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POSITIVE PERIODIC SOLUTIONS OF SUPERLINEAR SYSTEMS OF INTEGRAL EQUATIONS DEPENDING ON PARAMETERS

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Abstract. A class of superlinear system of integral equations depending on multi parameters is considered. It is shown that there are three mutually exclusive and exhaustive subsets Θ_1, Γ and Θ_2 of the parameter space such that there exist at least two positive periodic solutions associated with elements in Θ_1, at least one positive periodic solution associated with Γ and none associated with Θ_2.

1. Introduction

Coupled differential systems arise in a number of biological, ecological, economical and other models which describe interactions. In [3], a coupled differential system of the form

$$\begin{aligned} x'(t) &= -a(t)x(t) + \lambda k(t) f(x(t-\tau_1(t)), y(t-\sigma_1(t))), \\ y'(t) &= -b(t)y(t) + \nu h(t) g(x(t-\tau_2(t)), y(t-\sigma_2(t))), \end{aligned}$$

is studied and the existence of positive periodic solutions corresponding to different values of the parameters λ and ν are derived by transforming the above

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system into an equivalent coupled system of integral equations

$$x(t) = \lambda \int_t^{t+\omega} K(t,s)k(s)f(x(s-\tau_1(s)), y(s-\sigma_1(s)))\,ds, \tag{1.1}$$

$$y(t) = \nu \int_t^{t+\omega} H(t,s)h(s)g(x(s-\tau_2(s)), y(s-\sigma_2(s)))\,ds. \tag{1.2}$$

This prompts us to study more general coupled systems of integral equations. For this purpose, we follow some of the ideas developed by the authors in [2] in setting up our problem: First, $\mathbb{R}^N$ is the N-dimensional Euclidean space endowed with componentwise ordering $\le$. For any $u, v \in \mathbb{R}^N$, the interval $[u,v]$ is the set $\{x \in \mathbb{R}^N \mid u \le x \le v\}$. Let $T = (t_1, \ldots, t_N) \in \mathbb{R}^N$ with positive components and let $e^{(1)} = (1,0,\ldots 0), \ldots, e^{(N)} = (0,\ldots,0,1)$ be the standard orthonormal vectors in $\mathbb{R}^N$. Let G be a closed subset of $\mathbb{R}^N$ which has the following "periodic" structure: for each $x \in G$,

$$x + t_i e^{(i)} \in G,$$

and for each pair $y, z \in G$,

$$\mu([y, y+T] \cap G) = \mu([z, z+T] \cap G) > 0,$$

where μ is the Lebesgue measure, and we set

$$G(x) = [x, x+T] \cap G.$$

Examples of nontrivial G can be found in [2].

The system of integral equations of the form

$$\phi_j(x) = \lambda_j \int_{G(x)} K_j(x,s) f_j(s, \phi_1(s-\tau_{j1}(s)), \ldots, \phi_\omega(s-\tau_{j\omega}(s)))ds, \tag{1.3}$$

for $x \in G$, $j = 1, \ldots, \omega$, will be considered. Here, the functions K_j, f_j, τ_{jk}, where $j, k \in \{1, \ldots, \omega\}$, satisfy the following 'periodic' conditions:

- for $j \in \{1, \ldots, \omega\}$, $K_j \in C(G \times G, \mathbb{R}^+)$, $K_j(x + t_i e^{(i)}, y + t_i e^{(i)}) = K_j(x,y)$ for any $(x,y) \in G \times G$ and $i \in \{1, \ldots N\}$,
- for $j \in \{1, \ldots, \omega\}$, $f_j \in C(G \times \mathbb{R}^\omega, R)$, $f_j(x + t_i e^{(i)}, u_1, \ldots, u_\omega) = f_j(x, u_1, \ldots, u_\omega)$ for $i \in \{1, \ldots, N\}$ and any $x \in G$,
- for $j, k \in \{1, \ldots, \omega\}$, $\tau_{jk}: G \to G$ is continuous, $\tau_{jk}(x + t_i e^{(i)}) = \tau_{jk}(x)$ for any $x \in G$ and $i \in \{1, \ldots N\}$,

the boundedness conditions:

$$\inf_{x,y \in G(t), t \in G} K_j(x,y) \ge m_j > 0, \quad M_j = \sup_{x,y \in G(t), t \in G} K_j(x,y) < \infty,$$

for $j \in \{1, \ldots, \omega\}$, and the "superlinear" conditions:

(H1) for $j \in \{1, \ldots, \omega\}$, $f_j(x, 0, \ldots, 0) > 0$ for any $x \in G$, $f_j(x, u_1, \ldots, u_\omega)$ is nondecreasing on $(u_1, \ldots, u_\omega) \in [0,\infty) \times \ldots \times [0,\infty)$ (i.e. $f_j(x, u_1, \ldots, u_\omega) \le f_j(x, v_1, \ldots, v_\omega)$ for $0 \le u_j \le v_j$, $j \in \{1, \ldots, \omega\}$, and $x \in G$), (H2) for $j \in \{1, \dots, n\}$, $\lim_{u_1+\dots+u_\omega\to\infty} f_j(x, u_1, \dots, u_\omega)/(u_1 + \dots + u_\omega) = \infty$ uniformly with respect to all $x \in G$.

The numbers $\lambda_1, \dots, \lambda_\omega$ will be assumed to be nonnegative and treated as parameters. Note that when $\lambda_1 = \dots = \lambda_\omega = 0$, our system reduces to a system of decoupled equations. For this reason, the case $\lambda_1 = \dots = \lambda_\omega = 0$ will be avoided in our subsequent discussions. Therefore our system (1.3) may be regarded as a multi-state interactive model depending on the parameter vector $\lambda = (\lambda_1, \dots, \lambda_\omega)$ in the set

$$\Xi = \{(\lambda_1, \dots, \lambda_\omega) : \lambda_j \geq 0,\ j = 1, \dots, \omega\} \setminus \{(0, \dots, 0)\}.$$

For any $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ in $\mathbb{R}^m$, we will write $(a_1, \dots, a_m) \geq (b_1, \dots, b_m)$ if $a_j \geq b_j$ for $j \in \{1, \dots, m\}$. If $(a_1, \dots, a_m) \geq (b_1, \dots, b_m)$ and if $a_k > b_k$ or some $k \in \{1, \dots, m\}$, we will write $(a_1, \dots, a_m) > (b_1, \dots, b_m)$. A vector function $(\phi_1, \dots, \phi_\omega)\colon G \to \mathbb{R}^\omega$ is said to be positive if $(\phi_1(x), \dots, \phi_\omega(x)) \geq (0, \dots, 0)$ for all $x \in G$ and $(\phi_1(x_0), \dots, \phi_\omega(x_0)) > (0, \dots, 0)$ for some $x_0 \in G$. It is said to be T-periodic if $\phi_1, \dots, \phi_\omega$ are T-periodic, that is, $\phi_j(x + t_i e^{(i)}) = \phi_j(x)$ for $x \in G$, $j \in \{1, \dots, \omega\}$ and $i \in \{1, \dots, N\}$.

By a solution of (1.3) associated with the parameter vector $(\alpha_1 \dots, \alpha_\omega) \in \Xi$, we mean a continuous vector function $\phi\colon G \to \mathbb{R}^\omega$ which satisfies (1.3) for $\lambda_j = \alpha_j$ for $j \in \{1, \dots, \omega\}$. As in [3], we will prove there exists a continuous surface Γ splitting Ξ into disjoint subsets Θ_1, Γ and Θ_2 such that the system (1.3) has at least two, at least one, or no positive T-periodic solutions according whether λ is in Θ_1, Γ or Θ_2, respectively. We remark, however, that, the result in [3] is only good for the coupled system (1.1)–(1.2) which is much less general than our results below.

2. Some basic lemmas

Let X be the set of all real T-periodic continuous functions defined on G which is endowed with the usual linear structure as well as the norm

$$\|\psi\| = \sup_{x\in G(t),\ t\in G} |\psi(x)|.$$

Then X^ω is also a Banach space with the norm

$$\|(\phi_1, \dots, \phi_\omega)\| = \|\phi_1\| + \dots + \|\phi_\omega\|.$$

Furthermore, let Φ and Ω be defined respectively by

$$\Phi = \{(\phi_1, \dots, \phi_\omega) \in X^\omega : \phi_j(x) \geq 0,\ x \in G,\ j = 1, \dots, \omega\},$$
$$\Omega = \{(\phi_1, \dots, \phi_\omega) \in \Phi : \phi_1(x) + \dots + \phi_\omega(x) \geq \alpha^* \|(\phi_1, \dots, \phi_\omega)\|,\ x \in G\},$$

where $\alpha^* = \min_{j=1,\dots,\omega}\{m_j/M_j\}$. Then Φ and Ω are cones in X^ω.

Define, for each $\phi = (\phi_1, \dots, \phi_\omega) \in X^\omega$,

$$\mathbf{T}_\lambda(\phi)(x) = (A_{\lambda_1}(\phi)(x), \dots, A_{\lambda_\omega}(\phi)(x)),$$

where

$$A_{\lambda_j}(\phi)(x) = \lambda_j \int_{G(x)} K_j(x,s) f_j(s, \phi_1(s - \tau_{j1}(s)), \dots, \phi_\omega(s - \tau_{j\omega}(s)))\, ds,$$

for $j = 1, \dots, \omega$. Then our system (1.3) can be written as

$$\phi(x) = \mathbf{T}_\lambda(\phi)(x).$$

For the sake of convenience, we will set

$$f_j(s, \phi(*)) := f_j(s, \phi_1(s - \tau_{j1}(s)), \dots, \phi_\omega(s - \tau_{j\omega}(s)))$$

in the following discussions.

Let $\phi = (\phi_1, \dots, \phi_\omega) \in \Phi$. For each $j \in \{1, \dots, \omega\}$,

$$A_{\lambda_j}(\phi)(x) = \lambda_j \int_{G(x)} K_j(x,s) f_j(s, \phi(*))\, ds \le \lambda_j M_j \int_{G(x)} f_j(s, \phi(*))\, ds$$

so that

$$\frac{1}{M_j} \|A_{\lambda_j}(\phi)\| \le \lambda_j \int_{G(x)} f_j(s, \phi(*))\, ds$$

and

$$\begin{aligned} A_{\lambda_j}(\phi)(x) &= \lambda_j \int_{G(x)} K_j(x,s) f_j(s, \phi(*))\, ds \\ &\ge \lambda_j m_j \int_{G(x)} f_j(s, \phi(*))\, ds \ge \alpha^* \|A_{\lambda_j}(\phi)\|. \end{aligned}$$

That is, for each $\lambda \in \Xi$, $\mathbf{T}_\lambda \Phi$ is contained in Ω.

Furthermore, by standard arguments, we may also show that $\mathbf{T}_\lambda$ is completely continuous. To see this, we may assume for the sake of simplicity that G is a subset in $\mathbb{R}^2$. Recall that the interval $[u, v]$ is the set $\{x \in \mathbb{R}^2 \mid u \le x \le v\}$. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ in G. We consider the case where $(x_1, y_1) \le (x_2, y_2)$, while the other cases can similarly be treated. We set $C = (x_2, y_1)$, $D = (x_1, y_2)$, $E = (x_2, y_1 + t_2)$, $F = (x_1 + t_1, y_2 + t_2)$, $K = (x_1 + t_1, y_1 + t_2)$, $H = (x_1 + t_1, y_2)$, $I = (x_2 + t_1, y_1 + t_2)$, $J = (x_2 + t_1, y_2 + t_2)$, and $G_1 = [A, B]$, $G_2 = [D, E]$, $G_3 = [C, H]$, $G_4 = [B, K]$, $G_5 = [E, F]$, $G_6 = [H, I]$, $G_7 = [K, J]$.

We suppose that Δ is a bounded set of X^ω. Then there exists constant $\check{T} > 0$, such that $\|\phi\| \le \check{T}$ for any $\phi \in \Delta$. In view of the theorem of Arzela–Ascoli, we only need to show that $A_{\lambda_j}(\Delta)$ is equicontinuous for any $j \in \{1, \dots, \omega\}$. Indeed,

$$\begin{aligned}
A_{\lambda_j}(\phi)(B) - A_{\lambda_j}(\phi)(A) = \lambda_j \Big\{ \int_{G_7} + \int_{G_6} + \int_{G_5} \Big\} K_j(B,s) f_j(s,\phi(*))\, ds \\
+ \lambda_j \int_{G_4} [K_j(B,s) - K_j(A,s)] f_j(s,\phi(*))\, ds \\
- \lambda_j \Big\{ \int_{G_3} + \int_{G_2} + \int_{G_1} \Big\} K_j(A,s) f_j(s,\phi(*))\, ds.
\end{aligned}$$

Furthermore, $f_j \in C(G(x) \times [-\check{T}, \check{T}] \times \ldots \times [-\check{T}, \check{T}], R)$ and $f_j(x + t_i e_i, u_1, \dots, u_\omega) = f_j(x, u_1, \dots, u_\omega)$ for any $x \in G$, then there exists constant $\widehat{H}$, such that

$$|f_j(s,\phi(*))| \le \widehat{H}, \quad \text{for } s \in \bigcup_{j=1}^{7} G_j,$$

thus

$$\begin{aligned}
\Big| \lambda_j \int_{G_7} K_j(B,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} |x_2 - x_1| |y_2 - y_1|, \\
\Big| \lambda_j \int_{G_6} K_j(B,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} t_2 |x_2 - x_1|, \\
\Big| \lambda_j \int_{G_5} K_j(B,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} t_1 |y_2 - y_1|, \\
\Big| \lambda_j \int_{G_3} K_j(A,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} t_1 |y_2 - y_1|, \\
\Big| \lambda_j \int_{G_2} K_j(A,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} t_2 |x_2 - x_1|, \\
\Big| \lambda_j \int_{G_1} K_j(A,s) f_j(s,\phi(*))\, ds \Big| &\le \lambda_j M_j \widehat{H} |x_2 - x_1| |y_2 - y_1|,
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \lambda_j \int_{G_4} [K_j(B,s) - K_j(A,s)] f_j(s,\phi(*))\, ds \Big| \\
&\quad \le \lambda_j \widehat{H} \int_{G_4} |K_j(B,s) - K_j(A,s)|\, ds \le \lambda_j \widehat{H} \int_{G(B)} |K_j(B,s) - K_j(A,s)|\, ds.
\end{aligned}$$

In view of the uniformity of $K_j(x,y)$ in $G(B)$, for any $\varepsilon > 0$, there is δ which satisfies

$$0 < \delta < \min \Big\{ t_1, t_2, \frac{\varepsilon}{\lambda M_j \widehat{H} t_2}, \frac{\varepsilon}{\lambda M_j \widehat{H} t_1}, \sqrt{\frac{\varepsilon}{\lambda M_j \widehat{H}}} \Big\},$$

and for $0 < x_2 - x_1 < \delta$, $0 < y_2 - y_1 < \delta$, we have

$$|K_j(B,s) - K_j(A,s)| < \frac{\varepsilon}{\lambda_j \widehat{H} t_2 t_1}, \quad \text{for } s \in G(B).$$

Thus

$$\begin{aligned}|A_{\lambda_j}(\phi)(B) - A_{\lambda_j}(\phi)(A)| &\le \left|\lambda_j \int_{G_7} K_j(B,s) f_j(s,\phi(*))\,ds\right| \\ &+ \left|\lambda_j \int_{G_6} K_j(B,s) f_j(s,\phi(*))\,ds\right| + \left|\lambda_j \int_{G_5} K_j(B,s) f_j(s,\phi(*))ds\right| \\ &+ \left|\lambda_j \int_{G_4} [K_j(B,s) - K_j(A,s)] f_j(s,\phi(*))\,ds\right| \\ &+ \left|\lambda_j \int_{G_3} K_j(A,s) f_j(s,\phi(*))\,ds\right| + \left|\lambda_j \int_{G_2} K_j(A,s) f_j(s,\phi(*))\,ds\right| \\ &+ \left|\lambda_j \int_{G_1} K_j(A,s) f_j(s,\phi(*))\,ds\right| \le 7\varepsilon\end{aligned}$$

for any $\phi \in \Delta$. This means that $A_{\lambda_j}(\Delta)$ is equicontinuous.

LEMMA 2.1. *For any compact subset D of Ξ, there exists a constant $b_D > 0$ such that any positive T-periodic solution $\phi = (\phi_1, \ldots, \phi_\omega)$ of* (1.3) *associated with $\lambda = (\lambda_1, \ldots, \lambda_\omega) \in D$ will satisfy $\|\phi\| < b_D$.*

PROOF. Suppose to the contrary that there is a sequence

$$\{\phi^{(n)}\} = \{(\phi_1^{(n)}, \ldots, \phi_\omega^{(n)})\}_{n=1}^\infty$$

of positive T-periodic solutions of (1.3) associated with $\lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_\omega^{(n)})$ such that $\lambda^{(n)} \in D$ for all n and $\lim_{n\to\infty} \|\phi^{(n)}\| = \infty$.

Since $\phi^{(n)} = \mathbf{T}_{\lambda^{(n)}}(\phi^{(n)}) \in \Omega$, thus

$$\phi_1^{(n)}(x) + \ldots + \phi_\omega^{(n)}(x) \ge \alpha^* \|\phi^{(n)}\|$$

for $n \ge 1$. Since $\lambda^{(n)} \in D$ for all n, there is some k such that $\lambda_k^{(n)} > 0$ for all sufficiently large n. Then in view of (H2), we may choose $R_{f_k} > 0$, η_k and $n_0 \ge 1$ such that $f_k(x, u_1, \ldots, u_\omega) \ge \eta_k(u_1 + \ldots + u_\omega)$ for all nonnegative $u_1, \ldots, u_\omega$ and $x \in G$ which satisfy $u_1 + \ldots + u_\omega \ge R_{f_k}$, $\alpha^*(||\phi_1^{(n_0)}|| + \ldots + ||\phi_\omega^{(n_0)}||) \ge R_{f_k}$, and

$$\alpha^* \eta_k m_k \lambda_k^{(n_0)} \cdot \mu G(x) > 1.$$

Thus, we have

$$\begin{aligned}\|\phi_k^{(n_0)}\| &\ge \phi_k^{(n_0)}(x) \\ &= \lambda_k^{(n_0)} \int_{G(x)} K_k(x,s) f_k(s, \phi_1^{(n_0)}(s - \tau_{k1}(s)), \ldots, \phi_\omega^{(n_0)}(s - \sigma_{k\omega}(s)))\,ds \\ &\ge \alpha^* \eta_k m_k \lambda_k^{(n_0)} \cdot \mu G(x)(||\phi_1^{(n_0)}|| + \ldots + ||\phi_\omega^{(n_0)}||) > \|\phi_k^{(n_0)}\|.\end{aligned}$$

This is a contradiction. The proof is complete. □

LEMMA 2.2. . *If* (1.3) *has a positive* T*-periodic solution associated with* $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*) > (0, \ldots, 0)$*, then for any* $\lambda = (\lambda_1, \ldots, \lambda_\omega) \in \Xi$ *that satisfies* $\lambda \le \lambda^*$*, equation* (1.3) *also has a positive* T*-periodic solution associated with* λ*. The system* (1.3) *has a positive* T*-periodic solution associated with some* $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*)$ *satisfying* $\lambda_j^* > 0$ *for* $j = 1, \ldots, \omega$*.*

PROOF. Let $\phi^* = (\phi_1^*, \ldots, \phi_\omega^*)$ be a positive T-periodic solution of (1.3) associated with λ^*. Since $\lambda_j \le \lambda_j^*$, we have

$$\phi_j^*(x) = A_{\lambda_j^*}(\phi^*)(x) \ge A_{\lambda_j}(\phi^*)(x)$$

for $j \in \{1, \ldots, \omega\}$. Let $\phi^{(0)} = (\phi_1^*, \ldots, \phi_\omega^*)$ and

$$\phi^{(n+1)} = \mathbf{T}_\lambda(\phi^{(n)}), \quad \text{for } n = 0, 1, \ldots \tag{2.1}$$

Clearly, we have

$$\phi^{(0)}(x) \ge \phi^{(1)}(x) \ge \ldots \ge \phi^{(n)}(x) \ge (0, \ldots, 0).$$

Let $\phi(x) = \lim_{n\to\infty} \phi^{(n)}(x)$. In view of the Lebsegue dominated convergence theorem, we see from (2.1) that ϕ is a nonnegative T-periodic function that satisfies

$$\phi(x) = \mathbf{T}_\lambda(\phi)(x).$$

It will thus be a solution of (1.3) if we can show it is continuous. To see the proof, assume for the sake of simplicity that G is a subset of $\mathbb{R}^2$. Then we define $A, \ldots, J$ $G_1, \ldots, G_7$ as in the proof of the complete continuity of $\mathbf{T}_\lambda$. Then

$$\begin{aligned}\phi_j(B) - \phi_j(A) = {}& \lambda_j \bigg\{ \int_{G_5} + \int_{G_6} + \int_{G_7} \bigg\} K_j(B, s) f_j(s, \phi(*))\, ds \\ & + \lambda_j \int_{G_4} [K_j(B, s) - K_j(A, s)] f_j(s, \phi(*))\, ds \\ & - \lambda_j \bigg\{ \int_{G_1} + \int_{G_2} + \int_{G_3} \bigg\} K_j(A, s) f_j(s, \phi(*))\, ds\end{aligned}$$

for $j = 1, \ldots, \omega$. Since $\phi^{(0)}(x) \ge \phi^{(1)}(x) \ge \ldots \ge \phi^{(n)}(x) \ge (0, \ldots, 0)$, we see that $|\phi_j(x)| \le |\phi_j^*(x)| \le \|\phi^*\|$ for all $x \in G$. Furthermore, $f_j \in C(G(x) \times [-\|\phi^*\|, \|\phi^*\|] \times \ldots \times [-\|\phi^*\|, \|\phi^*\|], R)$ and $f_j(x + t_i e_i, u_1, \ldots, u_\omega) = f_j(x, u_1, \ldots, u_\omega)$ for any $x \in G$, thus there exists constant $\widehat{H}$, such that

$$|f_j(s, \phi(*))| \le \widehat{H}, \quad s \in \bigcup_{j=1}^{7} G_j,\ j = 1, \ldots, \omega.$$

By estimates similar to those in the proof of the complete continuity of $\mathbf{T}_\lambda$, we may then arrive at

$$|\phi_j(B) - \phi_j(A)| \le 7\varepsilon.$$

Now that we have shown ϕ is a solution of (1.3), we need to show it is positive. Indeed, since ϕ^* is positive, $\phi(x) \geq 0$ for $x \in G$. Since each $f_j(x, 0, \ldots, 0) > 0$ for $x \in G$ by our assumption, ϕ cannot be the trivial solution. Thus, ϕ is positive.

To show the existence of a positive periodic solution associated with some λ^*, let

$$\alpha_j(x) = \int_{G(x)} K_j(x,s)ds, \quad j = 1, \ldots, \omega,$$

and

$$M_{f_j} = \max_{x \in G(t), t \in G} f_j(x, \alpha_1(x - \tau_{j1}(x)), \ldots, \alpha_\omega(x - \tau_{j\omega}(x))), \quad j = 1, \ldots, \omega.$$

Then clearly $M_{f_j} > 0$ for $j \in \{1, \ldots, \omega\}$.

Let $(\lambda_1^*, \ldots, \lambda_\omega^*) = (1/M_{f_1}, \ldots, 1/M_{f_\omega})$. We have

$$\begin{aligned} \alpha_j(x) &= \int_{G(x)} K_j(x,s)\,ds \\ &\geq \lambda_j^* \int_{G(x)} K_j(x,s) f_j(s, \alpha_1(s - \tau_{j1}(s)), \ldots, \alpha_\omega(s - \tau_{j\omega}(s)))\,ds, \end{aligned}$$

for $j = 1, \ldots, \omega$. Now let $\phi^{(0)} = (\alpha_1(x), \ldots, \alpha_\omega(x))$ and $\phi^{(n+1)} = \mathbf{T}_{\lambda^*}(\phi^{(n)})(x)$ as in (2.1). Then the same argument shows that $\phi(x) = \lim_{n\to\infty} \phi^{(n)}(x)$ is a nonnegative T-periodic solution of (1.3) which satisfies $\phi(x) > (0, \ldots, 0)$. The proof is complete. □

Let Π be the subset of Ξ such that (1.3) has a positive T -periodic solution associated with $\lambda = (\lambda_1 \ldots, \lambda_\omega)$. Then by Lemma 2.2, Π contains some $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*)$ such that (1.3) has a positive T-periodic solution associated it, and hence it contains the subset

(2.2) $$\Pi_* = \{(\lambda_1, \ldots, \lambda_\omega) : (\lambda_1, \ldots, \lambda_\omega) > (0, \ldots, 0),\ \lambda_j \leq \lambda_j^*, j = 1, \ldots, \omega\}.$$

LEMMA 2.3. *The subset* Π *of* Ξ *is bounded.*

PROOF. Suppose to the contrary that there is a sequence

$$\phi^{(n)} = \{(\phi_1^{(n)}, \ldots, \phi_\omega^{(n)})\}$$

of positive T-periodic solutions of (1.3) associated with $\lambda^{(n)} = \{(\lambda_1^{(n)}, \ldots, \lambda_\omega^{(n)})\}$ such that $\lim_{n\to\infty} \lambda_j^{(n)} = \infty$ for some $k \in \{1, \ldots, \omega\}$. Then either there exists a subsequence $\phi^{(n_j)} = \{(\phi_1^{(n_j)}, \ldots, \phi_\omega^{(n_j)})\}$ such that $\|\phi^{(n_j)}\| \to \infty$ as $j \to \infty$ or there is $\overline{M} > 0$ such that $\|\phi^{(n)}\| \leq \overline{M}$ for all n. Since $\phi^{(n)} \in \Omega$, thus

$$\phi_1^{(n)}(x) + \ldots + \phi_\omega^{(n)}(x) \geq \alpha^* \|\phi^{(n)}\|.$$

By (H2), we may choose $R_{f_k} > 0$ such that $f_k(x, u_1, \ldots, u_\omega) \geq \eta_k(u_1 + \ldots + u_\omega)$ for all nonnegative numbers $u_1, \ldots, u_\omega$ and $x \in G$ which satisfy $u_1 + \ldots + u_\omega \geq R_{f_k}$ and some $\eta_k > 0$. In view of (H1), there exists $\delta_k > 0$ such that

$f_k(x,0,\dots,0) \geq \delta_k M_k$ for any $x \in G$. Let $\beta_k = \min\{\eta_k, \delta_k\}$. On the other hand, there exists a sequence $\{x^{(n)}\} \subset G(t)$, $t \in G$, such that $\phi_k^{(n)}(x^{(n)}) = \max_{x\in G(t),t\in G} \phi_k^{(n)}(x)$ by the periodicity and differentiability of $\{\phi_k^{(n)}(x)\}$. Thus, we have

$$\begin{aligned}\|\phi_k^{(n)}\| = \phi_k^{(n)}(x^{(n)}) = A_{\lambda_k^{(n)}}(\phi^{(n)})(x^{(n)}) &\geq \lambda_k^{(n)} m_k \beta_k \alpha^* \|\phi^{(n)}\| \cdot \mu G(x^{(n)}) \\ &\geq \lambda_k^{(n)} m_k \beta_k \alpha^* \|\phi^{(n)}\| \cdot \mu G(x^{(n)}) > \|\phi_k^{(n)}\|.\end{aligned}$$

But this is a contradiction. The proof is complete. □

3. Main theorem

We may now show that there exists a continuous surface Γ separating Ξ into two disjoint subsets Θ_1 and Θ_2 such that $(0,\dots,0)$ is a boundary point of Θ_1 and (1.3) has at least one positive T-periodic solution for $\lambda \in \Theta_1 \cup \Gamma$ and no positive T-periodic solution for $\lambda \in \Theta_2$. First let $e^{(1)},\dots,e^{(\omega)}$ be the standard orthonormal vectors in $\mathbb{R}^\omega$. Let Λ be the set of all convex combinations of $e^{(1)},\dots,e^{(\omega)}$, that is, Λ is the $(\omega-1)$-simplex in $\mathbb{R}^\omega$. For each $\mu \in \Lambda$, the half ray

$$L_\mu = \{\lambda \in \Xi : \lambda = t\mu,\ t > 0\}$$

has points which belong to Π_* defined by (2.2) and points outside Π (in view of Lemma 2.3). Thus the set $\{t > 0 : t\mu \in \Pi\}$ is nonempty and bounded above. Let

$$t_\mu^* = \sup\{t > 0 : t\mu \in \Pi\} \quad \text{and} \quad \lambda_\mu^* = t_\mu^* \mu.$$

Then for each $\mu \in \Lambda$, $\lambda_\mu^* \in \Pi$. Indeed, let $\{\lambda^{(n)}\}_{n=1}^\infty$ be a sequence which satisfies $\lambda^{(n)} < \lambda^{(n+1)}$ for $n \geq 1$ and converges to λ_μ^*. For each n, let $\phi^{(n)}$ be a positive T-periodic solution of (1.3) associated with $\lambda^{(n)}$. In view of Lemma 2.1, we know that the set $\{\phi^{(n)}\}$ is uniformly bounded in X^ω. Thus, the sequence $\{\phi^{(n)}\}$ has a subsequence converging to $\phi \in X^\omega$. Then we can easily show, by the Lebesgue dominated convergence theorem, that ϕ is a positive T-periodic solution of (1.3) at λ_μ^*.

Next, we let $\rho: \Lambda \to (0,\infty)$ be defined by

$$\rho(\mu) = t_\mu^* > 0.$$

Then we may assert that ρ is continuous. In order to see this, we will assume for the sake of simplicity that $\omega = 2$ and that $\zeta = (\zeta_1,\zeta_2) \in \Lambda$ such that $\zeta_1,\zeta_2 > 0$. Let $\lambda = (\lambda_1,\lambda_2)$ be a neighbouring vector of ζ in Λ such that $\lambda_1,\lambda_2 > 0$. Consider first the case $\lambda_1 < \zeta_1$ and $\lambda_2 > \zeta_2$. We will compare the vectors $t_\zeta^*\zeta = (\zeta_1^*,\zeta_2^*)$ and $t_\lambda^*\lambda = (\lambda_1^*,\lambda_2^*)$. Since Lemma 2.2 asserts that for each ξ inside

$$\{\xi \in \Xi : \xi \leq t_\zeta^*\zeta\},$$

there is a positive T-periodic solution of (1.3) associated with ξ, we see that

$$\frac{\lambda_1\zeta_2}{\zeta_1\lambda_2}\zeta_1^* \leq \lambda_1^* \quad \text{and} \quad \lambda_2^* \leq \frac{\zeta_1\lambda_2}{\lambda_1\zeta_2}\zeta_2^*.$$

If $\lambda_1 > \zeta_1$ and $\lambda_2 < \zeta_2$, by similar arguments, we may also show that

$$\lambda_1^* \leq \frac{\zeta_2\lambda_1}{\lambda_2\zeta_2}\zeta_1^* \quad \text{and} \quad \lambda_2^* \geq \frac{\zeta_1\lambda_2}{\lambda_1\zeta_2}\zeta_2^*.$$

In either cases, if $(\lambda_1, \lambda_2) \to (\zeta_1, \zeta_2)$, then $(\lambda_1^*, \lambda_2^*) \to (\zeta_1^*, \zeta_2^*)$ as required.

Hence by defining

$$\Gamma = \{\lambda : \lambda = \rho(\mu)\mu,\ \mu \in \Lambda\}, \tag{3.1}$$

we see that Γ is the desired continuous surface described above.

We intend to show that there are at least one more solution for each λ in Θ_1. To this end, we first recall the following lemmas for arguments involving the topological degree. One may refer to Guo and Lakshmikantham [1] for proofs and further discussion of the topological degree.

LEMMA 3.1. *Let X be a Banach space with cone K. Let Ω be a bounded and open subset in X. Let $0 \in \Omega$ and $\mathbf{T}\colon K \cap \overline{\Omega} \to K$ be condensing (or completely continuous). Suppose that $\mathbf{T}x \neq \xi x$ for all $x \in K \cap \partial\Omega$ and all $\xi \geq 1$. Then $i(\mathbf{T}, K \cap \Omega, K) = 1$.*

LEMMA 3.2. *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $\mathbf{T}\colon \overline{K}_r \to K$ is a compact map such that $\mathbf{T}x \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|\mathbf{T}x\|$ for $x \in \partial K_r$, then $i(\mathbf{T}, K_r, K) = 0$.*

Let ϕ^* be a positive T-periodic solution of (1.3) associated with $\lambda^* \in \Gamma$. Then for $\lambda < \lambda^*$ and $\lambda \in \Xi$, by the uniform continuity of f_j on compact sets, there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}\frac{f_j(s, 0, \dots, 0)(\lambda_j^* - \lambda_j)}{\lambda_j} > f_j(s, \phi_1^*(s - \tau_{j1}(s)) + \varepsilon, \dots, \phi_\omega^*(s - \tau_{j\omega}(s)) + \varepsilon)\\ - f_j(s, \phi_1^*(s - \tau_{j1}(s)), \dots, \phi_\omega^*(s - \tau_{j\omega}(s)))\end{aligned}$$

for $j \in \{1, \dots, \omega\}$, $s \in G$ and $0 < \varepsilon \leq \varepsilon_0$. Thus, we have

$$\begin{aligned}\lambda_j \int_{G(x)} &K_j(x,s) f_j(s, \phi_1^*(s - \tau_{j1}(s)) + \varepsilon, \dots, \phi_\omega^*(s - \tau_{j\omega}(s)) + \varepsilon)\, ds \\ &- \lambda_j^* \int_{G(x)} K_j(x,s) f_j(s, \phi_1^*(s - \tau_{j1}(s)), \dots, \phi_\omega^*(s - \tau_{j\omega}(s)))\, ds \\ = \lambda_j &\int_{G(x)} K_j(x,s)[f_j(s, \phi_1^*(s - \tau_{j1}(s)) + \varepsilon, \dots, \phi_\omega^*(s - \tau_{j\omega}(s)) + \varepsilon) \\ &- f_j(s, \phi_1^*(s - \tau_{j1}(s)), \dots, \phi_\omega^*(s - \tau_{j\omega}(s)))]\, ds \\ &- (\lambda_j^* - \lambda_j) \int_{G(x)} K_j(x,s) f_j(s, \phi_1^*(s - \tau_{j1}(s)), \dots, \phi_\omega^*(s - \tau_{j\omega}(s)))\, ds\end{aligned}$$

$$
\begin{aligned}
&< f_j(s,0,\dots,0)(\lambda_j^* - \lambda_j) \int_{G(x)} K_j(x,s)\,ds \\
&\quad - (\lambda_j^* - \lambda_j) \int_{G(x)} K_j(x,s) f_j(s, \phi_1^*(s-\tau_{j1}(s)),\dots,\phi_\omega^*(s-\tau_{j\omega}(s)))\,ds \\
&= (\lambda_j^* - \lambda_j) \int_{G(x)} K_j(x,s)[f_j(s,0,\dots,0) \\
&\quad - f_j(s,\phi_1^*(s-\tau_{j1}(s)),\dots,\phi_\omega^*(s-\tau_{j\omega}(s)))]\,ds \le 0
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_j &\int_{G(x)} K_j(x,s) f_j(s, \phi_1^*(s-\tau_{j1}(s)) + \varepsilon,\dots,\phi_\omega^*(s-\tau_{j\omega}(s)) + \varepsilon)\,ds \\
&\le \lambda_j^* \int_{G(x)} K_j(x,s) f_j(s,\phi_1^*(s-\tau_{j1}(s)),\dots,\phi_\omega^*(s-\tau_{j\omega}(s)))\,ds \\
&= \phi_j^*(x) < \phi_j^*(x) + \varepsilon.
\end{aligned}
$$

Let

$$\tilde{\phi}_j^*(x) = \phi_j^*(x) + \varepsilon, \quad \text{for } j = 1,\dots,\omega,$$

and

$$\Psi = \{(\phi_1,\dots,\phi_\omega) \in X^\omega : -\varepsilon < \phi_j(x) < \widetilde{\phi}_j^*(x),\ j = 1,\dots,\omega,\ x \in G\}.$$

Then Ψ is bounded and open in X^ω, $(0,\dots,0) \in \Psi$ and $\mathbf{T}_\lambda: \Omega \cap \overline{\Psi} \to \Omega$ is condensing (since it is completely continuous). Let $\phi = (\phi_1,\dots,\phi_\omega) \in \Omega \cap \partial\Psi$. Then there exists x_0 such that either $\phi_k(x_0) = \widetilde{\phi}_k^*(x_0)$ for some $k \in \{1,2,\dots,\omega\}$. Then, by (H1),

$$
\begin{aligned}
A_{\lambda_k}(\phi)(x_0) &= \lambda_k \int_{G(x_0)} K_k(x_0,s) f_k(s,\phi_1(s-\tau_{k1}(s)),\dots,\phi_k(s-\tau_{k\omega}(s)))\,ds \\
&\le \lambda_k \int_{G(x_0)} K_k(x_0,s) f_k(s,\widetilde{\phi}_1^*(s-\tau_{k1}(s)),\dots,\widetilde{\phi}_k^*(s-\tau_{k\omega}(s)))\,ds \\
&< \widetilde{\phi}_k^*(x_0) = \phi_k(x_0) \le \xi\phi_k(x_0)
\end{aligned}
$$

for all $\xi \ge 1$. Thus $\mathbf{T}_\lambda(\phi) \ne \xi\phi$ for $\phi \in \Omega \cap \partial\Psi$ and $\xi \ge 1$. In view of the properties of the fixed point index (see Lemma 3.1), we have $i(\mathbf{T}_\lambda, \Omega\cap\Psi, \Omega) = 1$.

By (H2), we may choose $R_{f_k} > 0$ such that $f_k(x,u_1,\dots,u_\omega) \ge \eta_k(u_1 + \dots + u_\omega)$ for all $u_1 + \dots + u_\omega \ge R_{f_k}$, where η_k satisfies

$$\alpha^* \eta_k m_k \lambda_k \cdot \mu G(x) > 1.$$

Let $R_k = \max\{b_D, R_{f_k}/\alpha^*, \|(\widetilde{\phi}_1^*,\dots,\widetilde{\phi}_\omega^*)\|\}$, where b_D is given in Lemma 2.1 with D a closed rectangle in Ξ containing λ. Let $\Omega_{R_k} = \{\phi \in \Omega : \|\phi\| < R_k\}$.

Then in view of Lemma 2.1, $\phi \neq \mathbf{T}_\lambda(\phi)$ for $\phi \in \partial\Omega_{R_k}$. Furthermore, if $\phi \in \partial\Omega_{R_k}$, then $\phi_1(x) + \ldots + \phi_\omega(x) \geq \alpha^*\|\phi\| \geq R_{f_k}$. Thus, we have

$$\begin{aligned} A_{\lambda_k}(\phi)(x) &= \lambda_k \int_{G(x)} K_k(x,s) f_k(s, \phi_1(s - \tau_{k1}(s)), \ldots, \phi_\omega(s - \tau_{k\omega}(s)))\, ds \\ &\geq \alpha^* \eta_k m_k \lambda_k \cdot \mu G(x)\|\phi\| > \|\phi\|. \end{aligned}$$

Therefore $\|\mathbf{T}_\lambda(\phi)\| \geq \|A_{\lambda_k}(\phi)\| > \|\phi\|$ and Lemma 3.2 then implies

$$i(\mathbf{T}_\lambda, \Omega_{R_k}, \Omega) = 0.$$

Consequently, by the additivity of the fixed point index,

$$0 = i(\mathbf{T}_\lambda, \Omega_{R_k}, \Omega) = i(\mathbf{T}_\lambda, \Omega \cap \Psi, \Omega) + i(\mathbf{T}_\lambda, \Omega_{R_k} \setminus \overline{\Omega \cap \Psi}, \Omega).$$

Since $i(\mathbf{T}_\lambda, \Omega \cap \Psi, \Omega) = 1$, $i(\mathbf{T}_{\lambda,\nu}, \Omega_{R_k} \backslash \overline{\Omega \cap \Psi}, \Omega) = -1$ and $\mathbf{T}_\lambda$ has a fixed point in $\Omega \cap \Psi$ and another in $\Omega_{R_k} \backslash \overline{\Omega \cap \Psi}$. Thus, we have the following result.

Theorem 3.3. *There exists a continuous surface* Γ *of the form* (3.1) *separating* Ξ *into two disjoint subsets* Θ_1 (*which is bounded*) *and* Θ_2 (*which is unbounded*) *such that* (1.3) *has at least two positive* T*-periodic solutions for* $\lambda \in \Theta_1$, *at least one positive* T*-periodic solution for* $\lambda \in \Gamma$, *and no positive* T*-periodic solution for* $\lambda \in \Theta_2$.

As our final remark, note that the surface Γ is defined by the shooting method. Therefore, numerical methods can be applied to calculate this surface.

References

[1] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, 1988.

[2] S. G. Kang, G. Zhang and S. S. Cheng, *Periodic solutions of a class of integral equations*, Topol. Methods in Nonlinear Anal. **22** (2003), 245–252.

[3] G. Zhang and S. S. Cheng, *Positive periodic solutions of coupled delay differential systems depending on two parameters*, Taiwanese J. Math. **8** (2004), 639–652.

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