

THE EFFECT OF THE DOMAIN'S CONFIGURATION SPACE ON THE NUMBER OF NODAL SOLUTIONS OF SINGULARLY PERTURBED ELLIPTIC EQUATIONS

THOMAS BARTSCH — TOBIAS WETH

Dedicated to the memory of Professor Olga A. Ladyzhenskaya

ABSTRACT. We prove a new multiplicity result for nodal solutions of the Dirichlet problem for the singularly perturbed equation $-\varepsilon^2 \Delta u + u = f(u)$ for $\varepsilon > 0$ small on a bounded domain $\Omega \subset \mathbb{R}^N$. The nonlinearity f grows superlinearly and subcritically. We relate the topology of the configuration space $C\Omega = \{(x, y) \in \Omega \times \Omega : x \neq y\}$ of ordered pairs in the domain to the number of solutions with exactly two nodal domains. More precisely, we show that there exist at least $\text{cupl}(C\Omega) + 2$ nodal solutions, where cupl denotes the cuplength of a topological space. We furthermore show that $\text{cupl}(C\Omega) + 1$ of these solutions have precisely two nodal domains, and the last one has at most three nodal domains.

1. Introduction

In this paper we are concerned with the singularly perturbed problem

$$(P_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + au = f(u) & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain, and $a > 0$ is fixed. The nonlinearity $f \in C^1(\mathbb{R})$ grows superlinearly and subcritically, but it does not have to be odd

2000 *Mathematics Subject Classification.* 35J65, 35B25, 47J30, 58E05.

Key words and phrases. Singularly perturbed problem, sign-changing solution, configuration space, Lusternik–Schnirelman category.

The first named author supported by DFG grant BA 1009/15-1.

The second named author supported by DFG grant WE 2821/2-1.

in u . A model nonlinearity is

$$(1.1) \quad f(u) = \sum_{i=1}^k c_i |u|^{p_i-2} u^+ + \sum_{j=1}^l d_j |u|^{q_j-2} u^-,$$

where $p_i, q_j \in (2, 2^*)$, $c_i, d_j > 0$ for $1 \leq i \leq k$, $1 \leq j \leq l$ and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Here 2^* denotes the critical Sobolev exponent, that is, $2^* := 2N/(N-2)$ for $N \geq 3$, and $2^* = \infty$ for $N = 2$.

There are quite a number of papers on the existence of positive solutions of (P_ε) under various hypotheses on the nonlinearity f and the domain Ω . Fascinating results have been achieved relating the topology of the domain Ω to the number and location of positive solutions. We refer the reader to [9], [10], [31], [18], [21] and the references therein where one can also find detailed information on the shape of positive solutions, in particular on the number and position of spikes.

Motivated by segregation phenomena for certain two competing species models, Dancer and Du [16], [17], as well as recently Conti, Terracini and Verzini [15], were led to investigate nodal solutions of semilinear Dirichlet problems. More precisely, they considered a coupled system of two elliptic equations where the coupling models the competition. If a coupling parameter α increases then certain solutions $u_\alpha > 0$ and $v_\alpha > 0$ of the system converge as $\alpha \rightarrow \infty$ towards functions w_+ , w_- , respectively, which have disjoint supports in Ω . Moreover, the difference $w_+ - w_-$ is a nodal solution of an associated scalar Dirichlet problem. On the other hand, given a nodal solution w of the limit problem one can find, for large competition parameters α , solutions $u_\alpha > 0$, $v_\alpha > 0$ of the system which converge towards the positive and the negative part of w , respectively. Since the mid nineties a considerable amount of work has been devoted to investigate the set of nodal solutions of Dirichlet problems. In addition to the existence of nodal solutions, estimates on the number of nodal domains and certain localization results have been obtained; cf. [3], [5]–[7], [11], [12], [19], [27]. These results should also be seen in the context of a long standing open question whether semilinear Dirichlet problems on a bounded domain with superlinear and subcritical nonlinearity always have infinitely many solutions. In order to settle the problem one needs to understand nodal solutions and their properties.

In this paper we show that the topology of the configuration space $C\Omega := \{(x, y) \in \Omega \times \Omega : x \neq y\}$ of ordered pairs in the domain plays an important role when one is interested in nodal solutions which have precisely two nodal domains. The role of $C\Omega$ is in fact similar to the role of Ω concerning the existence of positive solutions. A first result has been obtained in [7] where we proved that for $\varepsilon > 0$ small, (P_ε) has at least two nodal solutions with precisely two nodal domains plus a third nodal solution with either two or three nodal domains.

This result holds true for any bounded domain irrespective of its topology. It is a special case of our main theorem: If $\partial\Omega$ has a tubular neighbourhood then for $\varepsilon > 0$ small, (P_ε) has at least $\text{cupl}(C\Omega) + 1$ nodal solutions with precisely two nodal domains plus an additional nodal solution with either two or three nodal domains. We have a refined result for an arbitrary bounded domain which essentially involves the shape of $\partial\Omega$.

Our precise assumptions on the nonlinearity are as follows:

- (f₁) $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$.
- (f₂) There exists $p \in (2, 2^*)$ such that

$$|f'(t)| \leq C(1 + |t|^{p-2}) \quad \text{for all } t \in \mathbb{R}.$$

- (f₃) $f'(t) > f(t)/t$ for all $t \neq 0$.
- (f₄) There exists $\theta > 2$ such that $0 < \theta F(t) \leq tf(t)$ for all $t \in \mathbb{R}$.

Here $F(t) := \int_0^t f(s) ds$ is a primitive of f . These assumptions in particular hold for the model nonlinearity (1.1).

For $r > 0$ we set

$$\begin{aligned} \Omega_r &:= \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r\}, \\ \Omega^r &:= \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq r\}, \\ C_r\Omega &:= \{(x, y) \in \Omega_r \times \Omega_r : |x - y| \geq 2r\}, \end{aligned}$$

and we consider the inclusion

$$i_r = i_r(\Omega): (C_r\Omega \times I_0^2, C_r\Omega \times \partial I_0^2) \hookrightarrow (C\Omega^r \times \mathbb{R}^2, C\Omega^r \times (\mathbb{R}^2 \setminus \{0\}))$$

where $I_0 := [-1, 1]$ and $C\Omega^r$ is the configuration space of ordered pairs in Ω^r , that is, $C\Omega^r = \{(x, y) \in \Omega^r \times \Omega^r : x \neq y\}$.

THEOREM 1.1. *Suppose that (f₁)–(f₄) are satisfied. Then, for any bounded domain Ω and any $r > 0$, there is $\varepsilon_r > 0$ such that for $\varepsilon \in (0, \varepsilon_r]$ problem (P_ε) has at least $\text{cat}(i_r) + 1$ nodal solutions. Moreover, $\text{cat}(i_r)$ of these solutions have precisely two nodal domains, and the last one has at most three nodal domains.*

Here $\text{cat}(f)$ denotes the category of the map $f: (A, B) \rightarrow (A', B')$ as defined in [14]. This will be recalled in the next section. The category $\text{cat}(\text{id}_{(A,B)})$ of the identity map is just the usual relative category $\text{cat}(A, B)$ of the pair (A, B) of topological spaces. Here are two lower bounds for $\text{cat}(i_r)$.

PROPOSITION 1.2.

- (a) For any bounded domain Ω and $r > 0$ small we have $\text{cat}(i_r) \geq 2$.
- (b) If $\partial\Omega$ has a C^0 -tubular neighbourhood then

$$\text{cat}(i_r) \geq \text{cat}(j) \geq \text{cupl}(C\Omega) + 1 \geq \max\{2 + \text{cupl}(\Omega), 2\text{cupl}(\Omega)\}$$

holds for $r > 0$ small enough, where

$$j: (C\Omega \times I_0^2, C\Omega \times \partial I_0^2) \hookrightarrow (C\Omega \times \mathbb{R}^2, C\Omega \times (\mathbb{R}^2 \setminus \{0\}))$$

denotes the inclusion.

By a C^0 -tubular neighbourhood we mean a continuous embedding

$$\nu: \partial\Omega \times (-1, 1) \rightarrow \mathbb{R}^N$$

such that $\nu(x, 0) = x$ for all $x \in \partial\Omega$, $\nu^{-1}(\Omega) = \partial\Omega \times (-1, 0)$, and $\nu^{-1}(\mathbb{R}^N \setminus \bar{\Omega}) = \partial\Omega \times (0, 1)$.

Recall that the cuplength $\text{cupl}(A)$ of a topological space A is defined to be the largest integer k such that there exist elements $\alpha_1, \dots, \alpha_k \in \tilde{H}^*(A)$ in the reduced cohomology of A with nontrivial cup product:

$$\alpha_1 \smile \dots \smile \alpha_k \neq 0 \in H^*(A).$$

In Proposition 1.2 the cup product is defined using singular cohomology theory with coefficients in the field \mathbb{F}_2 of two elements.

Theorem 1.1 and Proposition 1.2 yield three nodal solutions of (P_ε) on *any* bounded domain. This particular result has already been established by the authors in [7]. It would be very interesting to investigate the shape of the solutions given by Theorem 1.1, and the nodal lines separating the nodal components. From our proof it will be clear that $\text{cat}(i_r)$ of the solutions have one positive and one negative spike. In the special case where $\partial\Omega$ is smooth and $f(u) = |u|^{p-2}u$, $2 < p < 2^*$, these two-spike solutions have already been found by Dancer and Yan [19]. Their method depends in an essential way on the non-obvious fact that the ground state solution of $-\Delta u + u = |u|^{p-2}u$ in $H^1(\mathbb{R}^N)$ is unique and nondegenerate modulo translations. Such a property is unknown for our general class of nonlinearities instead of $|u|^{p-2}u$, and we suspect that it is not true. Very few is known about the location of the spikes of the nodal solutions. For the nonlinearity $f(u) = |u|^{p-2}u$, $2 < p < 2^*$, Noussair and Wei [30] were able to localize the spikes of at least one nodal solution, namely the *least energy nodal solution*. Even for quite general nonlinearities the shape of the least energy nodal solution seems to be closely related to the geometry of Ω . In particular we proved in [8] that on a radially symmetric domain this solution is in general not radial but *foliated Schwarz symmetric*; see [32] for this notion and [1] for a proof of the fact that the least energy nodal solution is not radially symmetric. More symmetry information is available in the special case $f(u) = |u|^{p-2}u$, see [34]. It would also be interesting to find conditions and examples which yield the precise number of nodal domains (two or three) of the additional solution obtained in Theorem 1.1. In fact, on an arbitrary bounded domain Ω there is

not a single result which guarantees the existence of a solution with three or more nodal domains.

Throughout this paper, if B is a subset of a topological space A , we write $\text{int}_A(B)$, resp. $\text{clos}_A(B)$ resp. $\partial_A B$ for the interior, closure, boundary of B in A , respectively. If the ambient space is understood, we suppress the subscript and just write $\text{int}(B)$, $\text{clos}(B)$ and ∂B .

For $1 \leq p \leq \infty$ and a function $u \in L^p(\mathbb{R}^N)$, we denote by $|u|_p$ the usual L^p -norm of u .

Acknowledgements. Part of this paper was written while the second author was visiting the Department of Mathematics at the University of Minnesota, to which he is deeply grateful for its hospitality. The visit was supported by DFG Grant WE 2821/2-1 (Germany).

2. Preliminaries

In this section we discuss three different invariants for a map f between pairs of topological spaces. The category $\text{cat}(f)$, the excisive category $\text{ecat}(f)$ and the cuplength $\text{cupl}(f)$. All these invariants measure the topological complexity of the map f . $\text{ecat}(f)$ is the finest of these measures, and $\text{cupl}(f)$ is the easiest to calculate.

Let $B \subset A$ and $B' \subset A'$ be topological spaces and $f: (A, B) \rightarrow (A', B')$ be a continuous map, that is, $f: A \rightarrow A'$ is continuous and $f(B) \subset B'$. The category $\text{cat}(f)$ of f is the infimum of all integers $k \geq 0$ such that there exists a covering $A = A_0 \cup \dots \cup A_k$ of A by open sets $A_i \subset A$ with the following properties:

- (1) $B \subset A_0$ and there is a homotopy $h: (A_0 \times [0, 1], B \times [0, 1]) \rightarrow (A', B')$ satisfying $h(x, 0) = f(x)$ and $h(x, 1) \in B'$ for all $x \in A_0$.
- (2) For $i = 1, \dots, k$ the restriction $f|_{A_i}: A_i \rightarrow A'$ is homotopic to a constant map $A_i \rightarrow A'$.

If no such covering exists we define $\text{cat}(f) = \infty$.

This definition is due to [14] where a more general equivariant version has been introduced. If $f = \text{id}_{(A, B)}$ is the identity map, then we write $\text{cat}(A, B) = \text{cat}(\text{id}_{(A, B)})$. Two simple properties of the category of a map are:

LEMMA 2.1.

- (a) For any two continuous maps $f: (A, B) \rightarrow (A', B')$ and $f': (A', B') \rightarrow (A'', B'')$ we have $\text{cat}(f' \circ f) \leq \min\{\text{cat}(f), \text{cat}(f')\}$. In particular, $\text{cat}(f) \leq \min\{\text{cat}(A, B), \text{cat}(A', B')\}$.
- (b) If $f, g: (A, B) \rightarrow (A', B')$ are homotopic (as maps of pairs) then $\text{cat}(f) = \text{cat}(g)$.

A proof can be found in [14]. Observe that $\text{cat}_A(B) := \text{cat}(B \hookrightarrow A)$ is the usual category of B in A . Hence, $\text{cat}_A(B)$ is the smallest integer k such that

B can be covered by k open subsets of A each of which is contractible in A . We would like to make the reader aware of the fact that many authors define $\text{cat}_A B$ by coverings with *closed* sets. From the topological point of view it is more convenient to use open coverings. Both definitions coincide for ANRs. The following properties are immediate consequences of our definition.

LEMMA 2.2. *Let $B \subset A$ be such that every point in B has an open neighbourhood in A which is contractible in A . Then*

- (a) B contains at least $\text{cat}_A(B)$ points.
- (b) If $B \subset A$ is compact, then $\text{cat}_A(B) < \infty$, and there is a neighbourhood U of B such that $\text{cat}_A(U) = \text{cat}_A(B)$.

Next we introduce the excisive category $\text{ecat}(f)$ of $f: (A, B) \rightarrow (A', B')$. This number is defined as the smallest integer k such that there exists a covering $A = A_0 \cup \dots \cup A_k$ of A by open sets $A_i \subset A$ with the following properties:

- $B \subset A_0$ and there is a homotopy $h: A_0 \times [0, 1] \rightarrow A'$ satisfying
 - (a) $h(x, 0) = f(x)$ for $x \in A_0$.
 - (b) If $h(x, t) \in B'$, then $h(x, s) = h(x, t)$ for $s \geq t$.
 - (c) $h(x, 1) \in B'$ for all $x \in A_0$
- $f(A_i) \cap B' = \emptyset$ for $i = 1, \dots, k$, and the restriction $f|_{A_i}: A_i \rightarrow A' \setminus B'$ is homotopic to a constant map $A_i \rightarrow A' \setminus B'$.

Again we write $\text{ecat}(A, B) = \text{ecat}(\text{id}_{(A, B)})$. In [23] $\text{ecat}(A, B)$ is called strong category. This is however not consistent with the classical strong category $\text{Cat}(A)$ which is defined using coverings by sets which are contractible in itself. We observe two properties of ecat whose proofs are straightforward.

LEMMA 2.3. *For any two continuous maps $f: (A, B) \rightarrow (A', B')$ and $f': (A', B') \rightarrow (A'', B'')$ we have $\text{ecat}(f' \circ f) \leq \text{ecat}(f')$. In particular, $\text{ecat}(f) \leq \text{ecat}(A', B')$.*

LEMMA 2.4. *Let $B, C \subset A$ and $B', C' \subset A'$ be closed subsets such that $C \cup B = A$ and $C' \cup B' = A'$. Let $f: (C, C \cap B) \rightarrow (C', C' \cap B')$ be a continuous map, and let $\tilde{f}: (A, B) \rightarrow (A', B')$ be an arbitrary continuation of f . Then $\text{ecat}(f) = \text{ecat}(\tilde{f})$.*

As a corollary we get the excision property for the pair (A, B) (cf. [23, Proposition 3.8]).

CORROLARY 2.5. *Let $B, C \subset A$ be closed and such that $B \cup C = A$. Then $\text{ecat}(A, B) = \text{ecat}(C, C \cap B)$.*

We finally introduce the cuplength of $f: (A, B) \rightarrow (A', B')$. References for the algebraic topology which we use are [22], [28], [33]. Let H^* denote either singular or Alexander–Spanier cohomology with coefficients in the field \mathbb{F}_2 of

two elements. We recall that the cup product \smile turns $H^*(A)$ into a ring with unit 1_A , and it turns $H^*(A, B)$ into a module over $H^*(A)$. A continuous map $f: (A, B) \rightarrow (A', B')$ induces a homomorphism $f^*: H^*(A') \rightarrow H^*(A)$ of rings as well as a homomorphism $f^*: H^*(A', B') \rightarrow H^*(A, B)$ of abelian groups. The number $\text{cupl}(f)$ is defined as the largest integer $k \geq 0$ such that there exist elements $\alpha_1, \dots, \alpha_k \in \tilde{H}^*(A')$ and $\beta \in H^*(A', B')$ with

$$f^*(\alpha_1 \smile \dots \smile \alpha_k \smile \beta) = f^*(\alpha_1) \smile \dots \smile f^*(\alpha_k) \smile f^*(\beta) \neq 0 \in H^*(A, B).$$

It is 0 if $\tilde{H}^*(A') = 0$ and $f^* \neq 0$, and it is -1 if $f^* = 0$. Again we write $\text{cupl}(A, B) = \text{cupl}(\text{id}_{(A, B)})$. This is consistent with the definition of $\text{cupl}(A, B)$ given in [23]. We have the following.

LEMMA 2.6.

- (a) For any two continuous maps $f: (A, B) \rightarrow (A', B')$ and $f': (A', B') \rightarrow (A'', B'')$ we have $\text{cupl}(f' \circ f) \leq \min\{\text{cupl}(f), \text{cupl}(f')\}$. In particular, $\text{cupl}(f) \leq \min\{\text{cupl}(A, B), \text{cupl}(A', B')\}$.
- (b) If $f, g: (A, B) \rightarrow (A', B')$ are homotopic then $\text{cupl}(f) = \text{cupl}(g)$.
- (c) (Excision property) Let A, A' be paracompact Hausdorff spaces, and let $B, C \subset A$ and $B', C' \subset A'$ be closed subsets such that $C \cup B = A$ and $C' \cup B' = A'$. Let $f: (C, C \cap B) \rightarrow (C', C' \cap B')$ be a continuous map, and let $\tilde{f}: (A, B) \rightarrow (A', B')$ be a continuation of f . Then $\text{cupl}(f) = \text{cupl}(\tilde{f})$ if the cuplength is defined with Alexander–Spanier cohomology.

PROOF. (a) Set $k := \text{cupl}(f' \circ f)$ and let $\alpha_1, \dots, \alpha_k \in \tilde{H}^*(A'')$, $\beta \in H^*(A'', B'')$ satisfy $(f' \circ f)^*(\alpha_1 \smile \dots \smile \alpha_k \smile \beta) = f'^*(\alpha_1) \smile \dots \smile f'^*(\alpha_k) \smile f'^*(\beta) \neq 0 \in H^*(A'', B'')$. Then $f'^*(\alpha_1 \smile \dots \smile \alpha_k \smile \beta) \neq 0$, so $\text{cupl}(f') \geq k$, and $f^*(f'^*(\alpha_1) \smile \dots \smile f'^*(\alpha_k) \smile f'^*(\beta)) \neq 0$ which implies $\text{cupl}(f) \geq k$.

(b) is trivial because $f^* = g^*$.

(c) follows from the strong excision property of Alexander–Spanier cohomology which implies that the inclusions $(C, C \cap B) \hookrightarrow (A, B)$ and $(C', C' \cap B') \hookrightarrow (A', B')$ induce isomorphisms in cohomology. \square

The next lemma shows how ecat , cat and cupl are related.

LEMMA 2.7. $\text{ecat}(f) \geq \text{cat}(f) \geq \text{cupl}(f) + 1$.

PROOF. The first inequality is obvious by the definition of ecat and cat . The proof of the second inequality is a modification of the standard argument showing $\text{cat}(A, B) \geq \text{cupl}(A, B) + 1$. \square

We close this section with a brief discussion of semiflows and invariant sets. Let $\varphi: \mathcal{G} \subset [0, \infty) \times X \rightarrow X$ denote a continuous semiflow on a metric space X . Here $\mathcal{G} = \{(t, u) : u \in X, 0 \leq t < T(u)\}$, where $T(u)$ is the maximal existence

time for the trajectory $t \mapsto \varphi(t, u)$. We often write φ^t instead of $\varphi(t, \cdot)$. For a subset $Y \subset X$ we set

$$\begin{aligned} \text{Inv}(Y) &:= \{u \in Y : \varphi^t(u) \in Y \text{ for } 0 \leq t < T(u)\}, \\ \mathcal{A}(Y) &:= \{u \in X : \varphi^t(u) \in Y \text{ for some } t \in [0, T(u))\}. \end{aligned}$$

We say that Y is positively invariant if $\text{Inv}(Y) = Y$. We call Y *strictly positively invariant* if the following holds:

$$\text{if } u \in Y \text{ then } \varphi^t(u) \in \text{int}_X(Y) \text{ for } 0 \leq t < T(u).$$

Finally we define the Y -entrance time function $e_Y: X \rightarrow [0, \infty]$ by

$$e_Y(u) = \begin{cases} \inf\{t \geq 0 : \varphi^t(u) \in Y\} & \text{if } u \in \mathcal{A}(Y), \\ \infty & \text{else.} \end{cases}$$

LEMMA 2.8. *Suppose that $Y \subset X$ is closed and strictly positively invariant, and that $T(u) = \infty$ for all $u \in \text{Inv}(X \setminus Y)$. Then:*

- (a) *The map $e_Y: X \rightarrow [0, \infty]$ is continuous.*
- (b) *$\mathcal{A}(Y) = \{u \in X : e_Y(u) < \infty\}$ is open in X , and $\text{Inv}(X \setminus Y) = X \setminus \mathcal{A}(Y)$ is closed in X .*
- (c) *$\text{cat}_{X \setminus Y}(\text{Inv}(X \setminus Y)) \geq \text{ecat}(X, Y)$.*

PROOF. (a) is straightforward, and (b) follows immediately from (a). (c) Let $k = \text{cat}_{X \setminus Y}(\text{Inv}(X \setminus Y))$. Then there are open subsets $A_1, \dots, A_k \subset X \setminus Y$ such that $\text{Inv}(X \setminus Y) \subset \bigcup_i A_i$, and each A_i is contractible in $X \setminus Y$. Put $A_0 := \mathcal{A}(Y)$, and define $h: \mathcal{A}(Y) \times [0, 1] \rightarrow X$ by $h(u, t) = \varphi(te_Y(u), u)$. Then $h(\cdot, 0) = \text{id}_{A_0}$ and $h(A_0, 1) \subset Y$. Moreover, if $h(u, t) \in Y$ for $t < 1$, then already $u \in Y$ and hence $h(u, s) = h(u, t) = u$ for $t \leq s \leq 1$. We conclude that $\text{ecat}(X, Y) \leq k$. \square

3. Proof of the main theorem

Recall the map

$$i_R = i_R(\Omega): (C_R\Omega \times I_0^2, C_R\Omega \times \partial I_0^2) \hookrightarrow (C\Omega^R \times \mathbb{R}^2, C\Omega^R \times (\mathbb{R}^2 \setminus \{0\}))$$

from the introduction. We first give an easy estimate for $\text{cupl}(i_R)$ which we will need later on in the proof of our main theorem.

LEMMA 3.1. *If the bounded domain Ω contains the ball $B_{3R}(0)$, then $\text{cupl}(i_R) \geq 1$.*

PROOF. Put $S_{2R} := \{x \in \mathbb{R}^N : |x| = 2R\}$, and consider the maps

$$\begin{aligned} f: (S_{2R} \times I_0^2, S_{2R} \times \partial I_0^2) &\rightarrow (C_R\Omega \times I_0^2, C_R\Omega \times \partial I_0^2), \\ (y, s, t) &\mapsto (0, y, s, t) \end{aligned}$$

and

$$h: (C\Omega^R \times \mathbb{R}^2, C\Omega^R \times (\mathbb{R}^2 \setminus \{0\})) \rightarrow (S_{2R} \times \mathbb{R}^2, S_{2R} \times (\mathbb{R}^2 \setminus \{0\})),$$

$$(x, y, s, t) \mapsto \left(2R \frac{y-x}{|y-x|}, s, t \right).$$

The composition $\tilde{i} := h \circ i_R \circ f: (S_{2R} \times I_0^2, S_{2R} \times \partial I_0^2) \rightarrow (S_{2R} \times \mathbb{R}^2, S_{2R} \times (\mathbb{R}^2 \setminus \{0\}))$ is just the inclusion, and it induces an isomorphism in cohomology. Hence Lemma 2.6 yields

$$\text{cupl}(i_R) \geq \text{cupl}(\tilde{i}) \geq \text{cupl}(S_{2R} \times I_0^2, S_{2R} \times \partial I_0^2) \geq 1. \quad \square$$

For matters of convenience we now restate Theorem 1.1 in a rescaled version which does not contain the parameter ε .

THEOREM 3.2. *Suppose that (f₁)–(f₄) are satisfied. Then there is $R > 0$ such that for any bounded domain Ω with $B_{3R}(0) \subset \Omega$ the problem*

$$(3.1) \quad \begin{cases} -\Delta u + au = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least $\text{cat}(i_R) + 1$ nodal solutions. Moreover, $\text{cat}(i_R)$ of these solutions have precisely two nodal domains, and the last one has at most three nodal domains.

Theorem 1.1 easily follows from this version. Indeed, consider an arbitrary domain Ω and $r > 0$. We may assume that $0 \in \Omega$. We set

$$\varepsilon_r := \min \left\{ \frac{r}{R}, \frac{\text{dist}(0, \partial\Omega)}{3} \right\},$$

where R is given by Theorem 3.2. If $\varepsilon < \varepsilon_r$ then Theorem 3.2 applies to $\Omega_\varepsilon := \{x/\varepsilon : x \in \Omega\}$ and yields at least $\text{cat}(i_R(\Omega_\varepsilon)) + 1$ nodal solutions v_j of (3.1) on Ω_ε . Setting $u_j(x) := v_j(x/\varepsilon)$ we obtain solutions of (P $_\varepsilon$). Theorem 1.1 follows because $r/\varepsilon > R$ implies $\text{cat}(i_R(\Omega_\varepsilon)) \geq \text{cat}(i_{r/\varepsilon}(\Omega_\varepsilon)) = \text{cat}(i_r(\Omega))$. Here we used Lemma 2.1.

The remainder of this section is occupied with the proof of Theorem 3.2. Without loss we assume that $a = 1$, the general case follows by obvious modifications. We first look at equation (3.1) on $\Omega = \mathbb{R}^N$, and we fix some notation. Let $|\cdot|_2$ be the usual norm on $L^2(\mathbb{R}^N)$. For $u \in L^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ we define $x * u := u(\cdot - x) \in L^2(\mathbb{R}^N)$. We fix a continuous map

$$\beta: L^2(\mathbb{R}^N) \setminus \{0\} \rightarrow \mathbb{R}^N$$

which is equivariant with respect to translations and rotations, i. e. we have

$$(3.2) \quad \beta(x * u) = \beta(u) + x, \quad \beta(u \circ A^{-1}) = A(\beta(u)).$$

for every $x \in \mathbb{R}^N$, $A \in O(N)$ and $u \in L^2(\mathbb{R}^N) \setminus \{0\}$. Such a map β has been constructed by the authors in [7]. Note that every even function, and therefore

every radial function $u \in L^2(\mathbb{R}^N) \setminus \{0\}$ satisfies $\beta(u) = 0$. We call β a *generalized barycenter map*, since it shares property (3.2) with the standard barycenter map $u \mapsto \int_{\mathbb{R}^N} xu^2 / |u|_2^2$ which is not defined on all of $L^2(\mathbb{R}^N) \setminus \{0\}$.

Let $\|u\| = (|\nabla u|_2^2 + |u|_2^2)^{1/2}$ be the standard norm on $H^1(\mathbb{R}^N)$. It is well known that, as a consequence of (f₁) and (f₂), the functional

$$\Phi: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}, \quad \Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u(x)) dx$$

is of class C^2 , and that critical points of Φ are solutions of (3.1) with $a = 1$ on $\Omega = \mathbb{R}^N$. As usual we set

$$\Phi^\nu = \{u \in H^1(\mathbb{R}^N) : \Phi(u) \leq \nu\} \quad \text{for } \nu \in \mathbb{R}.$$

For $u \in H^1(\mathbb{R}^N)$ and $A \subset H^1(\mathbb{R}^N)$ we put

$$\text{dist}_1(u, A) = \inf_{v \in A} \|u - v\| \quad \text{and} \quad \text{dist}_2(u, A) = \inf_{v \in A} |u - v|_2,$$

and we define the closed neighbourhoods

$$\begin{aligned} U_\varepsilon^1(A) &= \{u \in H^1(\mathbb{R}^N) : \text{dist}_1(u, A) \leq \varepsilon\}, \\ U_\varepsilon^2(A) &= \{u \in H^1(\mathbb{R}^N) : \text{dist}_2(u, A) \leq \varepsilon\} \end{aligned}$$

for $\varepsilon > 0$. Consider the Nehari manifold

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \Phi'(u)u = 0\}.$$

which contains every nontrivial critical point of Φ . \mathcal{N} is a C^1 -submanifold of $H^1(\mathbb{R}^N)$ which is radially diffeomorphic to the unit sphere in $H^1(\mathbb{R}^N)$ (cf. [10, Lemma 2.2]). Moreover, for every $u \in \mathcal{N}$, there holds

$$\Phi(u) = \max_{t \geq 0} \Phi(tu).$$

We put

$$c_\pm = \inf\{\Phi(u) : u \in \mathcal{N}, \pm u \geq 0\}.$$

and

$$\begin{aligned} \mathcal{K}^\pm &= \{u \in \mathcal{N} : \pm u \geq 0, \Phi(u) = c_\pm\}, \\ \mathcal{K}_{\text{rad}}^\pm &= \{u \in \mathcal{K}^\pm : u \text{ is radially symmetric}\}. \end{aligned}$$

It is well known that

$$\inf_{\mathcal{N}} \Phi = \min\{c_+, c_-\} > 0,$$

and that the sets \mathcal{K}^+ , \mathcal{K}^- consist of positive resp. negative critical points of Φ (see e.g. [10, Lemmas 2.2. and 2.3]). Moreover, every such critical point is radially symmetric around some point in \mathbb{R}^N ([25]). Therefore we have

$$\mathcal{K}^\pm = \mathbb{R}^N * \mathcal{K}_{\text{rad}}^\pm = \{x * u : x \in \mathbb{R}^N, u \in \mathcal{K}_{\text{rad}}^\pm\}.$$

We also note that the sets $\mathcal{K}_{\text{rad}}^\pm$ are compact and nonempty, which essentially follows from the compactness of the embeddings

$$H_{\text{rad}}^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ radially symmetric}\} \hookrightarrow L^s(\mathbb{R}^N), \quad s \in (2, 2^*)$$

(see e.g. [35]). Consider the set

$$\begin{aligned} \mathcal{M} &= \{u \in H^1(\mathbb{R}^N) : u^+, u^- \neq 0, \Phi'(u)u^+ = 0 = \Phi'(u)u^-\} \\ &= \{u \in H^1(\mathbb{R}^N) : u^+, u^- \in \mathcal{N}\}, \end{aligned}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. \mathcal{M} contains all sign changing critical points of Φ . It is easy to see that

$$c_0 := c_+ + c_- = \inf_{\mathcal{M}} \Phi.$$

We also define

$$\begin{aligned} \mathcal{U}_\varepsilon &= \{u \in H^1(\mathbb{R}^N) : u^+ \in U_\varepsilon^2(\mathcal{K}^+), u^- \in U_\varepsilon^2(\mathcal{K}^-)\}, \\ \mathcal{V}(\varepsilon, \delta) &= (\mathcal{U}_\varepsilon \setminus \mathcal{U}_{\varepsilon/2}) \cap \Phi^{c_0+\delta} \end{aligned}$$

for $\varepsilon, \delta > 0$. We need some quantitative results from [7] which we collect in the following proposition.

PROPOSITION 3.3. *There exists $\varepsilon > 0$ and $\delta \in (0, \min\{c_+, c_-\})$ with the following properties.*

- (a) $U_\varepsilon^2(\mathcal{K}^+ \cup \mathcal{K}^-) \subset H^1(\mathbb{R}^N) \setminus \{0\}$,
- (b) If $u \in U_\varepsilon^2(\mathcal{K}^+ \cup \mathcal{K}^-)$ satisfies $u \equiv 0$ on $B_1(0)$, then $\beta(u) \neq 0$.
- (c) $\beta(u^+) \neq \beta(u^-)$ for every $u \in \mathcal{U}_\varepsilon$
- (d) $\max_{i=\pm} |\Phi'(u)u^i|/\|u^i\| \geq 6\delta/\varepsilon$ for every $u \in \mathcal{V}(\varepsilon, \delta)$.
- (e) $\mathcal{M} \cap \Phi^{c_0+\delta} \subset \text{int}(\mathcal{U}_{\varepsilon/2})$.
- (f) There exist radially symmetric functions $w_1, w_2 \in \mathcal{N}$ with compact support, and there exists $0 < t_0 < 1$ satisfying:

$$\begin{aligned} w_1 &\geq 0, & w_2 &\leq 0, \\ (1+t)w_1 &\in U_\varepsilon^2(\mathcal{K}^+), & (1+t)w_2 &\in U_\varepsilon^2(\mathcal{K}^-) \quad \text{for } |t| \leq t_0, \\ \Phi(w_1) &< c_+ + \frac{\delta}{4}, & \Phi(w_2) &< c_- + \frac{\delta}{4}, \\ \Phi((1 \pm t_0)w_1) &\leq c_+ - 2\delta, & \Phi((1 \pm t_0)w_2) &\leq c_- - 2\delta. \end{aligned}$$

PROOF. Assertions (a) and (c)–(f) are quoted from [7, Proposition 3.3]. Moreover, the proof of [7, Proposition 3.3] shows that $\varepsilon > 0$ can be chosen arbitrary small, hence it only remains to prove (b) for $\varepsilon > 0$ small. Suppose by contradiction that there is a sequence of numbers $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$ and functions $u_n \in U_{\varepsilon_n}^2(\mathcal{K}^+ \cup \mathcal{K}^-)$ such that $u_n \equiv 0$ on $B_1(0)$ and $\beta(u_n) = 0$ for all $n \in \mathbb{N}$.

Choose $x_n \in \mathbb{R}^N$ such that $x_n * u_n \in U_{\varepsilon_n}^2(\mathcal{K}_{\text{rad}}^+ \cup \mathcal{K}_{\text{rad}}^-)$. Passing to a subsequence, we have $x_n * u_n \rightarrow u \in \mathcal{K}_{\text{rad}}^+ \cup \mathcal{K}_{\text{rad}}^-$. Hence $x_n = \beta(x_n * u_n) \rightarrow \beta(u) = 0$ as $n \rightarrow \infty$. Hence $u_n \rightarrow u$ as $n \rightarrow \infty$. Consequently $u \equiv 0$ on $B_1(0)$, which contradicts the fact that $u \in \mathcal{K}_{\text{rad}}^+ \cup \mathcal{K}_{\text{rad}}^-$. \square

We also need the following lemma.

LEMMA 3.4. *For any $w \in \mathcal{N}$ there exists a continuous map $\tau: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} \tau(s(x * w)) &= s \quad \text{for every } s \geq 0, x \in \mathbb{R}^N, \\ \tau(v) &= 1 \quad \text{if and only if } v \in \mathcal{N}. \end{aligned}$$

PROOF. We first define $\sigma: H^1(\mathbb{R}^N) \rightarrow [0, \infty)$ by

$$\sigma(v) = \begin{cases} \frac{\int_{\Omega} f(v)v}{\|v\|^2} & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Then σ is continuous, as follows easily from (f_1) , (f_2) and Sobolev embeddings. Moreover,

$$\sigma(v) = 1 \quad \text{if and only if } v \in \mathcal{N}.$$

Note also that the function $s \mapsto \xi(s) := \sigma(sw)$ is strictly increasing on $[0, \infty)$ by virtue of (f_3) , and $\xi(s) \rightarrow \infty$ as $s \rightarrow \infty$. Hence $\xi^{-1} \in C([0, \infty), [0, \infty))$ exists and is strictly increasing. We define

$$\tau: H^1(\mathbb{R}^N) \rightarrow [0, \infty), \quad \tau(v) = \xi^{-1}(\sigma(v))$$

Then $\tau(s(x * w)) = \tau(sw) = s$ for $s \geq 0$, $x \in \mathbb{R}^N$, and $\tau(v) = 1$ if and only if $v \in \mathcal{N}$. \square

We now start with the proof of Theorem 3.2. For this we choose $R > 1$ such that $B_{R/4}(0)$ contains the support of w_1 and w_2 from Proposition 3.3(f). We suppose that Ω is a bounded domain containing the ball $B_{3R}(0)$. We regard $H_0^1(\Omega)$ as a closed subspace of $H^1(\mathbb{R}^N)$ and consider the functional

$$\Psi = \Phi|_{H_0^1(\Omega)}: H_0^1(\Omega) \rightarrow \mathbb{R}, \quad \Psi(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u(x)) dx.$$

It is well known that weak solutions $u \in H_0^1(\Omega)$ of (3.1) (with $a = 1$) are critical points of Ψ and that Ψ satisfies the Palais-Smale condition (see [2]). As usual we set $\Psi^c = \{u \in H_0^1(\Omega) : \Phi(u) \leq c\}$ for $c \in \mathbb{R}$. Moreover, we denote by $\varphi: \mathcal{G} \subset \mathbb{R} \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ the negative gradient flow of Ψ defined by

$$(3.3) \quad \begin{cases} \frac{d}{dt}\varphi(t, u) = -\nabla\Psi(\varphi(t, u)), \\ \varphi(0, u) = u. \end{cases}$$

Here $\mathcal{G} := \{(t, u) : u \in H_0^1(\Omega), T^-(u) < u < T^+(u)\}$, where $T^+(u) > 0$ resp. $T^-(u) < 0$ is the maximal existence time of the trajectory $t \mapsto \varphi(t, u)$ in positive resp. negative direction. As in Section 2 we write $\varphi^t(u) = \varphi(t, u)$. Consider the convex cone of positive functions

$$\mathcal{P} = \{u \in H_0^1(\Omega) : u \geq 0\}$$

and the sets

$$D_\alpha = \{u \in H_0^1(\Omega) : \text{dist}_1(u, \mathcal{P} \cup -\mathcal{P}) \leq \alpha\}$$

for $\alpha > 0$. We are interested in critical points located in $H \setminus D_\alpha$ (for some α), since these are sign changing critical points. We first give an upper bound for the number of nodal domains.

LEMMA 3.5.

- (a) *Any critical point u of Ψ with $\Psi(u) < c_0 + \min\{c_+, c_-\}$ has at most two nodal domains.*
- (b) *Any critical point u of Ψ with $\Psi(u) < c_0 + 2 \min\{c_+, c_-\}$ has at most three nodal domains.*

PROOF. Every critical point u of Ψ is a continuous function on Ω . Hence, if Ω^* is a nodal domain of u , then $u\chi_{\Omega^*}$ defines an element of $H_0^1(\Omega)$ by [29, Lemma 1]; here χ_{Ω^*} denotes the characteristic function of Ω^* . Moreover,

$$\Psi'(u\chi_{\Omega^*})u\chi_{\Omega^*} = \Psi'(u)u\chi_{\Omega^*} = 0.$$

Thus $\pm u\chi_{\Omega^*} \geq 0$ implies $\Psi(u\chi_{\Omega^*}) = \Phi(u\chi_{\Omega^*}) \geq c_\pm$. Now if u has at least three nodal domains $\Omega_1, \Omega_2, \Omega_3$ such that $u > 0$ on Ω_1 and $u < 0$ on Ω_2 , then

$$c_0 + \min\{c_+, c_-\} \leq \Psi(u\chi_{\Omega_1 \cup \Omega_2}) + \Psi(u\chi_{\Omega_3}) \leq \Psi(u).$$

This proves (a), and (b) is proved by a similar argument. \square

LEMMA 3.6. *If $\alpha > 0$ is small enough, then*

- (a) $D_\alpha \cap \mathcal{M} = \emptyset$,
- (b) D_α is strictly positively invariant under the flow φ .

PROOF. (a) For $u \in \mathcal{M}$ we have by (f_2) and Sobolev embeddings

$$\begin{aligned} 0 < \inf_{v \in \mathcal{N}} \|v\|^2 &\leq \|u^+\|^2 = \int_{\Omega} f(u^+)u^+ \leq C(|u^+|_2^2 + |u^+|_p^p) \\ &\leq C \inf_{v \in -\mathcal{P}} (|u-v|_2^2 + |u-v|_p^p) \leq C_1 \inf_{v \in -\mathcal{P}} (\|u-v\|^2 + \|u-v\|^p), \end{aligned}$$

and similarly

$$\inf_{v \in \mathcal{N}} \|v\|^2 \leq C_1 \inf_{v \in \mathcal{P}} (\|u-v\|^2 + \|u-v\|^p),$$

where $C_1 > 0$ is a constant. Hence $D_\alpha \cap \mathcal{M} = \emptyset$ for $\alpha^2 + \alpha^p < C_1^{-1} \inf_{v \in \mathcal{N}} \|v\|^2$.

(b) Put $D_\alpha^\pm = \{u \in H_0^1(\Omega) : \text{dist}_1(u, \pm\mathcal{P}) \leq \alpha\}$. Then the sets D_α^\pm are closed and convex, and $D_\alpha = D_\alpha^+ \cup D_\alpha^-$. The gradient of Ψ has the form $\nabla\Psi = \text{id}_{H_0^1(\Omega)} - K$ with $K: H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ given by $K(u) = (-\Delta + 1)^{-1}f(u)$. Now precisely the same argument as in [4, Lemma 3.1] yields that

$$(3.4) \quad K(\partial D_\alpha^\pm) \subset \text{int}(D_\alpha^\pm),$$

for $\alpha > 0$ sufficiently small. Since D_α^+ is convex, this implies

$$u + \lambda(-\nabla\Psi(u)) = (1 - \lambda)u + \lambda K(u) \in D_\alpha^+ \quad \text{for } u \in D_\alpha^+, 0 \leq \lambda \leq 1.$$

Hence we infer from [20, Theorem 5.2] that

$$(3.5) \quad \varphi^t(u) \in D_\alpha^+ \quad \text{for } u \in D_\alpha^+, 0 \leq t < T^+(u).$$

We now suppose by contradiction that there is $u \in D_\alpha^+$ and $0 < t < T^+(u)$ such that $\varphi^t(u) \in \partial D_\alpha^+$. By Mazur's separation theorem, there exists a continuous linear functional $\ell \in (H_0^1(\Omega))^*$ and $\beta > 0$ such that $\ell(\varphi^t(u)) = \beta$ and $\ell(u) > \beta$ for $u \in \text{int}(D_\alpha^+)$. It follows from (3.4) that

$$\left. \frac{\partial}{\partial s} \right|_{s=t} \ell(\varphi^s(u)) = \ell(-\nabla\Psi(\varphi^t(u))) = \ell(K(\varphi^t(u))) - \beta > 0.$$

Hence there exists $t_1 < t$ such that $\ell(\varphi^s(u)) < \beta$ for $t_1 < s < t$. Consequently, $\varphi^s(u) \notin D_\alpha^+$ for $t_1 < s < t$ contradicting (3.5). Hence $\varphi^t(u) \in \text{int}(D_\alpha^+)$ for $u \in D_\alpha^+, t \in (0, T^+(u))$. The same argument shows that $\varphi^t(u) \in \text{int}(D_\alpha^-)$ for $u \in D_\alpha^-, t \in (0, T^+(u))$. We conclude that D_α^+, D_α^- and hence D_α are strictly positively invariant. \square

We now fix $\alpha > 0$ such that the statement of Lemma 3.6 holds, and for $T > 0$ we define

$$\mathcal{E}_T := \{u \in H_0^1(\Omega) : \varphi^t(u) \in D_\alpha \cup \Psi^{c_0 - \delta} \text{ for some } 0 \leq t \leq T, t < T^+(u)\}.$$

We note that

$$(3.6) \quad T^+(u) = \infty \quad \text{for every } u \in \text{Inv}(H_0^1(\Omega) \setminus \mathcal{E}_T).$$

This implies in particular that \mathcal{E}_T is a closed subset of $H_0^1(\Omega)$. Moreover we have:

LEMMA 3.7. *Let $T > 0, \nu \in \mathbb{R}$.*

- (a) \mathcal{E}_T is strictly positively invariant.
- (b) If $\Psi^{-1}(\nu) := \{u \in H_0^1(\Omega) : \Psi(u) = \nu\}$ contains no sign changing critical point, then $\Psi^\nu \cup \mathcal{E}_T$ is strictly positively invariant.

PROOF. (a) Let $u \in \partial \mathcal{E}_T$. Then $T^+(u) > T$, and $v := \varphi^T(u) \in \partial D_\alpha \cup \Psi^{c_0 - \delta}$. If $v \in D_\alpha$, then $\varphi^t(v) \in \text{int}(D_\alpha)$ for $0 < t < T^+(u) - T$, hence $\varphi^t(u) \in \text{int}(\mathcal{E}_T)$ for $0 < t < T^+(u)$. If $v \in \partial \Psi^{c_0 - \delta} \setminus D_\alpha = \Psi^{-1}(c_0 - \delta) \setminus D_\alpha$, then v is no critical point of Ψ , since all sign changing points of Ψ are contained in $\mathcal{M} \cap H_0^1(\Omega)$ and $\inf \Psi(\mathcal{M} \cap H_0^1(\Omega)) \geq c_0$. Hence $\Psi(\varphi^t(v)) < c_0 - \delta$ for $0 < t < T^+(u) - T$, and again we conclude that $\varphi^t(u) \in \text{int}(\mathcal{E}_T)$ for $0 < t < T^+(u)$.

The proof of (b) is similar. \square

LEMMA 3.8. *Let $T > 0$ and $\nu \in \mathbb{R}$ be such that $\Psi^{-1}(\nu)$ contains no sign changing critical point. Then the set $\Psi^\nu \setminus \mathcal{E}_T$ contains at least $\text{ecat}(\Psi^\nu \cup \mathcal{E}_T, \mathcal{E}_T)$ critical points.*

PROOF. We only sketch the proof, since the argument is fairly standard. Note that Lemma 2.8 and (3.6) yield

$$\text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(\text{Inv}(\Psi^\nu \setminus \mathcal{E}_T)) \geq \text{ecat}(\Psi^\nu \cup \mathcal{E}_T, \mathcal{E}_T) =: k.$$

We define $K_c := \{u \in H_0^1(\Omega) \setminus \mathcal{E}_T : \Psi(u) = c, \Psi'(u) = 0\}$ for $c \in \mathbb{R}$. Then $K_\nu = \emptyset$ by assumption. Since Ψ satisfies the Palais–Smale condition, we find that

$$\varphi^t(\text{Inv}(\Psi^\nu \setminus \mathcal{E}_T)) \subset \text{Inv}(\Psi^{\nu - \sigma_0} \setminus \mathcal{E}_T)$$

for some $\sigma_0 > 0$, $t > 0$. Hence

$$\text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(\text{Inv}(\Psi^{\nu - \sigma_0} \setminus \mathcal{E}_T)) = \text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(\text{Inv}(\Psi^\nu \setminus \mathcal{E}_T)) \geq k.$$

Now let $c < \nu$ so that $K_c \subset \text{Inv}(\Psi^c \setminus \mathcal{E}_T) \subset \text{int}(\Psi^\nu \setminus \mathcal{E}_T)$. Using Lemma 2.2(b) and the fact that Ψ satisfies the Palais–Smale condition, we obtain the following.

$$(3.7) \quad \begin{cases} K_c \text{ is compact, and there is a neighbourhood } U \subset \Psi^\nu \setminus \mathcal{E}_T \\ \text{of } K_c \text{ and } \sigma > 0, t > 0 \text{ such that } \text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(U) = \text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(K_c) \\ \text{and } \varphi^t(\text{Inv}(\Psi^{c+\sigma} \setminus \mathcal{E}_T) \setminus U) \subset \text{Inv}(\Psi^{c-\sigma} \setminus \mathcal{E}_T). \end{cases}$$

Put

$$c_j := \inf\{c \leq \nu : \text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(\text{Inv}(\Psi^c \setminus \mathcal{E}_T)) \geq j\}, \quad j = 1, \dots, k.$$

Then $c_1 \leq \dots \leq c_k \leq \nu - \sigma_0$. Moreover, from (3.7) it follows that $\text{cat}_{\Psi^\nu \setminus \mathcal{E}_T}(K_{c_j}) \geq l + 1$ whenever $c_j = \dots = c_{j+l}$ for some $j \leq k$, $0 \leq l \leq k - j$. By Lemma 2.2(a) this implies that K_{c_j} contains at least $l + 1$ critical points. We conclude that $\bigcup_{c < \nu} K_c$ contains at least k critical points, as claimed. \square

The next step is to derive a lower bound for $\text{ecat}(\Psi^\nu \cup \mathcal{E}_T, \mathcal{E}_T)$ for suitable values of T and ν . For this recall the notation introduced in Proposition 3.3, and recall also that $R > 0$ was fixed such that

$$\text{supp } w_i \subset B_{R/4}(0) \subset B_{3R}(0) \subset \Omega, \quad i = 1, 2.$$

We set $I := [1 - t_0, 1 + t_0]$, and we fix a value $\nu_1 \in (c_0 + \delta/2, c_0 + \delta)$ of Ψ such that $\Psi^{-1}(\nu_1)$ contains no sign changing critical points. If this does not exist then Ψ has infinitely many sign changing critical points in $\Psi^{c_0 + \delta}$. These have precisely two nodal domains by Lemma 3.5. We consider the map

$$g_T: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \rightarrow (\Psi^{\nu_1} \cup \mathcal{E}_T, \mathcal{E}_T),$$

$$g_T(x, y, s_1, s_2) := s_1(x * w_1) + s_2(y * w_2).$$

By Proposition 3.3 the map g_T is well defined as a map of pairs. By Lemma 2.3 we have

$$(3.8) \quad \text{ecat}(\Psi^{\nu_1} \cup \mathcal{E}_T, \mathcal{E}_T) \geq \text{ecat}(g_T).$$

The following estimates are the crucial step in the proof of Theorem 3.2.

PROPOSITION 3.9. *There is $T > 0$ such that*

- (a) $\text{ecat}(g_T) \geq \text{cat}(i_R)$,
- (b) $\text{cupl}(g_T) \geq \text{cupl}(i_R)$ if the cuplength is defined using Alexander–Spanier cohomology.

PROOF. (a) Note that by Proposition 3.3(e) the map g_T factorizes in the form

$$g_T: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \xrightarrow{\tilde{g}_T} (A, A \cap \mathcal{E}_T) \hookrightarrow (\Psi^{\nu_1} \cup \mathcal{E}_T, \mathcal{E}_T).$$

Here $A := \mathcal{U}_\varepsilon \cap \Psi^{\nu_1}$, and the second arrow is just the inclusion. We show

$$(3.9) \quad A \cup \mathcal{E}_T = \Psi^{\nu_1} \cup \mathcal{E}_T \quad \text{for some } T > 0.$$

As a consequence, the excision property (see Lemma 2.4) yields that $\text{ecat}(g_T) = \text{ecat}(\tilde{g}_T)$.

In order to prove (3.9), we just have to show that $\Psi^{\nu_1} \setminus \mathcal{E}_T \subset \mathcal{U}_\varepsilon$ for some $T > 0$. For this note that $\mathcal{M} \cap H_0^1(\Omega)$ contains all sign changing critical points of Ψ . Hence Proposition 3.3 (e) implies that the closed set $\Psi^{\nu_1} \setminus \text{int}_{H_0^1(\Omega)}([\mathcal{U}_{\varepsilon/2} \cap H_0^1(\Omega)] \cup D_\alpha)$ contains no critical point of Ψ . Consequently we have

$$(3.10) \quad \sigma_0 := \inf\{\|\Psi'(u)\| : u \in \Psi^{\nu_1} \setminus \text{int}_{H_0^1(\Omega)}([\mathcal{U}_{\varepsilon/2} \cap H_0^1(\Omega)] \cup D_\alpha \cup \Psi^0)\} > 0,$$

because Ψ satisfies the Palais–Smale condition. We now set $T := 3\delta/\sigma_0^2$, and we let $u \in \Psi^{\nu_1} \setminus \mathcal{E}_T$. Then $c_0 - \delta \leq \Psi(\varphi^T(u)) \leq \Psi(u) \leq c_0 + \delta$, and therefore

$$2\delta \geq \Psi(u) - \Psi(\varphi^T(u)) = \int_0^T \|\Psi'(\varphi^t(u))\|^2 dt \geq T \inf_{0 \leq t \leq T} \|\Psi'(\varphi^t(u))\|^2.$$

Hence there is $t \in [0, T]$ such that $\|\Psi'(\varphi^t(u))\| \leq \sqrt{2\delta/T} < \sigma_0$, which by (3.10) implies that $\varphi^t(u) \in \mathcal{U}_{\varepsilon/2}$. Now suppose by contradiction that $u \notin \mathcal{U}_\varepsilon$. Put

$t_2 := \inf\{s > 0 : \varphi^s(u) \in \mathcal{U}_{\varepsilon/2}\}$ and $t_1 := \sup\{s \in [0, t_2] : \varphi^s(u) \notin \mathcal{U}_\varepsilon\}$. Then

$$\begin{aligned} \|\varphi^{t_2}(u) - \varphi^{t_1}(u)\| &\geq |\varphi^{t_2}(u) - \varphi^{t_1}(u)|_2 \\ &\geq \max\{ |(\varphi^{t_2}(u))^+ - (\varphi^{t_1}(u))^+|_2, |(\varphi^{t_2}(u))^- - (\varphi^{t_1}(u))^-|_2 \} \geq \frac{\varepsilon}{2}. \end{aligned}$$

Moreover, $\varphi^t(u) \in (\mathcal{U}_\varepsilon \setminus \mathcal{U}_{\varepsilon/2}) \cap \Psi^{\nu_1}$ for $t_1 < t < t_2$. Note that Proposition 3.3(d) implies

$$\|\Psi'(u)\| \geq \max_{i=\pm} \frac{\Phi'(u)u^i}{\|u^i\|} \geq \frac{6\delta}{\varepsilon} \quad \text{for } u \in \mathcal{V}(\varepsilon, \delta) \cap H_0^1(\Omega) = (\mathcal{U}_\varepsilon \setminus \mathcal{U}_{\varepsilon/2}) \cap \Psi^{c_0+\delta}.$$

The definition of φ therefore yields

$$\begin{aligned} \frac{\varepsilon}{2} &\leq \|\varphi^{t_2}(u) - \varphi^{t_1}(u)\| \leq \int_{t_1}^{t_2} \|\Psi'(\varphi^t(u))\| dt \\ &\leq \frac{\varepsilon}{6\delta} \int_{t_1}^{t_2} \|\Psi'(\varphi^t(u))\|^2 dt = \frac{\varepsilon}{6\delta} \Psi(\varphi^{t_1}(u)) - \Psi(\varphi^{t_2}(u)). \end{aligned}$$

We conclude that

$$\Psi(\varphi^T(u)) \leq \Psi(\varphi^{t_2}(u)) \leq \Psi(\varphi^{t_1}(u)) - 3\delta \leq c_0 - 2\delta,$$

which contradicts our assumption that $u \notin \mathcal{E}_T$. This proves (3.9).

Next we show

$$(3.11) \quad \text{cat}(\tilde{g}_T) \geq \text{cat}(i_R)$$

For this we recall the functions $w_1, w_2 \in \mathcal{N}$ of Proposition 3.3 (f), and we let $\tau_1, \tau_2: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be the associated maps constructed in Lemma 3.4. We define $\tau: H_0^1(\Omega) \rightarrow \mathbb{R}^2$ by $\tau(u) = (\tau_1(u^+), \tau_2(u^-))$. Note that $\tau(u) = (1, 1)$ if and only if $u \in \mathcal{M} \cap H_0^1(\Omega)$. We put

$$h_t(u) := (\beta(u^+), \beta(u^-), \tau(\varphi^t(u)))$$

for $t \in [0, T]$ and $u \in A$. By Proposition 3.3(a) we have $\beta(u^+) \neq \beta(u^-)$ for all $u \in A \subset \mathcal{U}_\varepsilon$. Moreover, if $x \notin \Omega^R$ then $u|_{B_R(x)} \equiv 0$, hence $\beta(u^+) \neq x \neq \beta(u^-)$ by Proposition 3.3(b). Thus $h_t(A) \subset C\Omega^R \times \mathbb{R}^2$ for $t \in [0, T]$. Since also $h_T(A \cap \mathcal{E}_T) \subset C\Omega^R \times (\mathbb{R}^2 \setminus \{(1, 1)\})$ by Lemma 3.6(a) and the definition of τ , we find that h_T is a continuous map of pairs

$$h_T: (A, A \cap \mathcal{E}_T) \rightarrow (C\Omega^R \times \mathbb{R}^2, C\Omega^R \times (\mathbb{R}^2 \setminus \{(1, 1)\})).$$

Note furthermore that

$$h_t \circ \tilde{g}_T(C_R\Omega \times \partial I^2) \subset h_t(\mathcal{U}_\varepsilon \cap \Psi^{c_0-\delta}) \subset C\Omega^R \times (\mathbb{R}^2 \setminus \{(1, 1)\}) \quad \text{for } t \in [0, T].$$

Hence

$$h_T \circ \tilde{g}_T: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \rightarrow (C\Omega^R \times \mathbb{R}^2, C\Omega^R \times (\mathbb{R}^2 \setminus \{(1, 1)\}))$$

is homotopic to

$$\widetilde{i}_R := h_0 \circ \widetilde{g}_T: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \rightarrow (C\Omega^R \times \mathbb{R}^2, C\Omega^R \times (\mathbb{R}^2 \setminus \{(1,1)\}))$$

as a map of pairs. Note that \widetilde{i}_R is just the inclusion. By Lemma 2.1 this yields

$$\text{cat}(\widetilde{g}_T) \geq \text{cat}(h_T \circ \widetilde{g}_T) = \text{cat}(\widetilde{i}_R) = \text{cat}(i_R).$$

We conclude that

$$\text{ecat}(g_T) = \text{ecat}(\widetilde{g}_T) \geq \text{cat}(\widetilde{g}_T) \geq \text{cat}(i_R),$$

as claimed.

(b) This is proved precisely as (a), now using the properties of cupl stated in Lemma 2.6. \square

Next we define

$$\nu_2 := c_0 + \min\{c_+, c_-\} + \frac{3}{4}\delta > \nu_1,$$

and we show the following.

PROPOSITION 3.10. Ψ has a sign changing critical point u with

$$\nu_1 < \Psi(u) < \nu_2.$$

PROOF. Since Ψ has no sign changing critical points in $\Psi^{-1}(\nu_1)$, the set $Y := \Psi^{\nu_1} \cup \mathcal{E}_T$ is strictly positively invariant by Lemma 3.7(b). We suppose by contradiction that $\Psi^{\nu_2} \setminus (\Psi^{\nu_1} \cup \mathcal{E}_T)$ contains no critical point. Then every $u \in \Psi^{\nu_2} \cup \mathcal{E}_T$ has a finite entrance time $0 \leq e_Y(u) < \infty$ in Y via the flow φ . Consider

$$h: \Psi^{\nu_2} \cup \mathcal{E}_T \rightarrow \Psi^{\nu_1} \cup \mathcal{E}_T, \quad h(u) = \varphi^{e_Y(u)}(u),$$

Then h is continuous by Lemma 2.8(a). Moreover, the inclusion

$$i: (\Psi^{\nu_1} \cup \mathcal{E}_T, \mathcal{E}_T) \rightarrow (\Psi^{\nu_2} \cup \mathcal{E}_T, \mathcal{E}_T)$$

is a homotopy equivalence with homotopy inverse h . Setting

$$g = i \circ g_T: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \rightarrow (\Psi^{\nu_2} \cup \mathcal{E}_T, \mathcal{E}_T),$$

Lemma 3.1 and Proposition 3.9 yield

$$(3.12) \quad \text{cupl}(g) = \text{cupl}(g_T) \geq \text{cupl}(i_R) \geq 1.$$

We fix $b > 1 + t_0$ and put $I_b := [0, b]^2$. Then g factorizes in the form

$$g: (C_R\Omega \times I^2, C_R\Omega \times \partial I^2) \hookrightarrow (C_R\Omega \times I_b^2, C_R\Omega \times [I_b^2 \setminus \text{int}(I^2)]) \xrightarrow{g_b} (\Psi^{\nu_2} \cup \mathcal{E}_T, \mathcal{E}_T).$$

Here the first arrow is the inclusion which is a homotopy equivalence, and g_b is defined by

$$g_b(x, y, s_1, s_2) = s_1(x * w_1) + s_2(y * w_2).$$

It follows that $\text{cupl}(g) = \text{cupl}(g_b) = \text{cupl}(g_b \circ j_b)$, where the inclusion

$$j_b: (C_R\Omega \times I_b^2, C_R\Omega \times \partial I_b^2) \hookrightarrow (C_R\Omega \times I_b^2, C_R\Omega \times [I_b^2 \setminus \text{int}(I^2)])$$

is also a homotopy equivalence. Without loss we assume from now on that $c_+ \leq c_-$. We fix $x_0 \in \Omega$ with $\text{dist}(x_0, \partial\Omega) = R/4$, and we define

$$h_b: C_R\Omega \times I_b^2 \times [0, 2] \rightarrow H_0^1(\Omega)$$

by

$$h_b(x, y, s_1, s_2, t) = \begin{cases} s_1[t(x_0 * w_1) + (1-t)x * w_1] + s_2(y * w_2), & t \in [0, 1], \\ s_1(x_0 * w_1) + s_2[(t-1)w_2 + (2-t)(y * w_2)], & t \in [1, 2]. \end{cases}$$

Recalling that $\text{supp } w_i \subset B_{R/4}(0)$, we find that

$$h_b(C_R\Omega \times I_b^2 \times [0, 2]) \subset \Psi^{2c_+ + c_- + 3/4\delta} = \Psi^{\nu_2}.$$

Moreover, if b is chosen large enough, then

$$h_b(C_R\Omega \times \partial I_b^2 \times [0, 2]) \subset \Psi^{c_0 - \delta} \cup D_\alpha \subset \mathcal{E}_T.$$

Indeed, this follows from the well known fact that

$$\lim_{b \rightarrow \infty} \sup_{w \in \mathcal{C}} \Psi(bw) = -\infty \quad \text{for every compact subset } \mathcal{C} \subset H_0^1(\Omega) \setminus \{0\}.$$

We conclude that $g_b \circ j_b$ and $f_b := h_b(\cdot, 1)$ are homotopic as maps

$$(C_R\Omega \times I_b^2, C_R\Omega \times \partial I_b^2) \rightarrow (\Psi^{\nu_2} \cup \mathcal{E}_T, \mathcal{E}_T).$$

Note that f_b factorizes in the form

$$f_b: (C_R\Omega \times I_b^2, C_R\Omega \times \partial I_b^2) \rightarrow (I_b^2, \partial I_b^2) \xrightarrow{\tilde{f}} (\Psi^{\nu_2} \cup \mathcal{E}_T, \mathcal{E}_T),$$

where the first arrow is the canonical projection and \tilde{f} is given by $\tilde{f}(s_1, s_2) = s_1(x_0 * w_1) + s_2 w_2$. Applying Lemma 2.6 we get

$$\text{cupl}(g) = \text{cupl}(g_b) = \text{cupl}(g_b \circ j_b) = \text{cupl}(f_b) \leq \text{cupl}(I_b^2, \partial I_b^2) = 0,$$

which contradicts (3.12). \square

PROOF OF THEOREM 3.2 (completed). Combining Lemma 3.8, (3.8) and Proposition 3.9, we find that $\Psi^{\nu_1} \setminus \mathcal{E}_T$ contains at least $\text{cat}(i_R)$ critical points. Note that every such critical point u is a continuous sign changing function on Ω with precisely two nodal domains by Lemma 3.5. The critical point obtained in Proposition 3.10 has two or three nodal domains, again by Lemma 3.5. \square

4. Lower bounds for the category

In this section we prove Proposition 1.2. Part (a) of this proposition just follows from Lemma 2.7 and Lemma 3.1. We now prove (b) and recall the inclusions

$$\begin{aligned} j: (C\Omega \times I_0^2, C\Omega \times \partial I_0^2) &\hookrightarrow (C\Omega \times \mathbb{R}^2, C\Omega \times (\mathbb{R}^2 \setminus \{0\})), \\ i_r: (C_r\Omega \times I_0^2, C_r\Omega \times \partial I_0^2) &\hookrightarrow (C^r\Omega \times \mathbb{R}^2, C^r\Omega \times (\mathbb{R}^2 \setminus \{0\})), \end{aligned}$$

We need to show the inequalities

$$(4.1) \quad \text{cat}(i_r) \geq \text{cat}(j) \geq \text{cupl}(C\Omega) + 1 \geq \max\{2 + \text{cupl}(\Omega), 2\text{cupl}(\Omega)\},$$

for $r > 0$ sufficiently small. In this section we use singular cohomology because we do not need the strong excision property and because it has better product structures. First of all, observe that j induces an isomorphism in cohomology by the 5-lemma. Hence Lemma 2.7 implies

$$\text{cat}(j) \geq \text{cupl}(j) + 1 = \text{cupl}(C\Omega \times I_0^2, C\Omega \times \partial I_0^2) + 1 = \text{cupl}(C\Omega) + 1$$

The last equality is a consequence of

LEMMA 4.1. $\text{cupl}(A \times I_0^n, A \times \partial I_0^n) = \text{cupl}(A)$ for any topological space A and any $n \in \mathbb{N}$.

PROOF. Given two pairs (A, B) and (A', B') of topological spaces we write

$$\times: H^k(A, B) \times H^l(A', B') \rightarrow H^{k+l}(A \times A', A \times B' \cup B \times A')$$

for the exterior cohomology (or cross) product. It is defined, for instance, if $B = \emptyset$ or $B' = \emptyset$ which is the only case that appears in our argument. Let $1_n \in H^0(I_0^n)$ be the unit and $\iota \in H^n(I_0^n, \partial I_0^n) \cong \mathbb{F}_2$ be the generator. The exterior cohomology product induces an isomorphism

$$(4.2) \quad H^k(A) \xrightarrow{\cong} H^{k+n}(A \times I_0^n, A \times \partial I_0^n), \quad \alpha \mapsto \alpha \times \iota,$$

by the Künneth theorem [22, Proposition VII.7.6]. For $\alpha_1, \dots, \alpha_k, \beta \in H^*(A)$ we have by [22, Proposition VII.8.16]:

$$(4.3) \quad (\alpha_1 \times 1_n) \smile \dots \smile (\alpha_k \times 1_n) \smile (\beta \times \iota) = (\alpha_1 \smile \dots \smile \alpha_k \smile \beta) \times \iota$$

If $\text{cupl}(A) = k$ there exist $\alpha_1, \dots, \alpha_k \in \tilde{H}^*(A)$ with $\alpha_1 \smile \dots \smile \alpha_k \neq 0$. Thus (4.3) with $\beta = 1_A$ and (4.2) imply $\text{cupl}(A \times I_0^n, A \times \partial I_0^n) \geq k$. If on the other hand $\text{cupl}(A \times I_0^n, A \times \partial I_0^n) = k$ then there exist $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \in \tilde{H}^*(A \times I_0^n)$ and $\tilde{\beta} \in H^*(A \times I_0^n, A \times \partial I_0^n)$ with $\tilde{\alpha}_1 \smile \dots \smile \tilde{\alpha}_k \smile \tilde{\beta} \neq 0$. Clearly there exist $\alpha_1, \dots, \alpha_k \in \tilde{H}^*(A)$ with $\tilde{\alpha}_j = \alpha_j \times 1_n$. Moreover, by (4.2) there exists $\beta \in H^*(A)$ with $\tilde{\beta} = \beta \times \iota$. Applying (4.3) once more we obtain $\text{cupl}(A) \geq k$.

Now it remains to prove

$$(4.4) \quad \text{cat}(i_r) \geq \text{cat}(j),$$

$$(4.5) \quad \text{cupl}(C\Omega) \geq \max\{\text{cupl}(\Omega) + 1, 2\text{cupl}(\Omega) - 1\}.$$

For this let $\nu: \partial\Omega \times (-1, 1) \rightarrow \mathbb{R}^N$ be a \mathcal{C}^0 -tubular neighbourhood of $\partial\Omega$ and set

$$D := \Omega \setminus \nu(\partial\Omega \times [-1/2, 0]), \quad U := \Omega \cup \nu(\partial\Omega \times (-1, 1)).$$

Then there exist homeomorphisms $h_1: \Omega \rightarrow D$ and $h_2: U \rightarrow \Omega$ which are homotopy inverse to the inclusions $D \hookrightarrow \Omega$ resp. $\Omega \hookrightarrow U$. Indeed, h_1 is induced from a homeomorphism $(-1, 0) \rightarrow (-1, -1/2)$ which is homotopy inverse to the inclusion $(-1, -1/2) \hookrightarrow (-1, 0)$, and similarly h_2 is induced from a homeomorphism $(-1, 1) \rightarrow (-1, 0)$ which is homotopy inverse to the inclusion $(-1, 0) \hookrightarrow (-1, 1)$. This implies that the induced inclusions $CD \hookrightarrow C\Omega$ and $C\Omega \hookrightarrow CU$ between the configuration spaces are homotopy equivalences (with homotopy inverses $Ch_1: C\Omega \rightarrow CD$ and $Ch_2: CU \rightarrow C\Omega$ induced by h_1, h_2 , respectively). For $r > 0$ small we have

$$D \subset \Omega_{2r} \subset \Omega \subset \Omega^r \subset U.$$

We fix r with this property and we first complete the proof of (4.5). Consider the maps

$$g: D \times S^{N-1} \rightarrow C\Omega, \quad g(x, \theta) := (x, x + r\theta),$$

and

$$h: C\Omega \rightarrow \Omega \times S^{N-1}, \quad h(x, y) := \left(x, \frac{y-x}{|y-x|} \right).$$

These maps are well defined and continuous. Since the composition $h \circ g: D \times S^{N-1} \rightarrow \Omega \times S^{N-1}$ is just the inclusion and D is a deformation retract of Ω , $h \circ g$ is a homotopy equivalence. Therefore the induced homomorphism h^* on cohomology level is injective. Hence the definition of cupl implies

$$\text{cupl}(C\Omega) \geq \text{cupl}(h) = \text{cupl}(\Omega \times S^{N-1}) = \text{cupl}(\Omega) + 1.$$

In order to see $\text{cupl}(C\Omega) \geq 2\text{cupl}(\Omega) - 1$ let $\Delta \subset \Omega \times \Omega$ be the diagonal and $\Delta_\varepsilon := \{(x, y) \in \Omega \times \Omega : |x - y| \leq \varepsilon\}$. We consider the following commutative diagram:

$$\begin{array}{ccccc} H^*(\Omega \times \Omega, \Delta_\varepsilon) & \longrightarrow & H^*(\Omega \times \Omega) & \longrightarrow & H^*(\Delta_\varepsilon) \\ \cong \downarrow & & \downarrow & & \downarrow \\ H^*(C\Omega, \Delta_\varepsilon \setminus \Delta) & \longrightarrow & H^*(C\Omega) & \longrightarrow & H^*(\Delta_\varepsilon \setminus \Delta) \end{array}$$

Here all homomorphisms are induced by inclusions, the first vertical arrow is an excision isomorphism. Given an inclusion $i: (C, D) \hookrightarrow (A, B)$ and $\zeta \in H^*(A, B)$ we use the notation $\zeta|_{(C, D)} := i^*(\zeta) \in H^*(C, D)$. For $k := \text{cupl}(\Omega)$ there exist $\xi_1, \dots, \xi_k \in \tilde{H}^*(\Omega)$ so that $\xi := \xi_1 \smile \dots \smile \xi_k \neq 0 \in H^*(\Omega)$. We define

$x_i := \xi_i \times 1 \in H^*(\Omega \times \Omega)$, $y_i := 1 \times \xi_i \in H^*(\Omega \times \Omega)$, $i = 1, \dots, k$, and $z_k := x_k - y_k$. Here $1 \in H^0(\Omega)$ denotes the unit with respect to the cup product. Let $\delta: \Omega \rightarrow \Omega \times \Omega$ be the diagonal map. Then $\delta^*(x_k) = \xi_k \smile 1 = 1 \smile \xi_k = \delta^*(y_k)$, hence $\delta^*(z_k) = 0$ and therefore $z_k|_{\Delta} = 0$. Since the inclusion $\Delta \hookrightarrow \Delta_\varepsilon$ is a homotopy equivalence for ε small enough we obtain $z_k|_{\Delta_\varepsilon} = 0$. This implies that there exists $z \in H^*(\Omega \times \Omega, \Delta_\varepsilon)$ with $z|_{(\Omega \times \Omega)} = z_k$. The multiplicativity formula [22, VII.8.16] yields

$$\begin{aligned} x_1 \smile \cdots \smile x_k \smile y_1 \smile \cdots \smile y_{k-1} \smile z_k \\ = -x_1 \smile \cdots \smile x_k \smile y_1 \smile \cdots \smile y_k = -\xi \times \xi \neq 0 \end{aligned}$$

in $H^*(\Omega \times \Omega)$ because $x_1 \smile \cdots \smile x_k \smile x_k = 0$. As a consequence we obtain

$$x_1 \smile \cdots \smile x_k \smile y_1 \smile \cdots \smile y_{k-1} \smile z \neq 0 \in H^*(\Omega \times \Omega, \Delta_\varepsilon)$$

and hence

$$(x_1 \smile \cdots \smile x_k \smile y_1 \smile \cdots \smile y_{k-1} \smile z)|_{(C\Omega, \Delta_\varepsilon \setminus \Delta)} \neq 0 \in H^*(C\Omega, \Delta_\varepsilon \setminus \Delta)$$

by excision. Using the naturality property [22, VII.8.6] of the cup product we deduce that

$$\begin{aligned} (x_1|_{C\Omega}) \smile \cdots \smile (x_k|_{C\Omega}) \smile (y_1|_{C\Omega}) \smile \cdots \smile (y_{k-1}|_{C\Omega}) \\ = (x_1 \smile \cdots \smile x_k \smile y_1 \smile \cdots \smile y_{k-1})|_{C\Omega} \end{aligned}$$

is nontrivial in $H^*(C\Omega)$ and thus $\text{cupl}(C\Omega) \geq 2k - 1 = 2\text{cupl}(\Omega) - 1$, as desired. Thus we have proved (4.5).

Finally we prove (4.4). For this we consider the commutative diagram

$$\begin{array}{ccc} (C_r\Omega \times I_0^2, C_r\Omega \times \partial I_0^2) & \xrightarrow{i_r} & (C\Omega^r \times \mathbb{R}^2, C\Omega^r \times (\mathbb{R}^2 \setminus \{0\})) \\ k_r \downarrow & & \downarrow l_r \\ (C\Omega \times \mathbb{R}^2, C\Omega \times (\mathbb{R}^2 \setminus \{0\})) & \xrightarrow[\simeq]{} & (C\Omega \times \mathbb{R}^2, C\Omega \times (\mathbb{R}^2 \setminus \{0\})) \end{array}$$

where all maps are inclusions. The lower horizontal map is a homotopy equivalence since the inclusion $C\Omega \hookrightarrow C\Omega$ is a homotopy equivalence. Lemma 2.1 yields

$$(4.6) \quad \text{cat}(i_r) \geq \text{cat}(l_r \circ i_r) = \text{cat}(k_r).$$

Hence it remains to prove

$$(4.7) \quad \text{cat}(k_r) \geq \text{cat}(j).$$

Recall that by our preceding considerations there exists a homotopy

$$h_t: C\Omega \rightarrow C\Omega, \quad 0 \leq t \leq 1,$$

with $h_0 = \text{id}$ and $h_1(C\Omega) \subset CD$. We define a homotopy

$$g_t: CD \rightarrow C\Omega, \quad 0 \leq t \leq 1,$$

with g_0 being the inclusion and $g_1(CD) \subset C_r\Omega$ as follows. For $(x, y) \in CD$ and $0 \leq t \leq 1$ we set

$$x_t := (1-t)x + t\left(\frac{x+y}{2} + \frac{r(x-y)}{|x-y|}\right), \quad y_t := (1-t)y + t\left(\frac{x+y}{2} - \frac{r(x-y)}{|x-y|}\right),$$

and define

$$g_t(x, y) := \begin{cases} (x, y) & \text{if } |x-y| \geq 2r, \\ (x_t, y_t) & \text{if } |x-y| \leq 2r. \end{cases}$$

One easily checks that this map is well defined and continuous. Observe that $x_t, y_t \in \Omega_r$ because $D \subset \Omega_{2r}$ and $|x - x_t|, |y - y_t| \leq r$. Clearly $|x_1 - y_1| \geq 2r$ for all $(x, y) \in CD$, hence $g_1(CD) \subset C_r\Omega$. It follows that the identity on $C\Omega$ is homotopic to the map $g_1 \circ h_1$. Let

$$H_t: (C\Omega \times I_0^2, C\Omega \times \partial I_0^2) \rightarrow (C\Omega \times I_0^2, C\Omega \times \partial I_0^2), \quad 0 \leq t \leq 1,$$

be the induced homotopy keeping the I_0^2 -component fixed. Then H_0 is the identity, and $H_1(x, y, s, t) = (g_1 \circ h_1(x, y), s, t)$. Thus H_1 maps the pair $(C\Omega \times I_0^2, C\Omega \times \partial I_0^2)$ into $(C_r\Omega \times I_0^2, C_r\Omega \times \partial I_0^2)$, and

$$k_r \circ H_1: (C\Omega \times I_0^2, C\Omega \times \partial I_0^2) \rightarrow (C\Omega \times \mathbb{R}^2, C\Omega \times \mathbb{R}^2 \setminus \{0\})$$

is homotopic to the inclusion j . Applying Lemma 2.1 once more we obtain (4.7):

$$\text{cat}(k_r) \geq \text{cat}(k_r \circ H_1) = \text{cat}(j).$$

Now (4.4) follows from (4.6) and (4.7). The proof is finished. \square

REFERENCES

- [1] A. AFTALION AND F. PACELLA, *Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains*, preprint.
- [2] A. AMBROSETTI AND P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
- [3] T. BARTSCH, *Critical point theory on partially ordered Hilbert spaces*, J. Funct. Anal. **186** (2001), 117–152.
- [4] T. BARTSCH, Z. L. LIU AND T. WETH, *Sign changing solutions to superlinear Schrödinger equations*, Comm. Partial Differential Equations **29** (2004), 25–42.
- [5] T. BARTSCH AND Z.-Q. WANG, *On the existence of sign changing solutions for semilinear Dirichlet problems*, Topol. Methods Nonlinear Anal. **7** (1996), 115–131.
- [6] T. BARTSCH AND T. WETH, *A note on additional properties of sign changing solutions to superlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. **22** (2003), 1–14.
- [7] ———, *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann. Inst. H. Poincaré, Anal. Non Linéaire (to appear).

- [8] T. BARTSCH, T. WETH AND M. WILLEM, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. (to appear).
- [9] V. BENCI AND G. CERAMI, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), 79–93.
- [10] ———, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), 29–48.
- [11] A. CASTRO, J. COSSIO AND J. M. NEUBERGER, *A sign-changing solution for a super-linear Dirichlet problem*, Rocky Mountain J. Math. **27** (1997), 1041–1053.
- [12] ———, *A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic BVPs*, Electronic J. Differential Equations **2** (1998), 18.
- [13] K. C. CHANG, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser Boston, 1993.
- [14] M. CLAPP AND D. PUPPE, *Critical point theory with symmetries*, J. Reine Angew. Math. **418** (1991), 1–29.
- [15] M. CONTI, S. TERRACINI AND G. VERZINI, *Nehari’s problem and competing species systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 871–888.
- [16] E. DANCER AND Y. DU, *Existence of sign-changing solutions for some semilinear problems with jumping nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 1165–1176.
- [17] ———, *On sign changing solutions of certain semi-linear elliptic problems*, Appl. Anal. **56** (1995), 193–206.
- [18] E. DANCER AND J. WEI, *On the effect of domain topology in a singular perturbation problem*, Topol. Methods Nonlinear Anal. **11** (1998), 227–248.
- [19] E. N. DANCER AND S. YAN, *A singularly perturbed elliptic problem in bounded domains with nontrivial topology*, Adv. Differential Equations **4** (1999), 347–368.
- [20] K. DEIMLING, *Ordinary differential equations in Banach spaces*, Lecture Notes in Mathematics, vol. 596, Springer, Berlin, 1978.
- [21] M. DEL PINO, P. L. FELMER AND J. C. WEI, *On the role of distance function in some singular perturbation problems*, Comm. Partial Differential Equations **25** (2000), 155–177.
- [22] A. DOLD, *Algebraic Topology*, Springer-Verlag, Berlin, 1972.
- [23] G. FOURNIER AND M. WILLEM, *Relative category and the calculus of variations*, Variational methods (Paris, 1988), vol. 4 of Progr. Nonlinear Differential Equations Appl., Birkhäuser, Boston, MA, 1990, pp. 95–104.
- [24] ———, *Multiple solution of the forced double pendulum equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), 259–281.
- [25] B. GIDAS, W. M. NI AND L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
- [26] S. J. LI AND Z.-Q. WANG, *Mountain pass theorems in order intervals and multiple solutions for semilinear elliptic Dirichlet problems*, J. Anal. Math. **81** (2001), 373–396.
- [27] ———, *Lusternik–Schnirelman theory in partially ordered Hilbert spaces*, Trans. Amer. Math. Soc. **354** (2002), 3207–3227.
- [28] W. S. MASSEY, *Homology and Cohomology Theory*, Marcel Dekker, New York, 1978.
- [29] E. MÜLLER-PFEIFFER, *On the number of nodal domains for eigenfunctions of elliptic differential operators*, J. London Math. Soc. (2) **31** (1985), 91–100.
- [30] E. S. NOUSSAIR AND J. C. WEI, *On the effect of the domain geometry on the existence of nodal solutions in singular perturbations problems*, Indiana Univ. Math. J. **46** (1996), 1255–1271.

- [31] W. M. NI AND J. C. WEI, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. **48** (1995), 731–768.
- [32] D. SMETS M. AND WILLEM, *Partial symmetry and asymptotic behaviour for some elliptic variational problems*, Calc. Var. Partial Differential Equations **18** (2003), 57–75.
- [33] E. H. SPANIER, *Algebraic topology*, McGraw Hill, New York, 1966.
- [34] J. C. WEI AND M. WINTER, *Symmetry of nodal solutions for singularly perturbed elliptic problems on a ball*, Indiana Univ. Math. J. (to appear).
- [35] M. WILLEM, *Minimax Theorems*, Birkhäuser, Boston, 1996.

Manuscript received August 30, 2004

THOMAS BARTSCH AND TOBIAS WETH

Mathematisches Institut

Universitaet Giessen

Arndtstr. 2

D-35392 Giessen, GERMANY

E-mail address: Thomas.Bartsch@math.uni-giessen.de
Tobias.Weth@math.uni-giessen.de