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PERIODIC SOLUTIONS FOR EVOLUTION COMPLEMENTARITY SYSTEMS: A METHOD OF GUIDING FUNCTIONS

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Abstract. A guiding function method for a class of variational inequalities is developed.

1. Introduction

The problem of the existence of periodic solutions have been extensively studied for differential equations of the form

$$\frac{du}{dt} = f(t, u(t)), \tag{1.1}$$

where $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field. In particular, M. A. Krasnosel'skiĭ (see e.g. [8], [9]) has developed an approach using the Brouwer topological degree method applied to the Poincaré translation operator. Sufficient conditions on f for the degree applied to the Poincaré operator P_T to be different from zero are needed to prove the existence of T-periodic solutions. Such conditions can be obtained by using the guiding function method. A function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ is called a guiding function for (1.1) provided that there exists $r_0 > 0$ such that $\langle \nabla V(x), f(t,x)\rangle > 0$, for all $x \in \mathbb{R}^n$, $\|x\| \geq r_0$ and all $t \in [0,T]$.

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The original approach reduces the computation of the degree of P_T to the one of $f(0,\,\cdot\,)$ by using the homotopy

$$(\lambda, x) \to h(\lambda, x) := \frac{x - P_{\lambda T}(x)}{\lambda}.$$

We have indeed $h(1,x) = x - P_T(x)$ and $h(0,x) = -Tf(0,x)$. Moreover, it is also clear that for large x, the qualitative behavior of the vector field f is similar to that of ∇V. The details can be found in the expository article of J. Mawhin [10].

The original approach of M. A. Krasonel'skiĭ has later been generalized so as to obtain a continuation method for differential inclusions of the form

$$\frac{du}{dt} \in \varphi(t, u(t)), \tag{1.2}$$

where φ is a Caratheodory multivalued map with compact and convex values and linear growth. We refer the reader to the expository article [7] of L. Górniewicz for details and references.

In this paper, we consider the problem of existence of a solution $u(\,\cdot\,)$ to the following periodic problem:

$$u(t) \in C \qquad \text{for } t \in [0,T], \tag{1.3}$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), u(t) \right\rangle = 0 \qquad \text{a.e. } t \in [0,T], \tag{1.4}$$

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in C^* \qquad \text{a.e. } t \in [0,T], \tag{1.5}$$

$$u(0) = u(T) \tag{1.6}$$

where $C \subset \mathbb{R}^n$ is a nonempty closed convex cone, C^* denotes the dual cone of C, i.e.

$$C^* := \{h \in \mathbb{R}^n : \langle h, v\rangle \geq 0 \text{ for all } v \in C\},$$

$F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a given map and $f(\,\cdot\,)$ a given function.

The problem (1.3)–(1.5) is equivalent with that of the evolution variational inequality

$$u(t) \in C, \quad \text{for } t \in [0,T],$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \right\rangle \geq 0, \quad \text{for all } v \in C, \ \text{a.e. } t \in [0,T],$$

i.e. with that of the differential inclusion

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in -\partial\Psi_C(u(t)), \quad \text{a.e. } t \in [0,T], \tag{1.7}$$

where Ψ_C is the indicator function of C and $\partial\Psi_C$ denotes the convex subdifferential of Ψ_C.

However, the results obtained for problem (1.2) cannot be applied to the problem in (1.7) since the multivalued map $\partial\Psi_C$ has not a linear growth and for $x \in \partial C$, has not compact values.

In Section 2 of this paper, we show that a Poincaré operator $S(T)$ can also be defined for problem (1.7). In Section 3, the concept of guiding function is generalized for problem (1.7) and a continuation method applicable to problem (1.7) is developed.

2. The Poincaré operator

In the sequel the scalar product on $\mathbb{R}^n$ is denoted by $\langle \cdot\, , \cdot \rangle$ (with the associated norm $\| \cdot \|$). Let us first recall some general existence and uniqueness result.

THEOREM 2.1. *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex subset and $F: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator such that for some $\omega \geq 0$, $F + \omega I$ is monotone. Suppose that $f: [0, \infty) \to \mathbb{R}^n$ satisfies*

$$f \in C^0([0, \infty); \mathbb{R}^n), \qquad \frac{df}{dt} \in L^1_{\mathrm{loc}}(0, \infty; \mathbb{R}^n).$$

Let $y \in C$ and $0 < T < \infty$ be given. There exists a unique $u \in C^0([0, T]; \mathbb{R}^n)$ such that

$$\frac{du}{dt} \in L^\infty(0, T; \mathbb{R}^n); \tag{2.1}$$

$$u \text{ is right-differentiable on } [0, T); \tag{2.2}$$

$$u(t) \in C, \quad t \in [0, T]; \tag{2.3}$$

$$u(0) = y; \tag{2.4}$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \right\rangle \geq 0, \quad \text{for all } v \in C, \ \text{a.e. } t \in [0, T]. \tag{2.5}$$

Theorem 2.1 is a direct consequence of a Kato's result (we refer the reader to Brezis [2], [3] for the Kato's result and to [5] for the proof of Theorem 2.1).

REMARK 2.2. Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$F(x) = Ax + \Phi'(x) + F_1(x), \quad \text{for all } x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{n \times n}$ is a real matrix, $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$ is convex and F_1 is Lipschitz continuous, i.e.

$$\|F_1(x) - F_1(y)\| \leq k\|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n,$$

for some constant $k > 0$. Then F is continuous and $F + \omega I$ is monotone provided that $\omega \geq 0$ is great enough, i.e.

$$\omega \geq \sup_{\|x\|=1} \langle -Ax, x \rangle + k.$$

Remark 2.3. It follows from Theorem 2.1 that the unique solution of (2.1)–(2.5) is Lipschitz continuous on $[0,T]$.

Theorem 2.1 enable us to define the one parameter family $\{S(t) : 0 \leq t \leq T\}$ of operators from C into C, as follows:

$$S(t)y = u(t) \in C, \quad \text{for all } y \in C, \tag{2.6}$$

u being the unique solution on $[0,T]$ of the evolution problem (2.1)–(2.5). Note that

$$S(0)y = y \quad \text{for all } y \in C.$$

Lemma 2.4 (see e.g. [11]). *Let $T > 0$ be given and let $a, b \in L^1(0,T;\mathbb{R})$ with $b(t) \geq 0$ a.e. $t \in [0,T]$. Let the absolutely continuous function $w\colon [0,T] \to \mathbb{R}_+$ satisfy*

$$(1-\alpha)\frac{dw}{dt}(t) \leq a(t)w(t) + b(t)w^{\alpha}(t), \quad \text{a.e. } t \in [0,T],$$

where $0 \leq \alpha < 1$. Then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(0)e^{\int_0^t a(s)\,ds} + \int_0^t e^{\int_s^t a(q)\,dq} b(s)\,ds, \quad \textit{for all } t \in [0,T].$$

Theorem 2.5. *Suppose that the assumptions of Theorem 2.1 hold. Then*

$$\|S(t)y - S(t)z\| \leq e^{\omega t}\|y - z\|,$$

for all $y, z \in C$, $t \in [0,T]$.

Proof. Let $y, z \in C$ be given. We know that

$$\left\langle \frac{d}{dt}S(t)y + F(S(t)y) - f(t), v - S(t)y \right\rangle \geq 0, \tag{2.7}$$

for all $v \in C$, a.e. $t \in [0,T]$ and

$$\left\langle \frac{d}{dt}S(t)z + F(S(t)z) - f(t), h - S(t)z \right\rangle \geq 0, \tag{2.8}$$

for all $h \in C$, a.e. $t \in [0,T]$. Setting $v = S(t)z$ in (2.7) and $h = S(t)y$ in (2.8), we obtain the relations:

$$-\left\langle \frac{d}{dt}S(t)y + F(S(t)y) - f(t), S(t)z - S(t)y \right\rangle \leq 0,$$

a.e. $t \in [0,T]$ and

$$\left\langle \frac{d}{dt}S(t)z + F(S(t)z) - f(t), S(t)z - S(t)y \right\rangle \leq 0,$$

a.e. $t \in [0,T]$. It results that

$$\begin{aligned}\left\langle \frac{d}{dt}(S(t)z - S(t)y), S(t)z - S(t)y \right\rangle &\leq \langle \omega S(t)z - \omega S(t)y, S(t)z - S(t)y\rangle \\ &\quad - \langle [F+\omega I](S(t)z) - [F+\omega I](S(t)y), S(t)z - S(t)y\rangle\end{aligned}$$

a.e. $t \in [0,T]$.

Our hypothesis ensure that $F + \omega I$ is monotone. It results that

$$\frac{d}{dt}\|S(t)z - S(t)y\|^2 \leq 2\omega\|S(t)z - S(t)y\|^2, \quad \text{a.e. } t \in [0,T].$$

Let us first set $w(\,\cdot\,) := \|S(\,\cdot\,)z - S(\,\cdot\,)y\|^2$ and note that according to Remark 2.3 this function is absolutely continuous on $[0,T]$. Using then Lemma 2.4 with $a(\,\cdot\,) := 2\omega$, $b(\,\cdot\,) = 0$ and $\alpha = 0$, we get

$$\|S(t)z - S(t)y\|^2 \leq \|z - y\|^2 e^{2\omega t}, \quad \text{for all } t \in [0,T].$$

The conclusion follows. □

REMARK 2.6. It follows from theorem 2.5 that the unique solution of the evolution problem (2.1)–(2.5) is Lipschitz continuously depending on the initial data:

$$\max_{t\in[0,T]} \|S(t)y - S(t)z\| \leq e^{\omega T}\|y - z\|, \quad \text{for all } y, z \in C.$$

REMARK 2.7. Let us now consider the Poincaré operator $S(T): C \to C$; $y \to S(T)y$. Theorem 2.5 ensures that $S(T)$ is Lipschitz continuous on C, i.e.

$$\|S(T)y - S(T)z\| \leq e^{\omega T}\|y - z\|, \quad \text{for all } y, z \in C.$$

REMARK 2.8. Note that if F is monotone and continuous then Theorem 2.5 holds with $\omega = 0$. In this case, the Poincaré operator $S(T)$ is nonexpansive on C, i.e.

$$\|S(T)y - S(T)z\| \leq \|y - z\|, \quad \text{for all } y, z \in C.$$

For such case, the fixed point theory of nonexpansive operators can be used so as to give conditions ensuring the existence of periodic solutions for problem (1.7). We refer the reader to [6] for details.

According to (2.6), the unique solution of the problem (2.1)–(2.5) satisfies, in addition, the periodicity condition

$$u(0) = u(T)$$

if and only if y is a fixed point of $S(T)$, that is

$$S(T)y = y.$$

Thus the problem of the existence of a periodic solution for the evolution problem (2.1)–(2.3), (2.5) is reduced to that of the existence of a fixed point for $S(T)$.

3. A method of guiding functions for complementarity systems

Suppose that $\Omega \subset \mathbb{R}^n$ is open but possibly unbounded. Let $\widetilde{C}(\overline{\Omega})$ be the set of all $\varphi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ such that $\varphi^{-1}(0) := \{x \in \overline{\Omega} : \varphi(x) = 0\}$ is compact.

Remark 3.1. If $\varphi \in C^0(\overline{\Omega})$ and

$$\sup_{x \in \overline{\Omega}} \|x - \varphi(x)\| < \infty.$$

then $\varphi \in \widetilde{C}(\overline{\Omega})$.

Let $\varphi \in \widetilde{C}(\overline{\Omega})$ be given. If $0 \notin \varphi(\partial\Omega)$ then the topological degree of φ with respect to Ω and 0 is well-defined by (see e.g. [4]):

$$\deg(\varphi, \Omega, 0) := \deg_B(\varphi, \Omega \cap \Omega_0, 0)$$

where $\deg_B$ denotes the Brouwer degree and Ω_0 is any open bounded set that contains $\varphi^{-1}(0)$. Recall that deg has all properties of $\deg_B$ and coincides with $\deg_B$ as soon as Ω is bounded.

Let us now recall some properties of the topological degree we will use later in this section.

(1) If $0 \notin \varphi(\partial\Omega)$ and $\deg(\varphi, \Omega, 0) \neq 0$ then there exist $x \in \Omega$ such that $\varphi(x) = 0$.
(2) Let $\Phi: [0,1] \times \Omega \to \mathbb{R}^n$, $(\lambda, x) \to \Phi(\lambda, x)$, $\Phi \in \widetilde{C}([0,1] \times \overline{\Omega})$ and such that, for each $\lambda \in [0,1]$, one has $0 \notin \Phi(\lambda, \partial\Omega)$, then the map $\lambda \to \deg(\Phi(\lambda, \cdot\,), \Omega, 0)$ is constant on $[0,1]$.
(3) Let us denote by $\mathrm{id}_{\mathbb{R}^n}$ the identity mapping on $\mathbb{R}^n$. If $0 \in \Omega$ then

$$\deg(\mathrm{id}_{\mathbb{R}^n}, \Omega, 0) = 1.$$

(4) If $0 \notin \varphi(\partial\Omega)$ and $\alpha > 0$ then

$$\deg(\alpha\varphi, \Omega, 0) = \deg(\varphi, \Omega, 0), \quad \deg(-\alpha\varphi, \Omega, 0) = (-1)^n \deg(\varphi, \Omega, 0).$$

Let C be a nonempty closed convex cone, i.e.

$$0 \in C, \quad \lambda C \subset C, \quad \text{for all } \lambda > 0, \quad C + C \subset C.$$

In this case, the problem in (1.3) and (1.5) is equivalent to the following complementarity system:

$$u(t) \in C, \qquad t \in [0,T], \tag{3.1}$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), u(t) \right\rangle = 0, \qquad \text{a.e. } t \in [0,T], \tag{3.2}$$

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in C^*, \quad \text{a.e. } t \in [0,T]. \tag{3.3}$$

The projection operator $P_C\colon \mathbb{R}^n \to C$; $x \to P_C(x)$ is well-defined as the unique solution of the variational inequality:

$$\langle P_C(x) - x, v - P_C(x)\rangle \geq 0, \quad \text{for all } v \in C.$$

For $r > 0$, we set

$$\Omega_{C,r} := \{x \in \mathbb{R}^n : \|P_C(x)\| < r\}.$$

Then

$$\overline{\Omega}_{C,r} := \{x \in \mathbb{R}^n : \|P_C(x)\| \leq r\}$$

and

$$\partial\Omega_{C,r} := \{x \in \mathbb{R}^n : \|P_C(x)\| = r\}.$$

If $\varphi \in \widetilde{C}(\Omega_{C,r})$ and $0 \notin \varphi(\partial\Omega_{C,r})$ then $\deg(\varphi, \Omega_{C,r}, 0)$ is well-defined. Moreover, if there exists $r_0 > 0$ such that for every $r \geq r_0$, $\varphi \in \widetilde{C}(\Omega_{C,r})$ and $0 \notin \varphi(\partial\Omega_{C,r})$ then $\deg(\varphi, \Omega_{C,r}, 0)$ is constant for $r \geq r_0$ and one defines the index of φ at infinity $\operatorname{ind}(\varphi, \infty)$ by

$$\operatorname{ind}(\varphi, \infty) := \deg(\varphi, \Omega_{C,r}, 0), \quad \text{for all } r \geq r_0.$$

If in addition $\varphi(C) \subset C$ then we define

$$\mathrm{I}_C(\varphi, \infty) := \operatorname{ind}(\mathrm{id}_{\mathbb{R}^n} - P_C - \varphi \circ P_C, \infty).$$

Note that $\mathrm{I}_C(\varphi, \infty)$ is well-defined. First, because if $x \in \overline{\Omega}_{C,r}$ then $P_C x \in \overline{\Omega}_{C,r}$ and φ is continuous on $\overline{\Omega}_{C,r}$, it follows that the mapping $x \in \overline{\Omega}_{C,r} \mapsto x - P_C x - \varphi(P_C x)$ is continuous on $\overline{\Omega}_{C,r}$. Moreover, $(\mathrm{id}_{\mathbb{R}^n} - P_C - \varphi \circ P_C)^{-1}(0)$ is compact and $0 \notin \partial\Omega_{C,r}$. Indeed,

$$\|x - (x - P_C(x) - \varphi(P_C(x)))\| = \|P_C(x) + \varphi(P_C(x))\|$$

and thus

$$\sup_{x \in \overline{\Omega}_{C,r}} \|x - (x - P_C(x) - \varphi(P_C(x)))\| < \infty$$

so that $\mathrm{id}_{\mathbb{R}^n} - P_C - \varphi \circ P_C \in \widetilde{C}(\overline{\Omega}_{C,r})$. Moreover, $0 \notin (\mathrm{id}_{\mathbb{R}^n} - P_C - \varphi \circ P_C)(\partial\Omega_{C,r})$. Indeed, suppose that there exists $x \in \partial\Omega_{C,r}$ such that $x - P_C(x) - \varphi(P_C(x)) = 0$. Then $x = P_C(x) + \varphi(P_C(x)) \in C$ since C is cone. It results that $\varphi(x) = 0$ which is a contradiction since $0 \notin \varphi(\partial\Omega_{C,r})$.

We say that $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is a guiding function for (3.1)–(3.3) provided that there exists $R > 0$ such that

$$\langle F(x) - f(t), \nabla V(x)\rangle < 0, \quad \text{for all } x \in C,\ \|x\| \geq R,\ t \in [0, T].$$

Theorem 3.2. *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex cone. Suppose that $f\colon [0,\infty) \to \mathbb{R}^n$ satisfies $f \in C^0([0,\infty);\mathbb{R}^n)$, $df/dt \in L^1_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the conditions of Theorem 2.1. Suppose in addition that F has linear growth, i.e. there exist $C_1 > 0$, $C_2 \geq 0$ such that*

$$\|F(x)\| \leq C_1\|x\| + C_2, \quad \textit{for all } x \in C.$$

Suppose that there exists $V \in C^1(\mathbb{R}^n,\mathbb{R})$ and $R > 0$ such that

(a) *$\langle F(x) - f(t), \nabla V(x)\rangle < 0$, for all $x \in C$, $\|x\| \geq R$, $t \in [0,T]$;*
(b) *$\nabla V(x) \in C$, for all $x \in C$, $\|x\| \geq R$.*

Then there exists $r_0 > R$ such that

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(T)\,\cdot\,, \Omega_{C_r}, 0) = \mathrm{I}_C(\nabla V, \infty), \quad \textit{for all } r \geq r_0.$$

Proof. Let us set

$$r_0 := Re^{C_1T} + \frac{C_2}{C_1}(e^{C_1T} - 1) + \int_0^T \|f(s)\|e^{C_1 s}\,ds.$$

Part 1. We claim that if $y \in C$, $\|y\| = r$ with $r \geq r_0$, then

$$\|S(t)y\| \geq R, \quad \text{for all } t \in [0,T].$$

Suppose by contradiction that there exists $t^* \in [0,T]$ such that $\|S(t^*)y\| < R$. We know that $u(\,\cdot\,) \equiv S(\,\cdot\,)y \in C$ satisfies

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in -\partial\Psi_C(u(t)), \quad \text{a.e. } t \in [0,T],$$

and thus

$$\frac{du}{dt}(t^* - t) + F(u(t^* - t)) - f(t^* - t) \in -\partial\Psi_C(u(t^* - t)), \quad \text{a.e. } t \in [0,t^*].$$

Setting

$$Y(t) = u(t^* - t), \quad t \in [0,t^*],$$

we get

$$-\frac{dY}{dt}(t) + F(Y(t)) - f(t^* - t) \in -\partial\Psi_C(Y(t)), \quad \text{a.e. } t \in [0,t^*].$$

Thus

$$\left\langle -\frac{dY}{dt}(t), v - Y(t)\right\rangle \geq \langle -F(Y(t)) + f(t^* - t), v - Y(t)\rangle,$$

for all $v \in C$, a.e. $t \in [0,t^*]$. Recalling that C is a cone, we may set $v = 2Y(t) \in C$ to obtain

$$\begin{aligned}
\left\langle \frac{dY}{dt}(t), Y(t)\right\rangle &\leq \langle F(Y(t)) - f(t^* - t), Y(t)\rangle \\
&\leq (C_1\|Y(t)\| + C_2)\|Y(t)\| + \|f(t^* - t)\|\|Y(t)\| \\
&= C_1\|Y(t)\|^2 + C_2\|Y(t)\| + \|f(t^* - t)\|\|Y(t)\|,
\end{aligned}$$

a.e. $t \in [0, t^*]$. Thus

$$\frac{1}{2}\frac{d}{dt}\|Y(t)\|^2 \leq C_1\|Y(t)\|^2 + C_2\|Y(t)\| + \|f(t^* - t)\|\|Y(t)\|, \quad \text{a.e. } t \in [0, t^*].$$

Using Lemma 2.4 with $w(\,\cdot\,) := \|Y(\,\cdot\,)\|^2$, $a(\,\cdot\,) := C_1$, $b(\,\cdot\,) := C_2 + \|f(t^* - \,\cdot\,)\|$ and $\alpha := 1/2$, we obtain

$$\|Y(t)\| \leq \|Y(0)\|e^{C_1 t} + \int_0^t C_2 e^{C_1(t-s)}\, ds + \int_0^t \|f(t^* - s)\|e^{C_1(t-s)}\, ds,$$

for all $t \in [0, t^*]$. Since $Y(t^*) = u(0) = S(0)y = y$ and $Y(0) = u(t^*) = S(t^*)y$, we get

$$\begin{aligned}\|y\| &\leq \|S(t^*)y\|e^{C_1 t^*} + \int_0^{t^*} C_2 e^{C_1(t^*-s)}\, ds + \int_0^{t^*} \|f(t^* - s)\|e^{C_1(t^*-s)}\, ds \\ &< Re^{C_1 T} + \frac{C_2}{C_1}(e^{C_1 T} - 1) + \int_0^T \|f(s)\|e^{C_1 s} ds = r_0.\end{aligned}$$

The contradiction $\|y\| < r_0$ has thus been obtained.

Let $r \geq r_0$ be given.

Part 2. We claim that there exists $\varepsilon > 0$ and $T^* \in (0, T]$ such that

$$\langle F(x) - f(t), \nabla V(y)\rangle < 0, \qquad \text{for all } x \in \mathbb{R}^n,\ y \in C,\ \|y\| = r,\ \|x - y\| \leq \varepsilon,\ t \in [0, T^*].$$

Indeed, the mapping $(t, x, y) \to \langle F(x) - f(t), \nabla V(y)\rangle$ is continuous on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and if $y \in C, \|y\| = r \geq r_0 > R$ then (by assumption (a)): $\langle F(y) - f(0), \nabla V(y)\rangle < 0$. Thus, for $t > 0$ closed to 0, let us say $t \leq T^*$ and x closed to y, let us say $\|x - y\| \leq \varepsilon$, $\varepsilon > 0$, small, we have $\langle F(x) - f(t), \nabla V(y)\rangle < 0$.

Part 3. We claim that there exists $\overline{T} \in (0, T^*]$ such that

$$\|S(t)y - y\| \leq \varepsilon, \quad \text{for all } y \in C,\ \|y\| = r \text{ and all } t \in [0, \overline{T}].$$

Indeed, if we suppose the contrary, then we may find sequences $t_n \in [0, T^*/n]$ ($n \in \mathbb{N}$, $n \geq 1$) and $y_n \in C$, $\|y_n\| = r$ such that $\|S(t_n)y_n - y_n\| > \varepsilon$. Along a subsequence, we may assume that $t_n \to 0+$ and $y_n \to y^* \in \partial\Omega_{C,r}$. On the other hand, we have

$$\begin{aligned}\|S(t_n)y_n - y_n\| &= \|S(t_n)y_n - S(t_n)y^* + S(t_n)y^* - y_n\| \\ &\leq \|S(t_n)y_n - S(t_n)y^*\| + \|S(t_n)y^* - y_n\|.\end{aligned}$$

Then using Theorem 2.5, we obtain

$$\|S(t_n)y_n - y_n\| \leq e^{wt_n}\|y_n - y^*\| + \|S(t_n)y^* - y_n\|.$$

Using the continuity of the map $t \to S(t)y$, we see that $\|S(t_n)y_n - y_n\| \to 0$ which is a contradiction.

(4) Let $H_{\overline{T}}\colon [0,1]\times\overline{\Omega}_{C,r}\to\mathbb{R}^n$; $(\lambda,y)\to H_{\overline{T}}(\lambda,y):=y-(1-\lambda)\nabla V(P_C(y))-S(\lambda\overline{T})P_C(y)$. We have

$$\begin{aligned}\sup_{(\lambda,y)\in[0,1]\times\overline{\Omega}_{C,r}} &\|y-H_{\overline{T}}(\lambda,y)\| \\ &= \sup_{(\lambda,y)\in[0,1]\times\overline{\Omega}_{C,r}} \|(1-\lambda)\nabla V(P_C(y))+S(\lambda\overline{T})P_C(y)\|<\infty.\end{aligned}$$

We claim that the homotopy $H_{\overline{T}}$ is such that $0\neq H_{\overline{T}}(\lambda,y)$, for all $y\in\partial\Omega_{C,r}$, $\lambda\in[0,1]$. By contradiction, suppose that there exists $y\in\mathbb{R}^n$, $\|P_C(y)\|=r$ and $\lambda\in[0,1]$ such that

$$y-(1-\lambda)\nabla V(P_C(y))-S(\lambda\overline{T})P_C(y)=0.$$

Then

$$y=(1-\lambda)\nabla V(P_C(y))+S(\lambda\overline{T})P_C(y)\in C$$

and thus $y=P_C(y)$. We obtain

$$S(\lambda\overline{T})y-y=-(1-\lambda)\nabla V(y)$$

and thus

$$\langle S(\lambda\overline{T})y-y,\nabla V(y)\rangle=-(1-\lambda)\|\nabla V(y)\|^2\leq 0. \tag{3.4}$$

On the other hand, we know that

$$\left\langle \frac{d}{dt}S(t)y, v-S(t)y\right\rangle \geq \langle -F(S(t)y)+f(t), v-S(t)y\rangle, \tag{3.5}$$

for all $v\in C$, a.e. $t\in[0,T]$.

We know that $y\in C$, $S(t)y\in C$, for all $t\in[0,T]$ and by assumption (b), $\nabla V(y)\in C$. Recalling that C is a cone, we may set $v=S(t)y+\nabla V(y)\in C$ in (3.5) to get

$$\left\langle \frac{d}{dt}S(t)y,\nabla V(y)\right\rangle\geq\langle -F(S(t)y)+f(t),\nabla V(y)\rangle,\quad \text{a.e. } t\in[0,T].$$

Thus

$$\left\langle \int_0^{\lambda\overline{T}}\frac{d}{ds}S(s)y\,ds,\nabla V(y)\right\rangle\geq\int_0^{\lambda\overline{T}}\langle -F(S(s)y)+f(s),\nabla V(y)\rangle\,ds.$$

Part 1 of this proof ensures that $\|S(t)y\|\geq R$, for all $t\in[0,\lambda\overline{T}]\subset[0,T]$. Part 3 of this proof garantees that $\|S(t)y-y\|\leq\varepsilon$, for all $t\in[0,\lambda\overline{T}]\subset[0,\overline{T}]$. Then using part (2) of this proof, we may assert that the map $s\to\langle -F(S(s)y)+f(s),\nabla V(y)\rangle$ is continuous and strictly positive on $[0,\lambda\overline{T}]$. Thus

$$\int_0^{\lambda\overline{T}}\langle -F(S(s)y)+f(s),\nabla V(y)\rangle\,ds>0$$

and we obtain

$$\langle S(\lambda\overline{T})y - y, \nabla V(y)\rangle = \left\langle \int_0^{\lambda\overline{T}} \frac{d}{ds} S(s)y\,ds, \nabla V(y)\right\rangle > 0.$$

This is a contradiction to (3.4).

Part 5. Thanks to Part 4 of this proof, we may use the invariance by homotopy property of the topological degree and see that

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - S(\overline{T})P_C, \Omega_{C,r}, 0) &= \deg(H_{\overline{T}}(1,\cdot), \Omega_{C,r}, 0) = \deg(H_{\overline{T}}(0,\cdot), \Omega_{C,r}, 0)\\ &= \deg(\mathrm{id}_{\mathbb{R}^n} - \nabla V \circ P_C - P_C, \Omega_{C,r}, 0) = \mathrm{I}_C(\nabla V, \infty).\end{aligned}$$

Part 6. Let $H\colon [0,1]\times\overline{\Omega}_{C,r} \to \mathbb{R}^n$; $(\lambda, y) \to H(\lambda, y) := y - S((1-\lambda)T + \lambda\overline{T})P_C(y)$. We have

$$\sup_{(\lambda,y)\in[0,1]\times\overline{\Omega}_{C,r}} \|y - H(\lambda,y)\| = \sup_{(\lambda,y)\in[0,1]\times\overline{\Omega}_{C,r}} \|S((1-\lambda)T + \lambda\overline{T})P_C(y)\| < \infty.$$

We claim that $H(\lambda, y) \neq 0$, for all $y \in \partial\Omega_{C,r}$, $\lambda \in [0,1]$. By contradiction, suppose that there exists $y \in \mathbb{R}^n$, $\|P_C(y)\| = r$ and $\lambda \in [0,1]$ such that $y = S((1-\lambda)T + \lambda\overline{T})P_C(y)$. Then $y \in C$ and $P_C(y) = y$. Let us now set $h := (1-\lambda)T + \lambda\overline{T}$. We have

$$y = S(h)y$$

and thus

$$V(y) = V(S(h)y). \tag{3.6}$$

On the other hand,

$$\left\langle \frac{d}{dt} S(t)y, v - S(t)y\right\rangle \geq \langle -F(S(t)y) + f(t), v - S(t)y\rangle, \tag{3.7}$$

for all $v \in C$, a.e. $t \in [0,T]$. We know that $y \in C$, $S(t)y \in C$, for all $t \in [0,T]$ and by assumption (b), $\nabla V(S(t)y) \in C$. Recalling that C is a cone, we may set $v = S(t)y + \nabla V(S(t)y) \in C$ in (3.7) to get

$$\left\langle \frac{d}{dt} S(t)y, \nabla V(S(t)y)\right\rangle \geq \langle -F(S(t)y) + f(t), \nabla V(S(t)y)\rangle, \tag{3.8}$$

for all $v \in C$, a.e. $t \in [0,T]$.

Part 1 of this proof ensures that $\|S(t)y\| \geq R$, for all $t \in [0,T]$. The map $s \to \langle -F(S(s)y) + f(s), \nabla V(S(s)y)\rangle$ is continuous and (by assumption (b)) strictly positive on $[0,T]$. Thus, using (3.8), we obtain

$$\begin{aligned}V(S(h)y) - V(y) &= \int_0^h \frac{d}{ds} V(S(s)y)\,ds = \int_0^h \left\langle \frac{d}{ds} S(s)y, \nabla V(S(s)y)\right\rangle\\ &\geq \int_0^h \langle -F(S(t)y) + f(t), \nabla V(S(t)y)\rangle > 0.\end{aligned}$$

This is a contradiction to (3.6).

Part 7. Thanks to Part 6 of this proof, we may use the invariance by homotopy property of the topological degree and see that

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - S(T)P_C, \Omega_{C,r}, 0) &= \deg(H(0,\,\cdot\,), \Omega_{C,r}, 0)\\ &= \deg(H(1,\,\cdot\,), \Omega_{C,r}, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(\overline{T})P_C, \Omega_{C,r}, 0).\end{aligned}$$

In conclusion, for all $r \geq r_0$, we have

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(T)P_C, \Omega_{C,r}, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(\overline{T})P_C, \Omega_{C,r}, 0)$$

and

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(\overline{T})P_C, \Omega_{C,r}, 0) = \mathrm{I}_C(\nabla V, \infty).$$

Thus

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(T)P_C, \Omega_{C,r}, 0) = \mathrm{I}_C(\nabla V, \infty). \qquad \square$$

Remark 3.3. In the case of differential equations, i.e. if $C = \mathbb{R}^n$, then one can easily prove the result by using the homotopy $h(\lambda, y) := (y - S(\lambda T)y)/\lambda$. Such homotopy cannot be used in the general case for our problem in (1.2) since $\lim_{\lambda \to 0+}(x - S(\lambda T)x)/\lambda \in -T(f(0) - F(x) - \partial\Psi_C(x))$ and the continuity of $h(0, x)$ is not ensured.

Remark 3.4. If there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ satisfying conditions (a) and (b) in Theorem 3.2 then necessarily $F(x) - f(t) \notin C^*$, for all $x \in C$, $\|x\| \geq R$, $t \in [0, T]$.

Corollary 3.5. *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex cone. Suppose that $f\colon [0,\infty) \to \mathbb{R}^n$ satisfies $f \in C^0([0,\infty); \mathbb{R}^n)$, $df/dt \in L^1_{\mathrm{loc}}(0,\infty; \mathbb{R}^n)$. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the conditions of Theorem 2.1. Suppose in addition that F has linear growth. Suppose that there exists $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and $R > 0$ such that*

(a) $\langle F(x) - f(t), \nabla V(x)\rangle < 0$, *for all $x \in C$, $\|x\| \geq R$, $t \in [0, T]$;*
(b) $\nabla V(x) \in C$, *for all $x \in C$, $\|x\| \geq R$;*
(c) $\mathrm{I}_C(\nabla V, \infty) \neq 0$.

Then there exists at least one $u \in C^0([0, T]; \mathbb{R}^n)$ such that $du/dt \in L^\infty(0, T; \mathbb{R}^n)$,

$$\begin{gathered} u(t) \in C, \quad t \in [0,T];\\ u(0) = u(T);\\ \left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), v - u(t) \right\rangle \geq 0, \quad \text{for all } v \in C,\ \text{a.e. } t \in [0,T]. \end{gathered}$$

Proof. Theorem 3.2 together with assumption (c) ensure that for $r > 0$ great enough, we have $\deg(\mathrm{id}_{\mathbb{R}^n} - S(T)\,\cdot\,, \Omega_{C,r}, 0) \neq 0$ and the existence of a fixed point for the Poincaré operator follows from the existence property of the topological degree. $\square$

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