H^1 \subset L^2$. For unbounded domains this embedding is no longer compact and so new ideas are needed to obtain the asymptotic compactness property.

In [8] Babin and Vishik considered an equation of the form

$$u_t + u - \Delta u = f(u) + g(x), \quad x \in \mathbb{R}^N,\, t \in [0,\infty[\,,$$

with f satisfying the dissipativeness condition $f(u)u \leq 0$ and the monotonicity condition $f'(u) \leq \ell$. They overcame the difficulties arising from the lack of compactness by introducing weighted Sobolew spaces. More recently, Wang considered the same equation in [19] and established the asymptotic compactness of the solutions in the space L^2, under the same hypotheses as those in [8]. To this end, he developed a technique based on tail-estimates of the solutions outside large balls. The simple remark in [15] shows that the solutions are actually asymptotically compact in the natural energy space H^1.

The equation studied in [8,19] has a very special form. In [6] Arrieta et al. considered the more general equation

$$\begin{aligned} u_t - \Delta u &= f(x,u), && x \in \Omega,\ t \in [0,\infty[\,, \\ u(x,t) &= 0, && x \in \partial\Omega,\ t \in [0,\infty[\,. \end{aligned}$$

In that paper Ω is an unbounded domain with uniformly C^2-boundary. The function f has the form $f(x,u) = m(x)u + f_0(x,u) + g(x)$ and satisfies the dissipativeness condition $f(x,u)u \leq C(x)|u|^2 + D(x)|u|$, where C is such that the semigroup generated by $\Delta + C(x)$ decays exponentially. The operator Δ could be replaced by a general second order differential operator in divergence form like L, provided the coefficients a_{ij} are sufficiently smooth. The authors proved several results about existence of attractors in various Sobolew spaces, depending on the growth of f and on the summability properties of m, g, C and D. Their technique is based on the abstract comparison results of [5] and ultimately on the maximum principle for the heat equation. In order to apply the comparison results of [5], one needs to check that the nonlinear function f_0 satisfies the following property: for every $r > 0$ there exists a constant k such that the mapping $u(\,\cdot\,) \mapsto f_0(\,\cdot\,, u(\,\cdot\,)) + ku(\,\cdot\,)$ is increasing on the ball of radius r in the functional space in which the problem is set. In general this property is not satisfied, so one needs to 'prepare' the function f_0 before applying the comparison theorem. This means that one must first find some (local) L^∞-bound for the solutions and then modify f_0 so as to obtain a globally Lipschitzian function. Such L^∞-bounds are obtained through a bootstrapping argument which is possible only if $\partial\Omega$ and a_{ij} satisfy suitable smoothness assumptions.

In this paper we prove existence of global attractors for the parabolic equation (1.1) on an arbitrary unbounded domain Ω in $\mathbb{R}^3$, without smoothness assumptions on $a_{ij}(\,\cdot\,)$ and $\partial\Omega$. To this end we exploit the tail-estimate technique of Wang and the remarkable fact that the equation admits a natural Lyapunov functional. Our hypotheses on the function f are very general and, in particular, they cover the cases considered in [5]. Moreover, since our proof does not depend on the maximum principle, it works also for systems of equations with gradient nonlinearities.

In order to present our results in more detail, let us first describe the notation used in this paper.

Notation. Let Ω be an arbitrary open set in $\mathbb{R}^N$. Given any measurable function $v\colon \Omega \to \mathbb{R}$ and any $\nu \in [1, \infty[$ we set, as usual,

$$|v|_{L^\nu} = |v|_{L^\nu(\Omega)} := \bigg(\int_\Omega |v(x)|^\nu\,dx\bigg)^{1/\nu} \le \infty.$$

Moreover, for $v \in H^1_0(\Omega)$ we set $|v|_{H^1} = |v|_{H^1(\Omega)} := (|\nabla u|^2_{L^2} + |u|^2_{L^2})^{1/2}$.

We also use the common notation $\mathcal{D}(\Omega)$ resp. $\mathcal{D}'(\Omega)$ to denote the space of all test functions on Ω, resp. all distributions on Ω. If $w \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then we use the usual functional notation $w(\varphi)$ to denote the value of w at φ.

Given a function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\widehat{g}$ the (*Nemitski*) operator which associates with every function $u\colon \Omega \to \mathbb{R}$ the function $\widehat{g}(u)\colon \Omega \to \mathbb{R}$ defined by

$$\widehat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$

If X is a normed space and $u\colon I \subset \mathbb{R} \to X$ is differentiable into X at $t \in I$ then we often denote the derivative of u at t by $\partial(u; X)(t)$, in order to indicate its dependence on X.

Unless specified otherwise, all linear spaces considered in this paper are over the real numbers.

Definition 1.1. Let $w: \Omega \to \mathbb{R}$ be a measurable function and let $\gamma \in]0,1]$ be a real number. We say that $w \in \mathcal{E}_\gamma$ if and only if one of the following conditions is satisfied:

(a) $\gamma \in]0,1[$ and there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and all $u \in H_0^1(\Omega)$

$$\big| |w|^{1/2} u \big|_{L^2} \leq C(\gamma\varepsilon |u|_{H^1} + (1-\gamma)\varepsilon^{-\gamma/(1-\gamma)} |u|_{L^2}),$$

(b) $\gamma = 1$ and there exists $C > 0$ such that for all $u \in H_0^1(\Omega)$

$$\big| |w|^{1/2} u \big|_{L^2} \leq C |u|_{H^1}.$$

Remark. Denote by $L^\nu_{\mathrm{u}}(\mathbb{R}^N)$ the set of measurable functions $v: \mathbb{R}^N \to \mathbb{R}$ such that

$$|v|_{L^\nu_{\mathrm{u}}} := \sup_{y \in \mathbb{R}^N} \bigg(\int_{B(y)} |v(x)|^\nu \, dx \bigg)^{1/\nu} < \infty,$$

where, for $y \in \mathbb{R}^N$, $B(y)$ is the open unit cube in $\mathbb{R}^N$ centered at y. By carefully checking the proof of Lemma 2.2 in [6], we obtain that if the trivial extension $\widetilde{w}$ of w to $\mathbb{R}^N$ lies in $L^\nu_{\mathrm{u}}(\mathbb{R}^N)$ for some $\nu \in [(N/2), \infty[$ then $w \in \mathcal{E}_\gamma$, with $\gamma = (6\nu' - 2^*)/4\nu'$. Notice that if $\nu = N/2$ then $\gamma = 1$.

We make the following assumptions:

Hypothesis 1.2.

(a) $a_0, a_1 \in]0,\infty[$ *are constants and* $a_{ij}: \Omega \to \mathbb{R}$, $i,j = 1,\ldots,N$ *are functions in* $L^\infty(\Omega)$ *such that* $a_{ij} = a_{ji}$, $i,j = 1,\ldots,N$, *and for every* $\xi \in \mathbb{R}^N$ *and a.e.* $x \in \Omega$, $a_0|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \leq a_1|\xi|^2$. $A(x) := (a_{ij}(x))_{i,j=1}^N$, $x \in \Omega$.

(b) $\beta: \Omega \to \mathbb{R}$ *is a measurable function with the property that for every* $\varepsilon \in]0,\infty[$ *there is a* $C_\varepsilon \in [0,\infty[$ *with* $||\beta|^{1/2}u|_{L^2}^2 \leq \varepsilon |u|_{H^1}^2 + C_\varepsilon |u|_{L^2}^2$ *for all* $u \in H_0^1(\Omega)$ (*this is slightly less restrictive than the requirement that* $\beta \in \mathcal{E}_\gamma$ *for some* $\gamma \in]0,1[$).

(c) $\lambda_1 := \inf \bigg\{ \displaystyle\int_\Omega \bigg[\sum_{i,j=1}^N a_{ij}\partial_i u \partial_j u + \beta |u|^2 \bigg] dx \;\bigg|\; u \in H_0^1(\Omega),\ |u|_{L^2} = 1 \bigg\} > 0.$

Remark. Condition (c) roughly means that the ground state of the stationary Schrödinger equation

$$-\mathrm{L}u + \beta(x)u = 0$$

on Ω with potential β and with Dirichlet boundary condition has positive energy. In the special case of $\beta \in L^1(\Omega) + L^\infty(\Omega)$ with $\beta \geq 0$, condition (c) is equivalent to the condition that $\int_G \beta(x)\, dx = \infty$ for every domain $G \subset \Omega$ that contains arbitrary large balls. This was proved in [2, 3].

Hypothesis 1.3.

(a) $\overline{C}$ *and* $\overline{\rho} \in [0,\infty[$ *are constants with* $\overline{\rho} \in [2,4[$ *and* $c\colon \Omega \to [0,\infty[$ *is a function with* $c \in L^1(\Omega)$*;*
(b) $a\colon \Omega \to \mathbb{R}$ *is a measurable function with the property that* $a \in \mathcal{E}_\gamma$ *for some* $\gamma \in]0,1[$.
(c) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *is such that* $x \mapsto f(x,u)$ *is Lebesgue measurable for all* $u \in \mathbb{R}$ *and* $u \mapsto f(x,u)$ *is a* C^1*-function for a.e.* $x \in \Omega$*;*
(d) $|\partial_u f(x,u)| \le \overline{C}(a(x) + |u|^{\overline{\rho}})$ *for every* $u \in \mathbb{R}$ *and a.e.* $x \in \Omega$*;*
(e) $f(\,\cdot\,,0) \in L^2(\Omega)$*;*
(f) $f(x,u)u \le c(x)$ *and* $\int_0^u f(x,s)\,ds \le c(x)$ *for a.e.* $x \in \Omega$ *and every* $u \in \mathbb{R}$.

Under Hypothesis 1.2 the differential operator $u \mapsto -Lu + \beta(x)u$ defines a positive self-adjoint operator $\mathbf{A}\colon D(\mathbf{A}) \subset \mathbf{X} \to \mathbf{X}$ on the Hilbert space $\mathbf{X} = L^2(\Omega)$. $D(\mathbf{A})$ endowed with the graph norm of $\mathbf{A}$ is continuously included in $H_0^1(\Omega)$. The operator $\mathbf{A}$ generates the family $X^\alpha = D(\mathbf{A}^\alpha)$, $\alpha \in [0,\infty[$, of fractional power spaces. Setting $X^{-\alpha} = (X^\alpha)'$, $\alpha \in [0,\infty[$, we can construct a family $\mathbf{A}_{(\alpha)}$, $\alpha \in \mathbb{R}$, of self-adjoint operators, such that $\mathbf{A}_{(\alpha)}\colon X^\alpha \to X^{\alpha-1}$. Moreover, $D(\mathbf{A}_{(\alpha)}^\beta) = X^{\alpha+\beta-1}$ for all $\alpha,\beta \in \mathbb{R}$.

Under Hypothesis 1.3 one can find an $\alpha \in [0,1[$ such that the function f generates a locally Lipschitzian Nemitski operator $\mathbf{f}\colon H_0^1(\Omega) = D(\mathbf{A}_{(-\alpha+1)}^{\alpha+1/2}) \to X^{-\alpha}$. By general results on abstract parabolic equations (see e.g. [13]), (1.1) generates a local semiflow π on $H_0^1(\Omega)$. The choice of α depends on $\overline{\rho}$ and γ. The semiflow π does not depend on the choice of α.

The main result of this paper can now be stated as follows.

Theorem 1.4. *Assume Hypotheses* 1.2 *and* 1.3. *Then* π *is a global semiflow and it has a global attractor* $\mathcal{A}$. $\mathcal{A}$ *lies in* $X^{1-\alpha}$ *and is compact in the norm of* $X^{1-\alpha}$.

2. Preliminaries

We assume the reader's familiarity with attractor theory on metric spaces as expounded in e.g. [12] or, more recently, in [10] and we just collect here a few relevant concepts from that theory.

Definition. Let Y be a metric space. Recall that a *local semiflow* π *on* Y is, by definition, a continuous map from an open subset D of $[0,\infty[\times Y$ to Y such that, for every $x \in Y$ there is an $\omega_x = \omega_{\pi,x} \in]0,\infty]$ with the property that $(t,x) \in D$ if and only if $t \in [0,\omega_x[$, and such that (writing $x\pi t := \pi(t,x)$ for $(t,x) \in D$) $x\pi 0 = x$ for $x \in Y$ and whenever $(t,x) \in D$ and $(s, x\pi t) \in D$ then $(t+s,x) \in D$ and $x\pi(t+s) = (x\pi t)\pi s$. Given an interval I in $\mathbb{R}$, a map $\sigma\colon I \to Y$ is called a *solution* (*of* π) if whenever $t \in I$ and $s \in [0,\infty[$ are such that $t+s\in I$, then $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s=\sigma(t+s)$. If $I=\mathbb{R}$, then σ is called a *full solution (of* π*)*. A subset S of Y is called (π-)*invariant* if for every $x\in S$ there is a full solution σ with $\sigma(\mathbb{R})\subset S$ and $\sigma(0)=x$. A point $x\in Y$ is called an *equilibrium of* π if $x\pi t=x$ for all all $t\in[0,\omega_x[$.

Given a local semiflow π on Y and a subset N of Y, we say that π *does not explode in* N if whenever $x\in Y$ and $x\pi\,[0,\omega_x[\,\subset N$, then $\omega_x=\infty$. A *global semiflow* is a local semiflow with $\omega_x=\infty$ for all $x\in Y$.

Now let π be a global semiflow on Y. A subset A of Y is called a *global attractor* (*rel. to* π) if A is compact, invariant and if for every bounded set B in Y and every open neighborhood U of A there is a $t_{B,U}\in[0,\infty[$ such that $x\pi t\in U$ for all $x\in B$ and all $t\in[t_{B,U},\infty[$. It easily follows that a global attractor, if it exists, is uniquely determined.

A subset B of Y is called (π-)*ultimately bounded* if there is a $t_B\in[0,\infty[$ such the set $\{x\pi t\mid x\in B,\ t\in[t_B,\infty[\}$ is bounded.

π is called *asymptotically compact* if whenever $B\subset Y$ is ultimately bounded, $(x_n)_n$ is a sequence in B and $(t_n)_n$ is a sequence in $[0,\infty[$ with $t_n\to\infty$ as $n\to\infty$, then the sequence $(x_n\pi t_n)_n$ has a convergent subsequence.

The following result is well-known:

Proposition 2.1. *Let* π *be a global semiflow on a metric space* Y*. Suppose that*

(a) *π is asymptotically compact;*
(b) *every bounded subset of Y is ultimately bounded;*
(c) *the set of all equilibria of π is bounded;*
(d) *there is a continuous function $\mathcal{L}\colon Y\to\mathbb{R}$ which is bounded below, non-increasing along solutions of π and whenever $\mathcal{L}(x\pi t)=\mathcal{L}(x)$ for all $t\in[0,\infty[$ then x is an equilibrium of π.*

Under these assumptions, π has a global attractor.

Proof. This is just [10, Corollary 1.1.4 and Proposition 1.1.3]. □

Given a Banach space X and a sectorial operator $A\colon D(A)\subset X\to X$ in X with $\operatorname{re}\sigma(A)>0$ we know that $-A$ is the generator of an analytic semigroup $(e^{-At})_{t\in[0,\infty[}$ of linear operators on X. For $\alpha\in\,]0,\infty[$ we define, as usual, the operator $A^{-\alpha}\colon X\to X$ as

$$A^{-\alpha}u=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-At}u\,dt,\quad u\in X. \tag{2.1}$$

$A^{-\alpha}$ is injective and we define $X^\alpha=X^\alpha_A$ to be the range of $A^{-\alpha}$. We define $A^\alpha\colon X^\alpha\to X$ to be the inverse of $A^{-\alpha}$. We also set $X^0=X$ and $A^0=\mathrm{id}_X$. We call $A^{-\alpha}$, resp. A^α the *basic fractional power of A of order $-\alpha$*, resp. α and we call X^α the *fractional power space of A of order α*. X^α is a Banach space with respect to the norm

$$|u|_{X^\alpha} := |A^\alpha u|_X, \quad u \in X^\alpha.$$

If $\beta > \alpha$ then X^β is a dense subset of X^α.

Moreover,

$$A^{-\beta} A^{-\gamma} x = A^{-\beta-\gamma} x, \quad \alpha, \beta \in]0, \infty[, \ x \in X. \tag{2.2}$$

Now let X be a Hilbert space and $A: D(A) \subset X \to X$ be self-adjoint in X with $\operatorname{re} \sigma(A) > 0$. Then A is sectorial in X and, for $\alpha \in]0, \infty[$,

$$A^{-\alpha} = \int_0^\infty t^{-\alpha} \, dE(t), \tag{2.3}$$

where $(E(t))_{t \in \mathbb{R}}$ is the spectral measure defined by A. In this case the set X^α is a Hilbert space with respect to the scalar product

$$\langle u, v \rangle_{X^\alpha} := \langle A^\alpha u, A^\alpha v \rangle_X, \quad u, v \in X^\alpha.$$

For $\alpha \in]0, \infty[$ let $X^{-\alpha} = X_A^{-\alpha}$ be the dual space of X^α. We endow $X^{-\alpha}$ with the scalar product $\langle \cdot, \cdot \rangle_{X^{-\alpha}}$ dual to the scalar product $\langle \cdot, \cdot \rangle_{X^\alpha}$, i.e.

$$\langle u', v' \rangle_{X^{-\alpha}} = \langle R_\alpha^{-1} u', R_\alpha^{-1} v' \rangle_{X^\alpha}, \quad u', v' \in X^{-\alpha},$$

where $R_\alpha: X^\alpha \to X^{-\alpha}$ is the Fréchet–Riesz isomorphism $u \mapsto \langle \cdot, u \rangle_{X^\alpha}$.

$X^{-\alpha}$ is called the *fractional power space of A of order $-\alpha$*.

By φ denote the duality map from X to X', i.e.

$$\varphi(x) := \langle \cdot, x \rangle, \quad x \in X.$$

Let $\alpha, \beta \in \mathbb{R}$ be arbitrary. If $\beta \geq \alpha \geq 0$, then let $\varphi_{\beta,\alpha}: X^\beta \to X^\alpha$ be the inclusion map; if If $\beta \geq \alpha > 0$, then define the map $\varphi_{-\alpha,-\beta}: X^{-\alpha} \to X^{-\beta}$ by $\varphi_{-\alpha,-\beta}(y') = y'_{|X^\beta}$ for $y' \in X^{-\alpha}$ i.e. for $y': X^\alpha \to \mathbb{R}$ linear and bounded; if $\beta > 0$, define $\varphi_{0,-\beta}: X^0 = X \to X^{-\beta}$ as follows: if $x \in X$, then $\varphi_{0,-\beta}(x)$ is equal to the map $y': X^\beta \to \mathbb{R}$ such that $y'(y) = \langle y, x \rangle$ for all $y \in X^\beta$. It follows that $y' = \varphi_{0,-\beta}(x) \in X^{-\beta}$, so $\varphi_{0,-\beta}$ is defined. Finally, if $\alpha > 0$ and $\beta > 0$, then let $\varphi_{\beta,-\alpha} := \varphi_{0,-\alpha} \circ \varphi_{\beta,0}$.

We have the following basic result.

Proposition 2.2. *For all $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$ the map $\varphi_{\beta,\alpha}: X^\beta \to X^\alpha$ is defined, linear, bounded and injective. The set $\varphi_{\beta,\alpha}[X^\beta]$ is dense in the Hilbert space X^α. Moreover,*

$$\varphi_{\alpha,\alpha} = \mathrm{id}_{X^\alpha}, \quad \alpha \in \mathbb{R}$$

and

$$\varphi_{\gamma,\alpha} = \varphi_{\beta,\alpha} \circ \varphi_{\gamma,\beta}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \ \gamma \geq \beta \geq \alpha.$$

For all $\alpha, \gamma \in \mathbb{R}$, $\theta \in [0,1]$ *and* $x \in X^\gamma$ *with* $\alpha \le \gamma$ *and* $\beta = (1-\theta)\alpha + \theta\gamma$, *the interpolation inequality*

$$|\varphi_{\gamma,\beta} x|_{X^\beta} \le |\varphi_{\gamma,\alpha} x|_{X^\alpha}^{1-\theta} |x|_{X^\gamma}^\theta$$

holds.

For all $\alpha, \beta \in [0, \infty[$, $A_{(\alpha)}^{-\beta} := A^{-\beta}|X^\alpha \colon X^\alpha \to X^{\beta+\alpha}$ *is a linear bijective isometry.*

For every $\alpha \in]0, \infty[$, $\beta \in [0, \infty[$ *there is a unique continuous map* $A_{(-\alpha)}^{-\beta}$ *from* $X^{-\alpha}$ *to* $X^{\beta-\alpha}$ *with* $A_{(-\alpha)}^{-\beta} \circ \varphi_{0,-\alpha} = \varphi_{\beta,\beta-\alpha} \circ A^{-\beta}$. $A_{(-\alpha)}^{-\beta}$ *is a linear bijective isometry.*

For $\alpha \in \mathbb{R}$ *and* $\beta \in]0, \infty[$ *define the map* $A_{(\alpha)}^\beta \colon X^\alpha \to X^{-\beta+\alpha}$ *by*

$$A_{(\alpha)}^\beta = (A_{(-\beta+\alpha)}^{-\beta})^{-1}$$

and set $A_{(\alpha)} := A_{(\alpha)}^1$. *Then for all* $\gamma, \gamma' \in \mathbb{R}$ *with* $\gamma > \gamma'$ *and all* $\beta \in \mathbb{R}$,

$$\varphi_{-\beta+\gamma,-\beta+\gamma'} \circ A_{(\gamma)}^\beta = A_{(\gamma')}^\beta \circ \varphi_{\gamma,\gamma'}$$

and for all α, β *and* $\gamma \in \mathbb{R}$,

$$A_{(-\gamma+\alpha)}^\beta \circ A_{(\alpha)}^\gamma = A_{(\alpha)}^{\beta+\gamma}.$$

For $\alpha, \beta \in \mathbb{R}$ *with* $\beta \ge \alpha$ *the map* $\varphi_{\beta,\alpha}$ *is bijective from* X^β *to* $\varphi_{\beta,\alpha}[X^\beta]$. *For* $\alpha, \beta \in]0, \infty[$ *define the map*

$$\widetilde{A}_{(-\alpha)}^\beta := A_{(\beta-\alpha)}^\beta \circ \varphi_{\beta-\alpha,-\alpha}^{-1} \colon \varphi_{\beta-\alpha,-\alpha}[X^{\beta-\alpha}] \subset X^{-\alpha} \to X^{-\alpha}$$

and set $\widetilde{A}_{(-\alpha)} := \widetilde{A}_{(-\alpha)}^1$. *The map* $\widetilde{A}_{(-\alpha)}^\beta$ *is bijective and its inverse is* $\widetilde{A}_{(-\alpha)}^{-\beta} := \varphi_{\beta-\alpha,-\alpha} \circ A_{(-\alpha)}^{-\beta}$.

For every $\alpha \in]0, \infty[$ *the map* $B := \widetilde{A}_{(-\alpha)} \colon D(B) = \varphi_{1-\alpha,-\alpha}[X^{1-\alpha}] \subset X^{-\alpha} \to X^{-\alpha}$ *is self-adjoint in* $X^{-\alpha}$ *and* $\operatorname{re}\sigma(B) > 0$. *For* $\beta \in]0, \infty[$ *let* $B^{-\beta}$ *be the basic fractional power of* B *of order* $-\beta$ *and* X_B^β *be the corresponding fractional power space. Then*

$$B^{-\beta} = \widetilde{A}_{(-\alpha)}^{-\beta} \quad \textit{and} \quad X_B^\beta = \varphi_{\beta-\alpha,-\alpha}[X^{\beta-\alpha}].$$

The map $\varphi_{\beta-\alpha,-\alpha}$ *is an isometry of the Hilbert space* $X^{\beta-\alpha}$ *onto* X_B^β.

Finally, whenever $\alpha \in [0, (1/2)[$, $x \in X^{1-\alpha}$ *and* $v \in X^{1/2} \subset X^\alpha$, *then*

$$(A_{(1-\alpha)} x).v = \langle x, v \rangle_{X^{1/2}}.$$

Here, the dot '.' denotes function application between an element of $X^{-\alpha}$ *and* X^α.

Remark 2.3. In view of Proposition 2.2, for $\alpha, \beta \in \mathbb{R}$ with $\beta \ge \alpha$ one often regards $\varphi_{\beta,\alpha}$ as an inclusion map and X^β as a (dense) subset of X^α.

Sometimes (cf. e.g. [17]) the notation

$$H_\alpha := X^{\alpha/2}, \quad \alpha \in \mathbb{R}$$

is used. We then set

$$A_\alpha^\beta := A_{(\alpha/2)}^\beta, \quad \alpha, \beta \in \mathbb{R} \quad \text{and} \quad A_\alpha := A_\alpha^1, \quad \alpha \in \mathbb{R}.$$

Notice that $A_\alpha \colon H_\alpha \to H_{\alpha-2}$ for all $\alpha \in \mathbb{R}$.

Proposition 2.2 is well-known (see e.g. the book of Amann [1]) but it is not easy to find in the literature a proof that is both elementary and complete. Therefore, in the Appendix, we provide an elementary proof which presupposes only minimal knowledge of spectral measures.

3. Some results on semilinear parabolic equations

Proposition 3.1. *Let X be a Banach space and $A\colon D(A) \subset X \to X$ be sectorial. Let $\overline{k} \in [0,\infty[$ be such that $\operatorname{re}\sigma(A+\overline{k}I) > 0$ and X^β, $\beta \in [0,\infty[$, be the family of fractional power spaces generated by $A+\overline{k}I$. Let $\alpha \in [0,1[$, $g\colon X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α and π be the local semiflow on X^α generated by the solutions of the differential equation*

$$\dot{u} + Au = g(u). \tag{3.1}$$

Suppose $T \in]0,\infty[$, $u\colon [0,T] \to X^\alpha$ and $u_k\colon [0,T] \to X^\alpha$, $k \in \mathbb{N}$, are solutions of π such that, for some $R \in [0,\infty[$, $|u(t)|_{X^\alpha} \le R$ and $|u_k(t)|_{X^\alpha} \le R$ for all $k \in \mathbb{N}$ and $t \in [0,T]$. If $u_k(0) \to u(0)$ in X as $k \to \infty$, then, for every $T_0 \in]0,T]$, $u_k(t) \to u(t)$ in X^α as $k \to \infty$, uniformly for $t \in [T_0,T]$.

Proof. There is a constant $L = L(R) \in [0,\infty[$ such that

$$|g(v_1) - g(v_2)|_X \le L|v_1 - v_2|_{X^\alpha}$$

for all $v_1, v_2 \in X^\alpha$ with $|v_1|_{X^\alpha} \le R$ and $|v_2|_{X^\alpha} \le R$. Now, for every $k \in \mathbb{N}$ and $t \in]0,T]$,

$$u_k(t) - u(t) = e^{-At}(u_k(0) - u(0)) + \int_0^t e^{-A(t-s)}(g(u_k(s)) - g(u(s)))\, ds,$$

so

$$\begin{aligned} |u_k(t) - u(t)|_{X^\alpha} \le{}& Ct^{-\alpha}|u_k(0) - u(0)|_X \\ &+ C\int_0^t (t-s)^{-\alpha}|g(u_k(s)) - g(u(s))|_X\, ds \\ \le{}& Ct^{-\alpha}|u_k(0) - u(0)|_X + CL\int_0^t (t-s)^{-\alpha}|u_k(s) - u(s)|_{X^\alpha}\, ds, \end{aligned}$$

for some constant $C \in [0,\infty[$, depending only on α. By Henry's inequality, cf. [13, Theorem 7.1.1] or [10, Lemma 1.2.9], this implies that

$$|u_k(t) - u(t)|_{X^\alpha} \le C' t^{-\alpha}|u_k(0) - u(0)|_X, \quad k \in \mathbb{N},\ t \in]0,T]. \tag{3.2}$$

where $C' \in [0,\infty[$ is a constant which only depends on (α, C, L, T). Estimate (3.2) implies the assertion of the proposition. $\square$

THEOREM 3.2. *Let X be a Hilbert space and $A\colon D(A) \subset X \to X$ be selfadjoint and bounded from below. Let $\overline{k} \in [0,\infty[$ be such that $\operatorname{re}\sigma(A+\overline{k}I) > 0$ and X^β, $\beta \in \mathbb{R}$, be the family of fractional power spaces generated by $A+\overline{k}I$. Let $\alpha \in [0,1[$, $g\colon X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α and π be the local semiflow on X^α generated by the solutions of the differential equation*

$$\dot{u} + Au = g(u). \tag{3.3}$$

If $K \subset X^\alpha$ is a π-invariant set which is compact in X^α then $K \subset X^1 = D(A)$ and K is compact in X^1.

PROOF. By results in [13], $K \subset X^1$. Let $(\overline{u}_n)_n$ be an arbitrary sequence in K. Then there is a sequence $(u_n)_n$ of solutions of π lying in K such that $u_n(0) = \overline{u}_n$ for every $n \in \mathbb{N}$. Let $\beta \in]0,1[$ be such that $\beta > \alpha$. By [13, Theorem 3.5.2, and its proof] there is a constant $C \in]0,\infty[$ such that for every $n \in \mathbb{N}$, u_n is differentiable into X^β and

$$|v_n|_\beta \le C, \quad n \in \mathbb{N} \tag{3.4}$$

where $v_n = \partial(u_n; X^\beta)(0)$ for $n \in \mathbb{N}$. There is a strictly increasing sequence $(n_m)_m$ in $\mathbb{N}$ and a $\overline{u} \in K$ such that $\overline{u}_{n_m} \to \overline{u}$ in X^α as $m \to \infty$. Thus, using the notation of Proposition 2.2, $\varphi_{0,\alpha-1}(A+kI)\overline{u}_{n_m} \to \varphi_{0,\alpha-1}(A+kI)\overline{u}$ in $X^{\alpha-1}$ and $g(\overline{u}_{n_m}) \to g(\overline{u})$ in X^0 as $m \to \infty$ so $-\varphi_{0,\alpha-1}A\overline{u}_{n_m} + \varphi_{0,\alpha-1}g(\overline{u}_{n_m}) \to -\varphi_{0,\alpha-1}A\overline{u} + \varphi_{0,\alpha-1}g(\overline{u})$ in $X^{\alpha-1}$ as $m \to \infty$. Now

$$\partial(u_n; X^\alpha)(t) = -Au_n(t) + g(u_n(t)), \quad n \in \mathbb{N},\ t \in \mathbb{R}.$$

We thus conclude that $\varphi_{0,\alpha-1}v_{n_m} \to -\varphi_{0,\alpha-1}A\overline{u} + \varphi_{0,\alpha-1}g(\overline{u})$ in $X^{\alpha-1}$ as $m \to \infty$. This together with (3.4) and the interpolation inequality from Proposition 2.2 implies that $v_{n_m} \to -A\overline{u} + g(\overline{u})$ in X^0 as $m \to \infty$. Thus $-A\overline{u}_{n_m} + g(\overline{u}_{n_m}) \to -A\overline{u} + g(\overline{u})$ in X^0 as $m \to \infty$ and as $g(\overline{u}_{n_m}) \to g(\overline{u})$ in X^0 as $m \to \infty$ it follows that $A\overline{u}_{n_m} \to A\overline{u}$ in X^0 as $m \to \infty$. It follows that $(A+kI)\overline{u}_{n_m} \to (A+kI)\overline{u}$ in X^0 so $\overline{u}_{n_m} \to \overline{u}$ in X^1. The theorem is proved. $\square$

4. Some linear estimates

REMARK 4.1. Under Hypothesis 1.2 item (a) let the operator $\mathrm{L}\colon H_0^1(\Omega) \to \mathcal{D}'(\Omega)$ be defined by

$$\mathrm{L}u = \sum_{i,j=1}^N \partial_i(a_{ij}\partial_j u), \quad u \in H_0^1(\Omega).$$

The definition of distributional derivatives implies that

$$(4.1)\quad (\mathrm{L}u-\beta u)(v) = -\int_\Omega \bigg[\sum_{i,j=1}^N a_{ij}\partial_i u\partial_j v+\beta uv\bigg]\,dx, \quad u \in H_0^1(\Omega),\ v \in \mathcal{D}(\Omega).$$

It follows by density that

$$(4.2)\qquad \langle(\mathrm{L}u-\beta u), v\rangle_{L^2} = -\int_\Omega \bigg[\sum_{i,j=1}^N a_{ij}\partial_i u\partial_j v + \beta uv\bigg]\,dx$$

for $u, v \in H_0^1(\Omega)$ with $\mathrm{L}u - \beta u \in L^2(\Omega)$.

Lemma 4.2. *Assume Hypothesis* 1.2. *If* $\kappa \in [0, \lambda_1[$ *is arbitrary and if* $\overline{\varepsilon}$ *and* ρ *are chosen such that* $\overline{\varepsilon} \in]0, a_0[$, $\rho \in]0,1[$ *and* $c := \min(\rho(a_0-\overline{\varepsilon}), (1-\rho)(\lambda_1 - \kappa) - \rho(\overline{\varepsilon} + C_{\overline{\varepsilon}} + \kappa)) > 0$ *then*

$$\begin{aligned} c(|\nabla u|_{L^2}^2 + |u|_{L^2}^2) &\le \int_\Omega \bigg[\sum_{i,j=1}^N a_{ij}\partial_i u\partial_j u + (\beta-\kappa)|u|^2\bigg]\,dx \\ &\le C(|\nabla u|_{L^2}^2 + |u|_{L^2}^2), \quad u \in H_0^1(\Omega) \end{aligned}$$

where $C := \max(a_1 + \overline{\varepsilon}, \overline{\varepsilon} + C_{\overline{\varepsilon}})$.

Proof. This is just a simple computation. □

Lemma 4.3. *Assume Hypothesis* 1.2. *For* $u, v \in H_0^1(\Omega)$ *define*

$$(4.3)\qquad \langle u, v\rangle_1 = \int_\Omega \bigg[\sum_{i,j=1}^N a_{ij}\partial_i u\partial_j v + \beta uv\bigg]\,dx.$$

$\langle\,\cdot\,,\,\cdot\,\rangle_1$ *is a scalar product on* $H_0^1(\Omega)$ *and the norm defined by this scalar product is equivalent to the usual norm on* $H_0^1(\Omega)$.

Proof. This follows from Lemma 4.2. □

Lemma 4.4. *Suppose* $(Y, \langle\,\cdot\,,\,\cdot\,\rangle_Y)$ *and* $(X, \langle\,\cdot\,,\,\cdot\,\rangle_X)$ *are* (*real or complex*) *Hilbert spaces such that* $Y \subset X$, Y *is dense in* $(X, \langle\,\cdot\,,\,\cdot\,\rangle_X)$ *and the inclusion* $(Y, \langle\,\cdot\,,\,\cdot\,\rangle_Y) \to (X, \langle\,\cdot\,,\,\cdot\,\rangle_X)$ *is continuous. Then for every* $u \in X$ *there exists a unique* $w_u \in Y$ *such that*

$$\langle v, w_u\rangle_Y = \langle v, u\rangle_X \quad \text{for all } v \in Y.$$

The map $B\colon X \to X$, $u \mapsto w_u$ *is linear, symmetric and positive. Let* $B^{1/2}$ *be a square root of* B, *i.e.* $B^{1/2}\colon X \to X$ *linear, symmetric and* $B^{1/2} \circ B^{1/2} = B$. *Then* B *and* $B^{1/2}$ *are injective and* $R(B)$ *is dense in* Y. *Set* $X^{1/2} = X_B^{1/2} = R(B^{1/2})$ *and* $B^{-1/2}\colon X^{1/2} \to X$ *be the inverse of* $B^{1/2}$. *On* $X^{1/2}$ *the assignment*

$\langle u, v\rangle_{1/2} := \langle B^{-1/2}u, B^{-1/2}v\rangle_X$ *is a complete scalar product. We have* $Y = X^{1/2}$ *and* $\langle\,\cdot\,,\,\cdot\,\rangle_Y = \langle\,\cdot\,,\,\cdot\,\rangle_{1/2}$.

Proof. The function $v \mapsto \langle v, u\rangle_X$ is linear and continuous on Y. Thus Fréchet–Riesz theorem implies the existence and uniqueness of w and the linearity of B. Since, for u and $v \in X$

$$\langle Bu, v\rangle_X = \langle Bu, Bv\rangle_Y = \langle u, Bv\rangle_X \tag{4.4}$$

it follows that B is symmetric and positive. If $u \in X$ and $Bu = 0$ then $0 = \langle v, Bu\rangle_Y = \langle v, u\rangle_X$ for all $v \in Y$ and since Y is dense in X we see that $u = 0$ so B is injective. It follows that $B^{1/2}$ is injective as well. If $v \in Y$ and $\langle v, Bu\rangle_Y = 0$ for all $u \in X$ then $\langle v, u\rangle_X = 0$ for all $u \in X$ so $v = 0$. It follows that $R(B)$ is dense in Y. Clearly $\langle\,\cdot\,,\,\cdot\,\rangle_{1/2}$ is a complete scalar product on $X^{1/2}$. If $v \in X^{1/2}$ and $u \in X$ then

$$\langle v, Bu\rangle_{1/2} = \langle B^{-1/2}v, B^{1/2}u\rangle_X = \langle v, u\rangle_X.$$

Thus if $\langle v, Bu\rangle_{1/2} = 0$ for all $u \in X$, then $v = 0$. This shows that $R(B)$ is dense in $X^{1/2}$. We claim that

$$\langle u, v\rangle_Y = \langle u, v\rangle_{1/2}, \quad u, v \in R(B). \tag{4.5}$$

In fact, if u and $v \in R(B)$ then $u = B\widetilde{u}$ and $v = B\widetilde{v}$ for some u and $v \in X$. Thus

$$\langle u, v\rangle_Y = \langle u, B\widetilde{v}\rangle_Y = \langle u, \widetilde{v}\rangle_X$$

and

$$\langle u, v\rangle_{1/2} = \langle B^{1/2}\widetilde{u}, B^{1/2}\widetilde{v}\rangle_X = \langle B\widetilde{u}, \widetilde{v}\rangle_X = \langle u, \widetilde{v}\rangle_X.$$

The claim is proved.

Since $B^{1/2}$ is continuous from X to X with bound $|B^{1/2}|$ it follows that, for all $u \in X^{1/2}$,

$$|u|_X \le |B^{1/2}|\,|B^{-1/2}u|_X = |B^{1/2}|\,|u|_{1/2}$$

so the inclusion map $(X^{1/2}, |\cdot|_{1/2}) \to (X, |\cdot|_X)$ is continuous.

Now, if $u \in Y$, then there is a sequence $(u_n)_n$ in $R(B)$ converging to u in Y. It follows that $(u_n)_n$ is a Cauchy sequence in Y, so, by (4.5), it is a Cauchy sequence in $X^{1/2}$ and so it converges to a $v \in X^{1/2}$. By what we have proved so far, $(u_n)_n$ converges to v and to u in X. Thus $u = v$ so $u \in X^{1/2}$. It follows that $Y \subset X^{1/2}$. The same argument, with 'Y' and '$X^{1/2}$' exchanged with each other, proves that $X^{1/2} \subset Y$. The last statement of the lemma follows from (4.5) by density. □

Proposition 4.5. *Let $D(\mathbf{A})$ be the set of all $u \in H_0^1(\Omega)$ such that $Lu-\beta u \in L^2(\Omega)$. For $u \in D(\mathbf{A})$ define*

$$\mathbf{A}u = -Lu + \beta u.$$

Then $\mathbf{A}: D(\mathbf{A}) \to L^2(\Omega)$, selfadjoint in $X = L^2(\Omega)$ with $\operatorname{re}\sigma(\mathbf{A}) > 0$. Moreover, if X^α, $\alpha \geq 0$, is the family of fractional power spaces generated by $\mathbf{A}$, then $X^{1/2} = H_0^1(\Omega)$ and the scalar product on $X^{1/2}$ is identical to the scalar product $\langle \cdot\,,\,\cdot \rangle_1$ defined in Lemma 4.3.

Proof. Let $\langle \cdot\,,\,\cdot \rangle$ denote the scalar product of $L^2(\Omega)$. From (4.2) we conclude that

$$\langle -\mathbf{A}u, v\rangle = \langle v, -\mathbf{A}u\rangle, \quad u, v \in D(-\mathbf{A}). \tag{4.6}$$

Lemma 4.2 implies that

$$\langle -\mathbf{A}u, u\rangle \leq 0, \quad u \in D(-\mathbf{A}).$$

We have thus proved that $-A$ is symmetric and dissipative. We will now prove that $-\mathbf{A}$ is m-dissipative. To this end, we must prove that for every $\lambda \in \,]0,\infty[$ and every $g \in L^2(\Omega)$ there is a $u \in D(-\mathbf{A})$ such that

$$u + \lambda \mathbf{A}u = g.$$

Define the bilinear form $b: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$b(u,v) = \int_\Omega uv\,dx + \lambda \int_\Omega \left[\sum_{i,j=1}^N a_{ij}\partial_i u \partial_j v + \beta uv\right] dx, \quad u, v \in H_0^1(\Omega).$$

It follows from Hypothesis 1.2 and Lemma 4.2 that there are constants C and $c \in \,]0,\infty[$ such that, for $u, v \in H_0^1(\Omega)$

$$|b(u,v)| \leq C|u|_{H_0^1}|v|_{H_0^1} \quad \text{and} \quad b(u,u) \geq c|u|_{H_0^1}^2.$$

Thus Lax–Milgram theorem shows that for every $g \in L^2(\Omega)$ there is a $u \in H_0^1(\Omega)$ such that

$$b(u,v) = \langle g, v\rangle, \quad v \in H_0^1(\Omega).$$

In particular, $u + \lambda(-Lu + \beta u) = g$ in the distributional sense. It follows that $-Lu+\beta u \in L^2(\Omega)$ so $u \in D(\mathbf{A}) = D(-\mathbf{A})$ and $u+\lambda\mathbf{A}u = g$. Therefore, indeed, $-\mathbf{A}$ is m-dissipative. Now an application of the results of [9, Section 2.4] shows that $-\mathbf{A}$ is selfadjoint. Thus $\mathbf{A}$ is selfadjoint and $\operatorname{re}\sigma(\mathbf{A}) > 0$ by Lemma 4.2. To prove the last statement of the proposition, set $(X, \langle \cdot\,,\,\cdot \rangle_X) = (L^2(\Omega), \langle \cdot\,,\,\cdot \rangle)$ and $(Y, \langle \cdot\,,\,\cdot \rangle_Y) = (H_0^1(\Omega), \langle \cdot\,,\,\cdot \rangle_1$, where the scalar product $\langle \cdot\,,\,\cdot \rangle_1$ is defined in Lemma 4.3. Then Y is dense in X and the inclusion $Y \to X$ is continuous. Let

$\mathbf{B}: X \to X$ be the inverse of $\mathbf{A}$. Then for all $u \in X$, $\mathbf{B}u \in Y$ and (4.2) implies that for all $v \in Y$

$$\langle v, u \rangle_X = \langle v, \mathbf{B}u \rangle_Y.$$

Thus $\mathbf{B} = B$ where B is as in Lemma 4.4. Now Lemmas 4.4 and 4.3 imply the proposition. □

5. Some nonlinear estimates

In this section we assume that $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying Hypothesis 1.3.

Lemma 5.1. *Let X be a Banach space and $A: D(A) \subset X \to X$ be a sectorial operator with $\operatorname{re}\sigma(A) > 0$ generating the basic family X^α, $\alpha \in [0, \infty[$ of fractional power spaces. Suppose that $X^{1/2}$ is continuously included in $L^6(\Omega)$ and $X = X^0$ is continuously included in $L^2(\Omega)$. Then for every $p \in [2, 6[$ there is a $\overline{\beta} \in [0, 1/2[$ such that for all $\alpha \in]\overline{\beta}, 1[$ the space X^α is continuously included in $L^p(\Omega)$.*

Proof. If $u \in X^{1/2}$ then $u \in L^2(\Omega) \cap L^6(\Omega)$ so, by interpolation of Lebesgue spaces, $u \in L^p(\Omega)$ and

$$|u|_{L^p} \leq |u|_{L^2}^{\beta} |u|_{L^6}^{1-\beta}$$

where $\beta = (6-p)/(2p)$. Let B be the inclusion map from $X^{1/2}$ to $Y = L^p(\Omega)$. Since, by interpolation of fractional power spaces,

$$|u|_{X^{1/2}} \leq C |u|_{X^0}^{1/2} |u|_{X^1}^{1/2}, \quad u \in X^1$$

for some constant $C \in [0, \infty[$ we see that, for $u \in D(A) = X^1 \subset X^{1/2} = D(B)$,

$$\begin{aligned} |Bu|_Y &\leq C' |u|_{X^0}^{\beta} |u|_{X^{1/2}}^{1-\beta} \leq C' |u|_{X^0}^{\beta} (C |u|_{X^0}^{1/2} |u|_{X^1}^{1/2})^{1-\beta} \\ &= C'C |u|_{X^1}^{\overline{\beta}} |u|_{X^0}^{1-\overline{\beta}} = C'C |Au|_{X^0}^{\overline{\beta}} |u|_{X^0}^{1-\overline{\beta}} \end{aligned}$$

for some constant $C' \in [0, \infty[$, where $\overline{\beta} = (1/2)(1-\beta)$. By [13, p. 28, Exercise 11] we now obtain that for every $\alpha \in]\overline{\beta}, 1[$ the map $B \circ A^{-\alpha}$ is defined and continuous from X to Y. Thus $X^\alpha = R(A^{-\alpha})$ is continuously included in $L^p(\Omega)$, as claimed. □

Lemma 5.2. *Let X be a Banach space and $A: D(A) \subset X \to X$ be a sectorial operator with $\operatorname{re}\sigma(A) > 0$, generating the family X^α, $\alpha \in [0, \infty[$ of fractional power spaces. Suppose that $X^{1/2}$ is continuously included in $H_0^1(\Omega)$ and $X = X^0$ is continuously included in $L^2(\Omega)$. Let $w: \Omega \to \mathbb{R}$ be such that $w \in \mathcal{E}_\gamma$ for some $\gamma \in]0, 1[$. Then for all $\alpha \in](\gamma/2), 1]$, the mapping $u \mapsto |w|^{1/2} u$ defines a bounded linear fuction from X^α to $L^2(\Omega)$.*

Proof. It is easy to check that $w \in \mathcal{E}_\gamma$ if and only if there exists a constant C' such that

$$||w|^{1/2} u|_{L^2} \leq C' |u|_{H^1}^{\gamma} |u|_{L^2}^{1-\gamma}, \quad u \in H_0^1(\Omega).$$

Let B be the map from $X^{1/2}$ to $Y = L^2(\Omega)$ defined by the assignement $u \mapsto |w|^{1/2}u$. Since, by interpolation of fractional power spaces,

$$|u|_{X^{1/2}} \le C|u|_{X^0}^{1/2}|u|_{X^1}^{1/2}, \quad u \in X^1$$

for some constant $C \in [0,\infty[$ we see that, for $u \in D(A) = X^1 \subset X^{1/2} = D(B)$,

$$\begin{aligned} |Bu|_Y &\le C'|u|_{X^{1/2}}^{\gamma}|u|_{X^0}^{1-\gamma} \le C'|u|_{X^0}^{1-\gamma}(C|u|_{X^1}^{1/2}|u|_{X^0}^{1/2})^{\gamma} \\ &= C'C|u|_{X^1}^{\gamma/2}|u|_{X^0}^{1-\gamma/2} = C'C|Au|_{X^0}^{\gamma/2}|u|_{X^0}^{1-\gamma/2} \end{aligned}$$

for some constant $C' \in [0,\infty[$. By [13, p. 28, Exercise 11] we now obtain that for every $\alpha \in]\gamma/2, 1[$ the map $B \circ A^{-\alpha}$ is defined and continuous from X to Y. Thus B is a bounded linear map form $X^\alpha = R(A^{-\alpha})$ to $Y = L^2(\Omega)$, as claimed. □

Proposition 5.3. *Let $X = L^2(\Omega)$, $A: D(A) \subset X \to X$ a sectorial operator with* $\operatorname{re}\sigma(A) > 0$*, generating the family X^α, $\alpha \in [0,\infty[$ of fractional power spaces. Suppose that $X^{1/2}$ is continuously included in $H_0^1(\Omega)$. Let $q = (6/(\overline{\rho}+1))$ and $p = (q/(q-1))$. If $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$, then choose $\alpha \in]0,(1/2)[$ such that $\alpha > \max\{\gamma/2, (1-(6-p)/2p))/2\}$, so X^α is continuously included in $L^p(\Omega)$ (by Lemma 5.1) and the mapping $u \mapsto |a|^{1/2}u$ is bounded from X^α to X (by Lemma 5.2). If $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$ then let $\alpha = 0$.*

Let $F: \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x,u) = \int_0^u f(x,s)\,ds,$$

whenever $s \mapsto f(x,s)$ is continuous and $F(x,u) = 0$ otherwise. Then $\widehat{F}$ maps $X^{1/2}$ into $L^1(\Omega)$ and the operator $\widehat{F}: X^{1/2} \to L^1(\Omega)$ is Fréchet-differentiable with $D\widehat{F}(u).h = \widehat{f}(u) \cdot h$ for $u, h \in X^{1/2}$.

If $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$, then for every $u \in X^{1/2}$ the function $\mathbf{f}(u): X^\alpha \to \mathbb{R}$,

$$v \mapsto \int_\Omega f(x,u(x))v(x)\,dx,$$

is defined, linear and bounded, hence $\mathbf{f}(u) \in X^{-\alpha}$.

If $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$, then for every $u \in X^{1/2}$, let $\mathbf{f}(u) := \widehat{f}(u) \in X = X^{-\alpha}$.

For all $\overline{\rho} \in [2,4[$, the operator $\mathbf{f}: X^{1/2} \to X^{-\alpha}$ is Lipschitzian on bounded subsets of $X^{1/2}$.

Proof. If $u \in X^{1/2}$ then $u \in L^2(\Omega) \cap L^6(\Omega)$ so $u \in L^r(\Omega)$ for every $r \in [2,6]$. In particular, $\widehat{f}(u): \Omega \to \mathbb{R}$ and $\widehat{F}(u): \Omega \to \mathbb{R}$ are measurable.

Now, for $u \in X^{1/2}$ we have that

$$\begin{aligned} &\int_\Omega |F(x,u(x))|\,dx \\ &\quad \le \int_\Omega (|f(x,0)u(x)| + \overline{C}||a(x)|^{1/2}u(x)|^2/2 + \overline{C}|u(x)|^{\overline{\rho}+2}/(\overline{\rho}+2))\,dx < \infty \end{aligned}$$

as $\overline{\rho}+2 \in [2,6]$. Hence $\widehat{F}(u) \in L^1(\Omega)$. Moreover, for $u, h \in X^{1/2}$ we have

$$\begin{aligned}\int_\Omega &|f(x,u(x))h(x)|\,dx \\ &\le \int_\Omega (|f(x,0)h(x)| + \overline{C}|a(x)u(x)h(x)| + \overline{C}|u(x)|^{\overline{\rho}+1}|h(x)|\,dx) \\ &\le |\widehat{f}(0)|_{L^2}|h|_{L^2} + \overline{C}||a|^{1/2}u|_{L^2}||a|^{1/2}h|_{L^2} + \overline{C}|u|_{L^r}^{\overline{\rho}+1}|h|_{L^6},\end{aligned}$$

where $r = (6/5)(\overline{\rho}+1)$. It follows that for every $u \in X^{1/2}$ the map $h \mapsto \widehat{f}(u)\cdot h$ is linear and bounded from $X^{1/2}$ to $L^1(\Omega)$. Now, for $u, h \in X^{1/2}$,

$$\begin{aligned}&|\widehat{F}(u+h) - \widehat{F}(u) - \widehat{f}(u)\cdot h|_{L^1} \\ &= \int_\Omega |F(x,u(x)+h(x)) - F(x,u(x)) - f(x,u(x))h(x)|\,dx \\ &\le \overline{C}\int_\Omega (|a(x)h(x)h(x)| + \max(1,2^{\overline{\rho}-1})(|u(x)|^{\overline{\rho}} + |h(x)|^{\overline{\rho}})|h(x)||h(x)|)\,dx \\ &\le \overline{C}||a|^{1/2}h|_{L^2}^2 + \overline{C}\max(1,2^{\overline{\rho}-1})(|u|_{L^r}^{\overline{\rho}} + |h|_{L^r}^{\overline{\rho}})|h|_{L^6}^2,\end{aligned}$$

where $r = (6/4)\overline{\rho}$. This shows that the operator $\widehat{F}\colon X^{1/2} \to L^1(\Omega)$ is Fréchet-differentiable with $D\widehat{F}(u).h = \widehat{f}(u)\cdot h$ for $u, h \in X^{1/2}$.

Now suppose $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$. The fact that X^α is continuously embedded in $L^2(\Omega)$ and in $L^p(\Omega)$ (with a common embedding constant $C \in [0,\infty[$) implies that, for all $v \in X^\alpha$,

$$\begin{aligned}\int_\Omega &|f(x,u(x))v(x)|\,dx \\ &\le \int_\Omega (|f(x,0)v(x)| + \overline{C}|a(x)u(x)v(x)| + \overline{C}|u(x)|^{\overline{\rho}+1}|v(x)|)\,dx \\ &\le |\widehat{f}(0)|_{L^2}|v|_{L^2} + \overline{C}||a|^{1/2}u|_{L^2}||a|^{1/2}v|_{L^2} + |u|_{L^6}^{\overline{\rho}+1}|v|_{L^p} \\ &\le C(|\widehat{f}(0)|_{L^2} + \overline{C}||a|^{1/2}u|_{L^2} + |u|_{L^6}^{\overline{\rho}+1})|v|_{X^\alpha}.\end{aligned}$$

Thus, indeed, the function $\mathbf{f}(u)\colon X^\alpha \to \mathbb{R}$, $v \mapsto \int_\Omega f(x,u(x))v(x)\,dx$, is defined, linear and bounded, hence $\mathbf{f}(u) \in X^{-\alpha}$. Similarly, we obtain for $u, h \in X^{1/2}$ and $v \in X^\alpha$,

$$\begin{aligned}&\int_\Omega |(f(x,u(x)+h(x)) - f(x,u(x)))v(x)|\,dx \\ &\le \overline{C}\int_\Omega (|a(x)h(x)v(x)| + \max(1,2^{\overline{\rho}-1})(|u(x)|^{\overline{\rho}} + |h(x)|^{\overline{\rho}})|h(x)||v(x)|)\,dx \\ &\le \overline{C}||a|^{1/2}h|_{L^2}||a|^{1/2}v|_{L^2} + \overline{C}(|u|_{L^6}^{\overline{\rho}} + |h|_{L^6}^{\overline{\rho}})|h|_{L^6}|v|_{L^p} \\ &\le C(\overline{C}||a|^{1/2}h|_{L^2} + \overline{C}(|u|_{L^6}^{\overline{\rho}} + |h|_{L^6}^{\overline{\rho}})|h|_{L^6})|v|_{X^\alpha}.\end{aligned}$$

This shows that the operator $\mathbf{f}\colon X^{1/2} \to X^{-\alpha}$ is defined and Lipschitzian on bounded subsets of $X^{1/2}$.

Now suppose $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$. Then similar arguments show that

$$|\widehat{f}(u)|_{L^2} \leq |\widehat{f}(0)|_{L^2} + \overline{C}(|au|_{L^2} + |u|^{\overline{\rho}+1}_{L^{2(\overline{\rho}+1)}}),$$

$$|\widehat{f}(u+h) - \widehat{f}(u)|_{L^2} \\ \leq \overline{C}|ah|_{L^2} + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}} + |h|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}})|h|_{L^{2(\overline{\rho}+1)}},$$

and

$$|\widehat{F}(u+h) - \widehat{F}(u) - \widehat{f}(u)h|_{L^1} \\ \leq (\overline{C}|ah|_{L^2} + \overline{C}\max(1, 2^{\overline{\rho}-1})(|u|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}} + |h|^{\overline{\rho}}_{L^{2(\overline{\rho}+1)}})|h|_{L^{2(\overline{\rho}+1)}})|h|_{L^2}.$$

Again this shows that the operator $\mathbf{f}\colon X^{1/2} \to X^{-\alpha} = X$ is defined and Lipschitzian on bounded subsets of $X^{1/2}$. □

6. Tail estimates and the existence of attractors

Let $\mathbf{A}$ be the operator defined in Proposition 4.5 and X^α, $\alpha \in \mathbb{R}$, $\mathbf{A}_{(\alpha)}$, $\alpha \in \mathbb{R}$ and $\widetilde{\mathbf{A}}_{(-\alpha)}$, $\alpha \in]0, \infty[$, be the spaces and the operators defined in Proposition 2.2 with respect to $A = \mathbf{A}$.

If $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$, then, by what we have proved so far, the parabolic equation

$$\dot{u} = -\mathbf{A}u + \mathbf{f}(u)$$

defines a local semiflow π on $X^{1/2}$.

If $\overline{\rho} \in]2, 4[$ or $a^2 \notin \mathcal{E}_1$, then choose $\alpha \in]0, (1/2)[$ as in Proposition 5.3. Then the parabolic equation

$$\dot{\widetilde{u}} = -\widetilde{\mathbf{A}}_{(-\alpha)}\widetilde{u} + \mathbf{f}(\varphi^{-1}_{(1/2),-\alpha}\widetilde{u})$$

defines a local semiflow $\widetilde{\pi}$ on $\varphi_{(1/2),-\alpha}[X^{1/2}]$.

Let π be the local semiflow on $X^{1/2}$ which is conjugate to $\widetilde{\pi}$ via the conjugation $\varphi_{(1/2),-\alpha}\colon X^{1/2} \to \varphi_{(1/2),-\alpha}[X^{1/2}]$.

While the local semiflow $\widetilde{\pi}$ depends on α, it is not difficult to prove that

PROPOSITION 6.1. *If $\overline{\rho} \in]2, 4[$ or $a^2 \notin \mathcal{E}_1$, the local semiflow π is independent of the choice of α.*

From the definition of π we thus obtain, choosing $\alpha = 0$ if $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$, that

PROPOSITION 6.2. *Whenever $T \in]0, \infty]$ and $u\colon [0, T[\to X^{1/2}$ is a solution of π, then u is continuous on $[0, T[$, differentiable (into $X^{1/2}$) on $]0, T[$ and, for $t \in]0, T[$*

$$\varphi_{0,-\alpha}(\dot{u}(t)) = -\mathbf{A}_{1-\alpha}u(t) + \mathbf{f}(u(t)),$$

where $\dot{u}(t) := \partial(u; X^{1/2})(t) = \partial(u; X^0)(t)$.

Proposition 6.3. *Define the function* $\mathcal{L}\colon H^1_0(\Omega)\to\mathbb{R}$ *by*

$$\mathcal{L}(u)=\frac{1}{2}\langle u,u\rangle_{X^{1/2}}-\int_\Omega F(x,u(x))\,dx,\quad u\in H^1_0(\Omega).$$

Let $T\in\,]0,\infty]$ *and* $u\colon[0,T[\,\to H^1_0(\Omega)=X^{1/2}$ *be a solution of* π. *Then the function* $\mathcal{L}\circ u$ *is continuous on* $[0,T[$, *differentiable on* $]0,T[$ *and, for* $t\in\,]0,T[$,

$$(\mathcal{L}\circ u)'(t)=-|\dot u(t)|_{L^2}. \tag{6.1}$$

Proof. By Proposition 5.3 the function $\mathcal{L}$ is Fréchet differentiable and

$$D\mathcal{L}(w).v=\langle w,v\rangle_{X^{1/2}}-\int_\Omega \widehat f(w(x))v(x)\,dx,\quad w,v\in X^{1/2}.$$

Thus, by Proposition 6.2, $\mathcal{L}\circ u$ is continuous on $[0,T[$, differentiable on $]0,T[$ and, for $t\in\,]0,T[$,

$$(\mathcal{L}\circ u)'(t)=\langle u(t),\dot u(t)\rangle_{X^{1/2}}-\int_\Omega \widehat f(u(t)(x))\dot u(t)(x)\,dx.$$

If $\overline\rho>2$ or $a^2\notin\mathcal{E}_1$, then the last statement of Proposition 2.2 implies that

$$\langle u(t),\dot u(t)\rangle_{X^{1/2}}=(\mathbf{A}_{1-\alpha}u(t)).\dot u(t)$$

and so, by Proposition 5.3 and 6.2,

$$\begin{aligned}(\mathcal{L}\circ u)'(t)&=(\mathbf{A}_{1-\alpha}u(t)).\dot u(t)-\mathbf{f}(u(t)).\dot u(t)\\&=(\mathbf{A}_{1-\alpha}u(t)-\mathbf{f}(u(t))).\dot u(t)=(-\varphi_{0,-\alpha}\dot u(t)).\dot u(t)=-\langle \dot u(t),\dot u(t)\rangle_X\end{aligned}$$

This proves formula (6.1) when $\overline\rho\in\,]2,4[$ or $a^2\notin\mathcal{E}_1$. A similar but simpler argument proves (6.1) when $\overline\rho=2$ and $a^2\in\mathcal{E}_1$. □

Corollary 6.4. π *is a global semiflow on* $Y=H^1_0(\Omega)$ *and it satisfies properties* (b)–(d) *of Proposition* 2.1.

Proof. Proposition 6.3 together with Proposition 5.3 implies that $\mathcal{L}$ is continuous and nonincreasing along solutions of π. Thus, for all $u_0\in H^1_0(\Omega)$, writing $u(t)=u_0\pi t$ for $t\in[0,\omega_{u_0}[$, we obtain

$$\begin{aligned}\frac{1}{2}&\langle u(t),u(t)\rangle_{X^{1/2}}\\&\le\frac{1}{2}\langle u_0,u_0\rangle_{X^{1/2}}-\int_\Omega F(x,u_0(x))\,dx+\int_\Omega F(x,u(t)(x))\,dx\\&\le\frac{1}{2}\langle u_0,u_0\rangle_{X^{1/2}}-\int_\Omega F(x,u_0(x))\,dx+\int_\Omega c(x)\,dx,\quad t\in[0,\omega_{u_0}[\,.\end{aligned}\tag{6.2}$$

It follows that every solution of π is bounded in Y and so, as π does not explode in bounded subsets of Y, π is a global semiflow. Proposition 5.3 also implies that every bounded set B is π-ultimately bounded, with $t_B=0$. Moreover, $\mathcal{L}(u)\ge-\int_\Omega c(x)\,dx$ for every $u\in Y$ so $\mathcal{L}$ is bounded from below. If $u\colon[0,\infty[\,\to Y$

is a solution of π along which $\mathcal{L}$ is constant, then, by Proposition 6.3, $\dot{u}(t) = 0$ for all $t \in]0, \infty[$ and this clearly implies that u is a constant solution so $u(0)$ is an equilibrium of π. Let u_0 be an equilibrium of π. Suppose first that $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$. Then $\mathbf{A}_{(1-\alpha)}u_0$ is defined and $\mathbf{A}_{(1-\alpha)}u_0 = \mathbf{f}(u_0)$. It follows that

$$\begin{aligned}\langle u_0, u_0\rangle_{X^{1/2}} &= (\mathbf{A}_{(1-\alpha)}u_0).u_0 = (\mathbf{f}(u_0)).u_0 \\ &= \int_\Omega f(x, u_0(x))u_0(x)\,dx \le \int_\Omega c(x)\,dx\end{aligned}$$

so the set of all equilibria of π is bounded in Y. A similar but simpler argument shows that, if $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$, then again the set of all equilibria of π is bounded in Y. □

We can now state our basic result on tail estimates.

Theorem 6.5. *$\overline{\vartheta}\colon \mathbb{R} \to [0,1]$ be a function of class C^1 with $\overline{\vartheta}(s) = 0$ for $s \in]-\infty, 1]$ and $\overline{\vartheta}(s) = 1$ for $s \in [2, \infty[$. Let $\vartheta := \overline{\vartheta}^2$. For $k \in \mathbb{N}$ let the functions $\overline{\vartheta}_k\colon \mathbb{R}^N \to \mathbb{R}$ and $\vartheta_k\colon \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$\overline{\vartheta}_k(x) = \overline{\vartheta}(|x|^2/k^2) \quad \textit{and} \quad \vartheta_k(x) = \vartheta(|x|^2/k^2), \quad x \in \mathbb{R}^N.$$

Set $C_\vartheta = 2\sqrt{2}\sup_{y\in\mathbb{R}} |\vartheta'(y)|$ and $C_{\overline{\vartheta}} = 2\sqrt{2}\sup_{y\in\mathbb{R}} |\overline{\vartheta}'(y)|$. Let $\kappa \in]0, \lambda_1[$ be arbitrary, where λ_1 is defined in Hypothesis 1.2. *For every $k \in \mathbb{N}$ let*

$$b_k = \max(a_1 C_{\overline{\vartheta}}^2 k^{-2}, 2a_1 C_{\overline{\vartheta}} k^{-1}, a_1 C_\vartheta k^{-1})$$

and $c_k = \int_\Omega \vartheta_k(x)c(x)\,dx$. Then whenever $R \in [0, \infty[$, $\tau \in]0, \infty[$ and $u\colon [0, \infty[\to H_0^1(\Omega)$ is a solution of π such that $|u(t)|_{H_0^1} \le R$ for all $t \in [0, \tau]$, then, for every $t \in [0, \tau]$,

$$\int_\Omega \vartheta_k(x)|u(x)|^2\,dx \le R^2 e^{-2\kappa t} + (b_k R^2 + c_k)/\kappa. \tag{6.3}$$

Proof. Notice that, for all $x \in \mathbb{R}^N$, $\nabla\vartheta_k(x) = (2/k^2)\vartheta'(|x|^2/k^2)x$ and $\nabla\overline{\vartheta}_k(x) = (2/k^2)\overline{\vartheta}'(|x|^2/k^2)x$. It follows that that

$$\sup_{x\in\Omega} |\nabla\vartheta_k(x)| \le C_\vartheta/k \quad \text{and} \quad \sup_{x\in\Omega} |\nabla\overline{\vartheta}_k(x)| \le C_{\overline{\vartheta}}/k. \tag{6.4}$$

Assume first that $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$. We claim that, for all $v \in H_0^1(\Omega) = X^{1/2}$,

$$\begin{aligned}&(-\mathbf{A}_{(1-\alpha)}v).(\vartheta_k v) + \kappa\int_\Omega \vartheta_k(x)|v(x)|^2\,dx \\ &\quad\le \int_\Omega ((a_1 C_{\overline{\vartheta}}^2 k^{-2})|v|^2 + (2a_1 C_{\overline{\vartheta}} k^{-1})|v||\nabla v| + (a_1 C_\vartheta k^{-1})|v||\nabla v|)\,dx.\end{aligned} \tag{6.5}$$

To prove (6.5), we may suppose that $v \in X^1$ since the general case follows by the density of X^1 in $H_0^1(\Omega)$. Now, if $v \in X^1$, then

$$(-\mathbf{A}_{(1-\alpha)}v).(\vartheta_k v) = (-\varphi_{0,-\alpha}\mathbf{A}v).(\vartheta_k v) = \langle -\mathbf{A}v, \vartheta_k v\rangle_X$$

so, using (6.4), we obtain

$$
\begin{aligned}
&(-\mathbf{A}_{(1-\alpha)}v).(\vartheta_k v) + \kappa \int_\Omega \vartheta_k(x)|v(x)|^2\,dx \\
&= \int_\Omega \vartheta_k v(x)(-\mathbf{A}v)(x)\,dx + \kappa \int_\Omega \vartheta_k |v(x)|^2\,dx \\
&= \int_\Omega (-(A\nabla v)(x)\cdot\nabla(\vartheta_k v)(x) - \vartheta_k\beta(x)|v(x)|^2)\,dx + \kappa\int_\Omega \vartheta_k|v|^2\,dx \\
&= \int_\Omega (-(A\nabla(\overline{\vartheta}_k v))\cdot\nabla(\overline{\vartheta}_k v) - (\beta(x)-\kappa)|\overline{\vartheta}_k v|^2)\,dx \\
&\quad + \int_\Omega (|v|^2(A\nabla\overline{\vartheta}_k)\cdot\nabla\overline{\vartheta}_k + 2v\overline{\vartheta}_k(A\nabla\overline{\vartheta}_k)\cdot\nabla v - v(A\nabla\vartheta_k)\cdot\nabla v)\,dx \\
&\le \int_\Omega ((a_1C_{\overline{\vartheta}}^2k^{-2})|v|^2 + (2a_1C_{\overline{\vartheta}}k^{-1})|v||\nabla v| + (a_1C_\vartheta k^{-1})|v||\nabla v|)dx.
\end{aligned}
$$

This proves (6.5).

For $k\in\mathbb{N}$ define the function $V_k\colon H_0^1(\Omega)\to\mathbb{R}$ by

$$V_k(u) = (1/2)\int_\Omega \vartheta_k(x)|u(x)|^2\,dx, \quad u\in H_0^1(\Omega).$$

Then V_k is Fréchet differentiable and

$$DV_k(u)v = \int_\Omega \vartheta_k(x)u(x)v(x)\,dx, \quad u,v\in H_0^1(\Omega).$$

Let $u\colon[0,\infty[\,\to H_0^1(\Omega)$ be a solution of π. By Proposition 6.2, $V_k\circ u$ is differentiable on $]0,\infty[$ and, for $t\in\,]0,\infty[$,

$$(V_k\circ u)'(t) = \int_\Omega \vartheta_k(x)u(t)(x)\dot{u}(t)(x)\,dx = \langle \dot{u}(t), \vartheta_k u(t)\rangle_X.$$

Since $\vartheta_k u(t)\in H_0^1(\Omega) = X^{1/2}\subset X^\alpha$, it follows that, for $t\in\,]0,\infty[$,

$$\langle \dot{u}(t),\vartheta_k u(t)\rangle_X = (\varphi_{0,-\alpha}\dot{u}(t)).(\vartheta_k u(t)) = (-\mathbf{A}_{(1-\alpha)}u(t) + \mathbf{f}(u(t))).(\vartheta_k u(t))$$

so, using (6.5), we obtain

$$
\begin{aligned}
&(V_k\circ u)'(t) + 2\kappa(V_k\circ u)(t) = (-\mathbf{A}_{(1-\alpha)}u(t)).(\vartheta_k u(t)) \\
&\qquad + \int_\Omega \widehat{f}(u(t))(x)(\vartheta_k u(t))(x)\,dx + \kappa\int_\Omega \vartheta_k(x)|u(t)(x)|^2\,dx \\
&\quad \le (-\mathbf{A}_{(1-\alpha)}u(t)).(\vartheta_k u(t)) + \kappa\int_\Omega \vartheta_k(x)|u(t)(x)|^2\,dx + \int_\Omega \vartheta_k(x)c(x)\,dx, \\
&\quad \le \int_\Omega ((a_1C_{\overline{\vartheta}}^2k^{-2})|u(t)|^2 + (2a_1C_{\overline{\vartheta}}k^{-1})|u(t)||\nabla u(t)| \\
&\qquad + (a_1C_\vartheta k^{-1})|u(t)||\nabla u(t)|)\,dx + \int_\Omega \vartheta_k(x)c(x)\,dx
\end{aligned}
$$

Thus, whenever $R\in[0,\infty[$, $\tau\in\,]0,\infty[$ and $|u(t)|_{H_0^1}\le R$ for all $t\in[0,\tau]$ then

$$(V_k\circ u)'(t) + 2\kappa(V_k\circ u)(t) \le b_kR^2 + c_k, \quad t\in\,]0,\tau]\,.$$

This implies that

$$V_k(u(t)) \le e^{-2\kappa t} V_k(u(0)) + (b_k R^2 + c_k)/(2\kappa), \quad t \in [0,\tau]\,.$$

This clearly implies that (6.3) holds for every $t \in [0,\tau]$. This proves the theorem when $\overline{\rho} \in \,]2,4[$ or $a^2 \notin \mathcal{E}_1$. Similar, but simpler arguments prove the theorem when $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$. □

We can now prove

Theorem 6.6. *The semiflow π is asymptotically compact.*

Proof. Let B be an ultimately bounded subset of $Y = H_0^1(\Omega)$, $(v_n)_n$ be a sequence in B and $(t_n)_n$ be a sequence in $[0,\infty[$ with $t_n \to \infty$ as $n \to \infty$. We must show that there is a subsequence of $(v_n \pi t_n)_n$ which converges in Y.

There is a $t_B \in [0,\infty[$ and an $R \in [0,\infty[$ such that $|v\pi t|_{H_0^1} \le R$ for all $v \in B$ and $t \in [t_B,\infty[$. We may assume without loss of generality that $t_n \ge t_B + 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $s_n = t_n - t_B$ and $u_n\colon [0,\infty[\, \to H_0^1(\Omega)$ be defined by $u_n(s) = v_n \pi (t_B + s)$ for $s \in [0,\infty[$. Then, for $n \in \mathbb{N}$, $\tau_n := s_n - 1 \ge 0$ and u_n is a solution of π with $|u_n(s)|_{H_0^1} \le R$ for all $s \in [0,\infty[$ and $u_n(s_n) = v_n \pi t_n$.

Suppose that $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$. We claim that

(6.6) There is a strictly increasing sequence $(n_m)_m$ in $\mathbb{N}$ and $v \in H_0^1(\Omega)$ such that $(u_{n_m}(\tau_{n_m}))_m$ converges to v in $L^2(\Omega)$.

Let $\widetilde{u}_n = \varphi_{(1/2),-\alpha} \circ u_n$, $n \in \mathbb{N}$. Then $\widetilde{u}_n$ is a solution of $\widetilde{\pi}$ and $(\widetilde{u}_{n_m}(\tau_{n_m}))_m$ converges to $\widetilde{v} := \varphi_{(1/2),-\alpha} v$ in $\varphi_{0,-\alpha}[X]$. Thus $(\widetilde{u}_{n_m}(\tau_{n_m}))_m$ converges to $\widetilde{v}$ in $X^{-\alpha}$. Using Proposition 3.1 (with appropriately modified notation) we see that $(\widetilde{u}_{n_m}(\tau_{n_m}+1))_m$ converges to $\widetilde{v}\widetilde{\pi}1$ in $\varphi_{1/2,-\alpha}[X^{1/2}]$. This means that $(u_{n_m}(s_{n_m}))_m$ converges to $v\pi 1$ in $X^{1/2}$ and completes the proof of the theorem.

Thus we only have to prove (6.6). Let β_{K} be the Kuratowski measure of noncompactness on $X = L^2(\Omega)$. Then for every $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$,

$$\begin{aligned}\beta_{\mathrm{K}}\{u_n(\tau_n) \mid n \in \mathbb{N}\} &\le \beta_{\mathrm{K}}\{(1-\vartheta_k)u_n(\tau_n) \mid n \in \mathbb{N}\} + \beta_{\mathrm{K}}\{\vartheta_k u_n(\tau_n) \mid n \in \mathbb{N}\}\\ &= \beta_{\mathrm{K}}\{(1-\vartheta_k)u_n(\tau_n) \mid n \in \mathbb{N}\} + \beta_{\mathrm{K}}\{\vartheta_k u_n(\tau_n) \mid n \ge n_0\}.\end{aligned}$$

By Theorem 6.5 and the fact that $\tau_n \to \infty$ as $n \to \infty$, for every $\varepsilon \in \,]0,\infty[$ there are a $k \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that $|\vartheta_k u_n|_{L^2} < \varepsilon$ for all $n \ge n_0$. Thus, for every $\varepsilon \in \,]0,\infty[$, there is a $k \in \mathbb{N}$ such that

$$\beta_{\mathrm{K}}\{u_n(\tau_n) \mid n \in \mathbb{N}\} \le \beta_{\mathrm{K}}\{(1-\vartheta_k)u_n(\tau_n) \mid n \in \mathbb{N}\} + \varepsilon.$$

Since $1-\vartheta_k \in C_0^1(\mathbb{R}^N)$, the map $u \mapsto (1-\vartheta_k)u$ is compact from $H_0^1(\Omega)$ to $L^2(\Omega)$ and so

$$\beta_{\mathrm{K}}\{(1-\vartheta_k)u_n(\tau_n) \mid n \in \mathbb{N}\} = 0.$$

We therefore obtain that

$$\beta_{\mathrm{K}}\{u_n(\tau_n) \mid n \in \mathbb{N}\} = 0$$

and so there is a strictly increasing sequence $(n_m)_m$ in $\mathbb{N}$ and a $v \in L^2(\Omega)$ such that $(u_{n_m}(\tau_{n_m}))_m$ converges to v in $L^2(\Omega)$. Since $(u_{n_m}(\tau_{n_m}))_m$ is bounded in $H_0^1(\Omega)$, by taking a further subsequence if necessary, we may assume that $(u_{n_m}(\tau_{n_m}))_m$ converges weakly in $H_0^1(\Omega)$ (and hence in $L^2(\Omega)$) to some $w \in H_0^1(\Omega)$. Thus $w = v$ and (6.6) follows. This proves the theorem when $\overline{\rho} \in]2,4[$ or $a^2 \notin \mathcal{E}_1$. Similar (and simpler) arguments establish the proof when $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$. □

We may now prove the main result of this paper.

Proof of Theorem 1.4. In view of Corollary 6.4 and Theorem 6.6, Proposition 2.1 implies part (a) of the theorem. If $\overline{\rho} = 2$ and $a^2 \in \mathcal{E}_1$, then, noting that $\alpha = 0$ in this case, Theorem 3.2 implies part (b) of the theorem. If $\overline{\rho} > 2$ or $a^2 \notin \mathcal{E}_1$, then, by part (a), $\widetilde{\pi}$ is a global semiflow on $\varphi_{(1/2),-\alpha}[X^{1/2}]$ and it has a global attractor $\widetilde{\mathcal{A}} := \varphi_{(1/2),-\alpha}[\mathcal{A}]$. Thus, by Theorem 3.2, $\widetilde{\mathcal{A}}$ lies in $\varphi_{1-\alpha,-\alpha}[X^{1-\alpha}]$ and is compact in $\varphi_{1-\alpha,-\alpha}[X^{1-\alpha}]$. This implies part (b) of the theorem in this case. The proof is complete. □

7. Appendix

We will prove Proposition 2.2. We require the following simple and known result.

Lemma 7.1. *Let E_k and F_k, $k \in \{1,2\}$, be real or complex normed spaces with E_2 and F_2 complete. Let $e: E_1 \to E_2$, $f: F_1 \to F_2$ and $B_1: E_1 \to F_1$ be linear isometries with B_1 bijective. If $e[E_1]$ is dense in E_2 and $f[F_1]$ is dense in F_2, then there is a unique continuous map $B_2: E_2 \to F_2$ such that $B_2 \circ e = f \circ B_1$. B_2 is a linear bijective isometry.*

Proof of Proposition 2.2. It is immediate that for all $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$ the map $\varphi_{\beta,\alpha}: X^\beta \to X^\alpha$ is defined, linear and bounded. The density of X^δ in X^γ for all $\gamma, \delta \in]0,\infty[$ with $\delta > \gamma$ implies that

(7.1) for all $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$ the map $\varphi_{\beta,\alpha}$ is injective and $\varphi_{\beta,\alpha}[X^\beta]$ is dense in X^α.

Now formula (2.3) and an integration using Hölder inequality shows that

(7.2) $|A^\beta x|_X \leq |A^\alpha x|_X^{1-\theta} |A^\gamma x|_X^\theta$, for all $\alpha, \gamma \in \mathbb{R}$, $\delta \in [0,\infty[$, $\theta \in [0,1]$ and $x \in X^\delta$ with $\alpha \leq \gamma \leq \delta$ and $\beta = (1-\theta)\alpha + \theta\gamma$.

Formula (2.2) and the definition of the maps $\varphi_{\beta,\alpha}$ implies that

(7.3) For all $\alpha \in]0,\infty[$, $\beta \in [0,\infty[$ and $x \in X$ $|\varphi_{\beta,\beta-\alpha} A^{-\beta} x|_{X^{\beta-\alpha}} = |A^{-\alpha} x|_X$.

Using this formula (for $\beta = 0$) we see that

(7.4) $|\varphi_{\delta,\alpha}x|_{X^\alpha} = |A^\alpha x|_X$, for all $\alpha \in \mathbb{R}$, $\delta \in [0,\infty[$ and $x \in X^\delta$ with $\delta \geq \alpha$.

Now (7.1), (7.2) and (7.4) imply the interpolation inequality

(7.5) $|\varphi_{\gamma,\beta}x|_{X^\beta} \leq |\varphi_{\gamma,\alpha}x|^{1-\theta}_{X^\alpha}|x|^\theta_{X^\gamma}$, for all $\alpha, \gamma \in \mathbb{R}$, $\theta \in [0,1]$ and $x \in X^\gamma$ with $\alpha \leq \gamma$ and $\beta = (1-\theta)\alpha + \theta\gamma$.

A straightforward proof by cases also shows that

$$\varphi_{\alpha,\alpha} = \mathrm{id}_{X^\alpha}, \quad \alpha \in \mathbb{R}, \tag{7.6}$$

$$\varphi_{\gamma,\alpha} = \varphi_{\beta,\alpha} \circ \varphi_{\gamma,\beta}, \quad \alpha, \beta, \gamma \in \mathbb{R},\ \gamma \geq \beta \geq \alpha. \tag{7.7}$$

For all $\alpha, \beta \in [0,\infty[$, (2.2) implies that $A^{-\beta}[X^\alpha] = X^{\beta+\alpha}$ and for all $y \in X^\alpha$

$$|A^{-\beta}y|_{X^{\beta+\alpha}} = |A^{-(\beta+\alpha)}A^\alpha y|_{X^{\beta+\alpha}} = |A^\alpha y|_X = |y|_{X^\alpha}.$$

This implies that

(7.8) For all $\alpha, \beta \in [0,\infty[$, $A^{-\beta}_{(\alpha)} := A^{-\beta}|X^\alpha\colon X^\alpha \to X^{\beta+\alpha}$ is a linear bijective isometry.

For every $\alpha \in]0,\infty[$, $\beta \in [0,\infty[$ formula (2.2) implies that $\varphi_{\beta,\beta-\alpha}$ is an isometry from the space X^β endowed with the (in general incomplete) norm $x \mapsto n_{\beta-\alpha}(x) := |A^{\beta-\alpha}x|_X$ to the Hilbert space $X^{\beta-\alpha}$. The same formula with $\beta = 0$ shows that $\varphi_{0,-\alpha}$ is an isometry from the space $X = X^0$ endowed with the (in general incomplete) norm $x \mapsto n_{-\alpha}(x) = |A^{-\alpha}x|_X$ to the Hilbert space $X^{-\alpha}$. Since $A^{-\beta}$ is a bijective isometry from X endowed with the norm $n_{-\alpha}$ to $X^{\beta-\alpha}$ endowed with the norm $n_{\beta-\alpha}$ it follows from Lemma 7.1 that

(7.9) For every $\alpha \in]0,\infty[$, $\beta \in [0,\infty[$ there is a unique continuous map $A^{-\beta}_{(-\alpha)}\colon X^{-\alpha} \to X^{\beta-\alpha}$ with $A^{-\beta}_{(-\alpha)} \circ \varphi_{0,-\alpha} = \varphi_{\beta,\beta-\alpha} \circ A^{-\beta}$. $A^{-\beta}_{(-\alpha)}$ is a linear bijective isometry.

Now, for $\alpha \in \mathbb{R}$ and $\beta \in]0,\infty[$ we may define the map $A^\beta_{(\alpha)}\colon X^\alpha \to X^{-\beta+\alpha}$ by

$$A^\beta_{(\alpha)} = (A^{-\beta}_{(-\beta+\alpha)})^{-1}. \tag{7.10}$$

We also set $A_{(\alpha)} := A^1_{(\alpha)}$. The above definitions and simple density arguments show that:

(7.11) For all $\gamma, \gamma' \in \mathbb{R}$ with $\gamma > \gamma'$ and all $\beta \in \mathbb{R}$, $\varphi_{-\beta+\gamma,-\beta+\gamma'} \circ A^\beta_{(\gamma)} = A^\beta_{(\gamma')} \circ \varphi_{\gamma,\gamma'}$.

(7.12) For all α, β and $\gamma \in \mathbb{R}$, $A^\beta_{(-\gamma+\alpha)} \circ A^\gamma_{(\alpha)} = A^{\beta+\gamma}_{(\alpha)}$.

Since for $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$ the map $\varphi_{\beta,\alpha}$ is bijective onto its range, we may, for all $\alpha, \beta \in]0, \infty[$ define the map

$$\widetilde{A}^{\beta}_{(-\alpha)} := A^{\beta}_{(\beta-\alpha)} \circ \varphi^{-1}_{\beta-\alpha,-\alpha} \colon \varphi_{\beta-\alpha,-\alpha}[X^{\beta-\alpha}] \subset X^{-\alpha} \to X^{-\alpha}.$$

We also set $\widetilde{A}_{(-\alpha)} := \widetilde{A}^{1}_{(-\alpha)}$. It follows that $\widetilde{A}^{\beta}_{(-\alpha)}$ is bijective and its inverse is $\widetilde{A}^{-\beta}_{(-\alpha)} := \varphi_{\beta-\alpha,-\alpha} \circ A^{-\beta}_{(-\alpha)}$. We claim that $\widetilde{A}^{\beta}_{(\beta-\alpha)}$ is symmetric with respect to the scalar product on $X^{-\alpha}$. First notice that

$$R^{-1}_{\alpha}\varphi_{0,-\alpha}x = A^{-2\alpha}x, \quad \alpha \in]0, \infty[, \ x \in X. \tag{7.13}$$

This implies that, for u and $v \in X$

$$\langle \varphi_{0,-\alpha}u, \varphi_{0,-\alpha}v \rangle_{X^{-\alpha}} = \langle A^{-2\alpha}u, A^{-2\alpha}v \rangle_{X^{\alpha}} = \langle A^{-\alpha}u, A^{-\alpha}v \rangle_{X}$$

so

$$\langle \varphi_{0,-\alpha}u, \varphi_{0,-\alpha}v \rangle_{X^{-\alpha}} = \langle A^{-\alpha}u, A^{-\alpha}v \rangle_{X}, \quad \alpha \in]0, \infty[, \ u, v \in X. \tag{7.14}$$

Since $A^{\beta}_{(\beta-\alpha)}$ is the inverse of $A^{-\beta}_{(-\alpha)}$ and A^{β} is the inverse of $A^{-\beta}$, we obtain from (7.9)

$$A^{\beta}_{(\beta-\alpha)} \circ \varphi_{\beta,\beta-\alpha} = \varphi_{0,-\alpha} \circ A^{\beta}. \tag{7.15}$$

Hence, for $u \in X^{\beta}$, we obtain from (7.15),

$$\widetilde{A}^{\beta}_{(\beta-\alpha)}\varphi_{\beta-\alpha,-\alpha}\varphi_{\beta,\beta-\alpha}u = A^{\beta}_{(\beta-\alpha)}\varphi_{\beta,\beta-\alpha}u = \varphi_{0,-\alpha}A^{\beta}u.$$

Using (7.14), we thus obtain, for $u, v \in X^{\beta}$, $x = \varphi_{\beta-\alpha,-\alpha}\varphi_{\beta,\beta-\alpha}u$ and $y = \varphi_{\beta-\alpha,-\alpha}\varphi_{\beta,\beta-\alpha}v$

$$\langle \widetilde{A}^{\beta}_{(\beta-\alpha)}x, y \rangle_{X^{-\alpha}} = \langle A^{-\alpha}u, A^{-\alpha}A^{\beta}v \rangle_{X} = \langle A^{-\alpha}u, A^{\beta}A^{-\alpha}v \rangle_{X}$$

and in particular

$$\langle \widetilde{A}^{\beta}_{(\beta-\alpha)}x, x \rangle_{X^{-\alpha}} = \langle A^{-\alpha}u, A^{\beta}A^{-\alpha}u \rangle_{X} \tag{7.16}$$

for $u \in X^{\beta}$ and $x = \varphi_{\beta-\alpha,-\alpha}\varphi_{\beta,\beta-\alpha}u$. Similarly,

$$\langle x, \widetilde{A}^{\beta}_{(\beta-\alpha)}y \rangle_{X^{-\alpha}} = \langle A^{\beta}A^{-\alpha}u, A^{-\alpha}v \rangle_{X}.$$

Now the symmetry of A^{β} relative to the scalar product on X shows that

$$\langle \widetilde{A}^{\beta}_{(\beta-\alpha)}x, y \rangle_{X^{-\alpha}} = \langle x, \widetilde{A}^{\beta}_{(\beta-\alpha)}y \rangle_{X^{-\alpha}}. \tag{7.17}$$

for $x, y \in \varphi_{\beta-\alpha,-\alpha}[\varphi_{\beta,\beta-\alpha}[X^{\beta}]]$. Thus, by density, (7.17) holds for all $x, y \in \varphi_{\beta-\alpha,-\alpha}[X^{\beta}]$, proving our claim. This claim implies that the map $\widetilde{A}^{\beta}_{(\beta-\alpha)}$ is self-adjoint and so its inverse $\widetilde{A}^{-\beta}_{(-\alpha)}$ is symmetric on $X^{-\alpha}$. Moreover, (7.12) implies that

$$\widetilde{A}^{-\beta-\gamma}_{(-\alpha)} = \widetilde{A}^{-\beta}_{(-\alpha)} \circ \widetilde{A}^{-\gamma}_{(-\alpha)}, \quad \alpha, \ \beta, \ \gamma \in]0, \infty[. \tag{7.18}$$

In particular, $\widetilde{A}_{(-\alpha)}^{-\beta}$ is nonnegative.

Now, let $\alpha \in]0,\infty[$ be arbitrary. Since $\operatorname{re}\sigma(A) > 0$ there is a $\delta \in]0,\infty[$ such that

$$\langle Ax, x\rangle_X \geq \delta\langle x, x\rangle_X, \quad x \in X.$$

Hence, by (7.16), for $u \in X^1$ and $x = \varphi_{1-\alpha,-\alpha}\varphi_{1,1-\alpha}u$,

$$\begin{aligned}\langle \widetilde{A}_{(1-\alpha)}^1 x, x\rangle_{X^{-\alpha}} &= \langle A^{-\alpha}u, A^1A^{-\alpha}u\rangle_X \\ &\geq \delta\langle A^{-\alpha}u, A^{-\alpha}u\rangle_X = \delta\langle x, x\rangle_{X^{-\alpha}}.\end{aligned} \tag{7.19}$$

This implies, by density, that $\operatorname{re}\sigma(\widetilde{A}_{(1-\alpha)}^1) > 0$ so $B := \widetilde{A}_{(1-\alpha)}$ generates the family $B^{-\beta}$, $\beta \in]0,\infty[$, of basic fractional power spaces of B. By (2.2)

$$B^{-\beta-\gamma} = B^{-\beta} \circ B^{-\gamma}, \quad \beta,\ \gamma \in]0,\infty[. \tag{7.20}$$

We claim that

$$B^{-\beta} = \widetilde{A}_{(-\alpha)}^{-\beta}, \quad \beta \in]0,\infty[. \tag{7.21}$$

Let Z be the set of $\beta \in]0,\infty[$ with $B^{-\beta} = \widetilde{A}_{(-\alpha)}^{-\beta}$. Clearly, $1 \in Z$ and so induction on $m \in \mathbb{N}$ using (7.18) and (7.20) imply that Z contains all integers. Since nonnegative symmetric operators on a Hilbert space have unique nonnegative square roots, it follows by induction on $k \in \mathbb{N}$ (again using (7.18) and (7.20)) that Z contains all numbers of the form $m/2^k$ with $m, k \in \mathbb{N}$. The set Z_0 of such numbers is dense in $]0,\infty[$. Now let $\beta \in]0,\infty[$ be arbitrary and $(\beta_n)_n$ be a sequence in Z_0 converging to β. By formula (2.1) we have that

$$|B^{-\beta_n}x - B^{-\beta}x|_{X^{-\alpha}} \to 0, \quad x \in X^{-\alpha}$$

and

$$|A^{-\beta_n}x - A^{-\beta}x|_X \to 0, \quad x \in X.$$

In particular, using the fact that

$$\widetilde{A}_{-\alpha}^{-\gamma}\varphi_{0,-\alpha}u = \varphi_{0,-\alpha}A^{-\gamma}u, \quad \alpha,\ \gamma \in]0,\infty[,\ u \in X$$

we obtain that

$$\begin{aligned}|B^{-\beta_n}\varphi_{0,-\alpha}u - B^{-\beta}\varphi_{0,-\alpha}u|_{X^{-\alpha}} &\to 0, \quad u \in X, \\ |\widetilde{A}_{-\alpha}^{-\beta_n}\varphi_{0,-\alpha}u - \widetilde{A}_{-\alpha}^{-\beta}\varphi_{0,-\alpha}u|_{X^{-\alpha}} &\to 0, \quad u \in X.\end{aligned}$$

Thus

$$B^{-\beta}\varphi_{0,-\alpha}u = \widetilde{A}_{-\alpha}^{-\beta}\varphi_{0,-\alpha}u, \quad u \in X$$

so, by density, $B^{-\beta} = \widetilde{A}_{-\alpha}^{-\beta}$ and thus $\beta \in Z$. This proves our (7.21). We obtain from (7.21) that

$$X_B^\beta = \varphi_{\beta-\alpha,-\alpha}[X^{\beta-\alpha}].$$

We also claim that

(7.22) For all $\beta \in]0,\infty[$, $\varphi_{\beta-\alpha,-\alpha}$ is an isometry of $X^{\beta-\alpha}$ onto X_B^β.

To prove this claim, let $\widetilde{x},\widetilde{y} \in X^{\beta-\alpha}$ be arbitrary and let $x = \varphi_{\beta-\alpha,-\alpha}\widetilde{x}$, $y = \varphi_{\beta-\alpha,-\alpha}\widetilde{y}$. Suppose first that $\widetilde{x} = \varphi_{\beta,\beta-\alpha}u$, $\widetilde{y} = \varphi_{\beta,\beta-\alpha}v$ with $u,v \in X^\beta$. Then, by (7.15), $B^\beta x = \varphi_{0,-\alpha}A^\beta u$ and $B^\beta y = \varphi_{0,-\alpha}A^\beta v$. Therefore, using (7.14) we obtain

$$\begin{aligned}\langle x,y\rangle_{X_B^\beta} &= \langle B^\beta x, B^\beta y\rangle_{X^{-\alpha}} = \langle \varphi_{0,-\alpha}A^\beta u, \varphi_{0,-\alpha}A^\beta v\rangle_{X^{-\alpha}}\\ &= \langle A^{-\alpha}A^\beta u, A^{-\alpha}A^\beta v\rangle_X = \langle A^{\beta-\alpha}u, A^{\beta-\alpha}v\rangle_X.\end{aligned}$$

If $\beta-\alpha \geq 0$, then

$$\langle A^{\beta-\alpha}u, A^{\beta-\alpha}v\rangle_X = \langle u,v\rangle_{X^{\beta-\alpha}} = \langle \widetilde{x},\widetilde{y}\rangle_{X^{\beta-\alpha}}.$$

If $\beta-\alpha < 0$, then, by (7.14),

$$\begin{aligned}\langle A^{\beta-\alpha}u, A^{\beta-\alpha}v\rangle_X &= \langle A^{-(\alpha-\beta)}u, A^{-(\alpha-\beta)}v\rangle_X\\ &= \langle \varphi_{0,-(\alpha-\beta)}u, \varphi_{0,-(\alpha-\beta)}v\rangle_{X^{-(\alpha-\beta)}} = \langle \widetilde{x},\widetilde{y}\rangle_{X^{-(\alpha-\beta)}}.\end{aligned}$$

Thus, in both cases, $\langle x,y\rangle_{X_B^\beta} = \langle \widetilde{x},\widetilde{y}\rangle_{X^{\beta-\alpha}}$. Since the set $\varphi_{\beta,\beta-\alpha}[X^\beta]$ is dense in $X^{\beta-\alpha}$, (7.22) follows.

We further claim that

(7.23) Whenever $\alpha \in [0,(1/2)[$, $x \in X^{1-\alpha}$ and $v \in X^{1/2} \subset X^\alpha$, then $(A_{(1-\alpha)}x).v = \langle x,v\rangle_{X^{1/2}}$.

Here, the dot '.' denotes function application between an element of $X^{-\alpha}$ and X^α. To prove this claim, assume first that $x \in X^1$. Then, by (7.15), $A_{(1-\alpha)}x = \varphi_{0,-\alpha}Ax$ and so using the Fréchet–Riesz theorem and (7.13) we obtain

$$(A_{(1-\alpha)}x).v = \langle v, R_\alpha^{-1}\varphi_{0,-\alpha}Ax\rangle_{X^\alpha} = \langle v, A^{-2\alpha}Ax\rangle_{X^\alpha} = \langle v, Ax\rangle_X = \langle v,x\rangle_{X^{1/2}}$$

so the claim follows in this special case. The general case follows by the density of X^1 in $X^{1-\alpha}$ and in $X^{1/2}$. □

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Martino Prizzi
Università degli Studi di Trieste
Dipartimento di Matematica e Informatica
Via Valerio, 12
34127 Trieste, ITALY

E-mail address: prizzi@ dsm.univ.trieste.it

Krzysztof P. Rybakowski
Universität Rostock
Institut für Mathematik
Universitätsplatz 1
18055 Rostock, GERMANY

E-mail address: krzysztof.rybakowski@ mathematik.uni-rostock.de