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DIFFERENTIAL INCLUSIONS ON CLOSED SETS IN BANACH SPACES WITH APPLICATION TO SWEEPING PROCESS

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Abstract. This paper deals with the existence of absolutely continuous solutions of a differential inclusion with state constraint in a separable Banach space

$$x(0) = x_0, \quad x(t) \in C(t), \quad \dot{x}(t) \in F(t, x(t))$$

where $C: [0, a] \to X$ is a multifunction with closed graph G and $F: G \to X$ is a convex compact valued multifunction which is separately measurable in $t \in [0, a]$ and separately upper semicontinuous in $x \in X$. Application to a non convex sweeping process is also considered.

1. Introduction

Let X be a Banach space, $I = [0, a] \subset \mathbb{R}$, $t \mapsto C(t)$ a multifunction defined on I with closed graph G in $I \times X$. Let $F: G \to 2^X \setminus \emptyset$ be a multifunction defined on G with nonempty convex compact values on X such that $F(t, \cdot\,)$ is upper semicontinuous (u.s.c.) on $C(t)$ for every $t \in I$. In this paper, we consider the following problem: to find absolutely continuous solutions for the differential inclusion

$$\text{(P)} \qquad x(0) = x_0, \quad x(t) \in C(t) \quad \text{and} \quad x'(t) \in F(t, x(t)) \text{ a.e.}$$

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By absolutely continuous function we mean a function $x\colon I \to X$ such that $x(t) = x_0 + \int_0^t x'(s)\,ds$, $t \in I$, with $x' \in L^1_X(I)$. It is well known that problem (P) has applications in Evolution and Optimal Control Problems. Its origin comes from the work of Nagumo [35] when the second member of (P) is single-valued and $\dim(X) < \infty$. Most results for (P) concern the case when X is a finite dimensional space ([3], [23], [37]). Some papers deal with (P) when X is a general Banach space ([5], [9], [14], [27]) under various assumptions on tangential conditions and measurability assumption for the multifunction F. In ([10], [9], [27]), the authors consider the following tangential condition

(T_B) there exists a negligible set N of I such that
(a) $(\{1\} \times X) \cap T_G(t,x) \neq \emptyset$ for all $t \in N$ and $x \in C(t)$
(b) $(\{1\} \times F(t,x)) \cap T_G(t,x) \neq \emptyset$ for all $t \in I \setminus N$ and $x \in C(t)$
and restrictive measurability assumptions.

In the present paper, we prove the existence of solutions to (P) by assuming the following condition

(T) For every measurable selection σ of $C(\,\cdot\,)$, the multifunction Λ_σ from I to $\mathbb{R} \times X$ defined by

$$t \mapsto (\{1\} \times F(t,\sigma(t))) \cap T_G(t,\sigma(t))$$

is Lebesgue-a.e. nonempty valued and admits at least a measurable selection.

It is clear that condition (T) is weaker than those used in the literature. Furthermore, it turn out that condition (a) of (T_B), who first appeared in the work of Bothe ([10], [9]), is a topological/analytical property of the constraint C rather than a part of a tangential condition. In this work we replace it by the condition less restrictive

(T_C) For every $(t,x) \in G$ (with $t < a$), $\liminf_{h \to 0^+} d(x, C(t+h))/h < \infty$.

Condition (T_C) covers many known class of multifunctions including Lipschitzian and absolutely continuous multifunctions. In particular, we prove later in this paper that results such as those of Frankowska–Plaskacz–Rzeuchowski ([26], [25]) can be deduced from the Lipschitzian case by a simple change of variables. Moreover, our main result (Theorem 3.1) extends the results of Bothe ([10], [9]) and Gavioli ([27]) and is new even when X is of finite dimensional. Our work relies on several sophisticated techniques because we deal with a general Banach space and weaker assumption for both measurability and tangency condition (T). One of the main ingredient relies on a new extension of Scorza–Dragoni's theorem (Theorem 2.4) involving the use of the essential supremum for multifunctions that appears first in the pioneering work of Castaing–Marques ([13]). In this framework, consult ([9], [18], [4], [21], [36], [30]) for other related results concerning the Scorza–Dragoni property. We finish the paper by giving a sharp application of Theorem 3.1 to the study of nonconvex sweeping process.

2. Definitions and preliminaries results

Let $(T, \mathcal{F}, \mu)$ be a σ-finite complete measure space and let X be a Hausdorff topological space. We say that X is a *Polish* (resp. *Suslin*) space if it is metrizable, separable and complete (resp. if there exists a Polish space Y and a continuous onto mapping $s\colon Y \to X$). Let Γ be a multifunction from T to 2^X. We say that Γ is *weakly measurable* if for every open set U in X, the set $\Gamma^-(U) := \{t \in T : \Gamma(t) \cap U \neq \emptyset\}$ belongs to $\mathcal{F}$[1]. The multifunction Γ is said to be *graph-measurable* if the graph graph(Γ) of Γ belongs to $\mathcal{F} \otimes B(X)$, where $\mathcal{B}(X)$ denotes the Borel tribe of X. A function $\sigma\colon T \to X$ is called a *selection* of the multifunction Γ if $\sigma(t) \in \Gamma(t)$ for all $t \in T$. We denote by L^0_Γ the set of all measurable selections σ of Γ. For more about measurability of multifunctions, we refer to Castaing–Valadier [15] and [29]. In the set of all weakly-measurable (or graph-measurable) multifunctions Γ from T to X, we define a preorder $\preccurlyeq$ by setting: $\Gamma_1 \preccurlyeq \Gamma_2$ if and only if $\Gamma_1(t) \subseteq \Gamma_2(t)$ μ-almost everywhere; that is there exists a μ-negligible set N of T such that $\Gamma_1(t) \subseteq \Gamma_2(t)$ for all $t \in T \setminus N$.

We denote by $\mathrm{cl}(X)$ (resp. $\mathcal{K}(X)$) the set of all nonempty closed (resp. compact) subsets of X. Let (A_n) be a sequence of subsets of X. The *upper limit* (in the sense of Painlevé–Kuratowski) of the sequence (A_n) is defined by:

$$\mathrm{Ls}(A_n) := \bigcap_{p \in \mathbb{N}} \overline{\bigcup_{k \geq p} A_k}.$$

A multifunction F from a topological space Y to the subsets of X is said to be *upper semicontinuous* (shortly u.s.c.) on Y if for every closed subset U of X, the set $F^-(U) := \{y \in Y : F(y) \cap U \neq \emptyset\}$ is closed in Y. We can easily check that if X is a metric space, then the upper semicontinuity of F implies that for every $y \in Y$ and every sequence (y_n) of Y converging to y, we have $\mathrm{Ls}F(y_n) \subset \overline{F(y)}$.

Suppose that X is a metric space with distance d. For $A \subseteq X$ and $x \in X$, we set $d(x, A) := \inf\{d(x, a) : a \in A\}$ with the convention $d(x, \emptyset) := \infty$. For $C, D \subset X$, the *excess* of C over D is defined by $e(C, D) := \sup\{d(x, D) : x \in C\}$.

If E is a Banach space, we denote by $\mathrm{ck}(E)$ (resp. $\mathrm{cwk}(E)$) the set of all nonempty convex compact (resp. convex weakly compact) subsets of X. For $A \subset E$, we denote by $\mathrm{co}(A)$ (resp. $\overline{\mathrm{co}}(A)$; resp. $\delta^*(\,\cdot\,; A)$) the convex hull (resp. closed convex hull; resp. the support function) of A. We also set for any $A \subset E$, $|A| := \sup\{\|x\| : x \in A\}$.

[1]Note that this imply that the set $\{t \in T : \Gamma(t) = \emptyset\}$ is measurable.

We will need next the following general property of upper semicontinuous multifunctions (c.f. [8]).

Lemma 2.1. *Let X and Y be two Hausdorff topological spaces, X being first countable. Let F be a multifunction from X to the subsets of Y. We suppose that F has a closed graph in $X \times Y$ and that for every compact subset M of X, the image set $F(M)$ is relatively compact in Y. Then the multifunction F is upper semicontinuous on X.*

The following result is due to Valadier [40] (see also [13]). For the sake of completeness, we produce an alternative proof.

Proposition 2.2. *Let X be a separable and metrizable space and $\Sigma: T \to 2^X$ be a multifunction from T to the closed subsets of X. Then there exists a largest weakly measurable multifunction $\Sigma_0: T \to 2^X$ from T to the closed subsets of X such that $\Sigma_0(t) \subseteq \Sigma(t)$, for all $t \in T \setminus N$, where N is some negligible subset of T. The same result remains valid in the case of a Suslin space X; the multifunction Σ_0 is then graph-measurable instead of weakly measurable.*

Proof. (1) We suppose that X is metrizable and separable. Let $(x_n)_{n\geq 1}$ be a dense sequence in X. Denote by $\mathcal{M}$ the set of all weakly measurable multifunctions $C: T \to \mathrm{cl}(X) \cup \{\emptyset\}$ such that $C \preccurlyeq \Sigma$. We can set $\mathcal{M} = \{\Gamma_\alpha : \alpha \in A\}$ for some index set A. For $\alpha \in A$ and $n \geq 1$, we set

$$f_\alpha^n(t) := d(x_n, \Gamma_\alpha(t)), \quad t \in T.$$

Thus f_α^n is a $\mathcal{F}-$ measurable function from T to $[0, \infty]$. Let us put

$$f^n := \text{ess inf}\{f_\alpha^n : \alpha \in A\}.$$

Since the measure μ is σ-finite, there exists a sequence of indices $(\alpha_k^n)_{k\geq 1} \subset A$ such that

$$f^n(t) = \inf_{k\geq 1} f_{\alpha_k^n}^n(t), \quad t \in T.$$

Let $\Sigma_0(t) := \mathrm{cl} \bigcup_{n,k\geq 1} \Gamma_{\alpha_k^n}(t)$ for $t \in T$. Then obviously Σ_0 is a weakly measurable multifunction from T to the closed subsets of X such that $\Sigma_0(t) \subseteq \Sigma(t)$ μ-a.e. Let us prove that Σ_0 is the largest member of $\mathcal{M}$. Let $\Gamma_\alpha \in \mathcal{M}$. For any n we have, $f^n(t) \leq f_\alpha^n(t)$ μ-a.e. Then let N be a negligible set such that for every $t \in T \setminus N$ and every $n \geq 1$, $f^n(t) \leq f_\alpha^n(t) = d(x_n, \Gamma_\alpha(t))$. We shall prove that for every $t \in T \setminus N$, $\Gamma_\alpha(t) \subseteq \Sigma_0(t)$. Indeed, let us take $x \in \Gamma_\alpha(t)$ and $\varepsilon > 0$. There exists n_ε such that $d(x_{n_\varepsilon}, x) < \varepsilon/2$. We have

$$f^{n_\varepsilon}(t) = \inf_{k\geq 1} d(x_{n_\varepsilon}, \Gamma_{\alpha_k^{n_\varepsilon}}(t)) \leq d(x_{n_\varepsilon}, \Gamma_\alpha(t)) < \frac{\varepsilon}{2}.$$

So there exist $k_\varepsilon \geq 1$ and $y_\varepsilon \in \Gamma_{\alpha_{k_\varepsilon}^{n_\varepsilon}}(t)$ such that $d(x_{n_\varepsilon}, y_\varepsilon) < \varepsilon/2$. It follows that $d(x, y_\varepsilon) \leq d(x, x_{n_\varepsilon}) + d(x_{n_\varepsilon}, y_\varepsilon) < \varepsilon$. This proves that, for every $\varepsilon > 0$, $B(x, \varepsilon) \cap (\bigcup_{n,k\geq 1} \Gamma_{\alpha_k^n}(t)) \neq \emptyset$, thus $x \in \Sigma_0(t)$. Our claim is now proved.

(2) Suppose now that X is a Suslin topological space. Let X_0 be a Polish space and $s\colon X_0 \to X$ be a continuous onto mapping. Set $\Sigma_1(t) := s^{-1}(\Sigma(t))$ for $t \in T$. Then Σ_1 is a multifunction from T to the closed subsets of X_0. By step (1) there exists a largest weakly measurable multifunction $\Gamma_0\colon T \rightrightarrows X_0$ with closed values such that $\Gamma_0(t) \subseteq \Sigma_1(t)$ μ-a.e. Let $\Gamma(t) := s(\Gamma_0(t))$ for $t \in T$. For every open set V of X, we have $\Gamma^-(V) = \Gamma_0^-(s^{-1}(V)) \in \mathcal{F}$. So Γ is weakly measurable and obviously we have $\Gamma(t) \subseteq \Sigma(t)$ μ-a.e. Let Γ' be an other graph-measurable multifunction with closed values such that $\Gamma'(t) \subseteq \Sigma(t)$ μ-a.e. Then $\Gamma_1(t) := s^{-1}(\Gamma'(t))$, $t \in T$, is a graph-measurable multifunction (in fact $\mathrm{graph}(\Gamma_1) = \psi^{-1}(\mathrm{graph}(\Gamma'))$ where $\psi\colon T \times X_0 \to T \times X\colon (t,x) \mapsto (t, s(x))$) with closed values satisfying $\Gamma_1(t) \subseteq \Sigma_1(t)$ μ-a.e. Since Γ' is also weakly measurable ([15, Theorem III.30]), this implies that $\Gamma_1(t) \subseteq \Gamma_0(t)$ μ-a.e. It follows (since s is onto) that

$$\Gamma'(t) = s(s^{-1}(\Gamma'(t))) = s(\Gamma_1(t)) \subseteq s(\Gamma_0(t)) = \Gamma(t) \quad \mu\text{-a.e.} \tag{2.1}$$

In particular if we take $\Gamma' = \overline{\Gamma}$ (which is a graph-measurable multifunction) we obtain that $\Gamma(t) = \overline{\Gamma(t)}$ μ-a.e. Hence we can consider without loss of generality that the multifunction Γ is closed valued and so it is graph-measurable. Condition (2.1) shows also that Γ is the required multifunction. □

We suppose in what follows that T is a Hausdorff compact topological space, μ a positive Radon measure on T and $\mathcal{F} = \widehat{\mathcal{B}}(T)$ the μ-completion of the Borel tribe $\mathcal{B}(T)$. Recall the following version of Scorza–Dragoni theorem which is due to Castaing [11]:

PROPOSITION 2.3. *Let X be a Polish space. Let $\varphi\colon T \times X \to \mathbb{R}$ be a function such that $\varphi(\,\cdot\,, x)$ is μ-measurable for all $x \in X$, and $\varphi(t, \,\cdot\,)$ is continuous on X for all $t \in T$. Then for every $\varepsilon > 0$, there exists a compact set $T_\varepsilon \subset T$ with $\mu(T \setminus T_\varepsilon) < \varepsilon$ such that the restriction $\varphi_{|T_\varepsilon \times X}$ of φ to $T_\varepsilon \times X$ is continuous.*

Let us denote by τ_0 the topology of the compact space T. We consider another topology τ on T finer than τ_0 (i.e. $\tau_0 \subseteq \tau$) and that is *first countable*. When T is equipped with the topology τ, we will denote it by T_τ (note that this is not necessarily a compact space).

We are now ready to state the following general multivalued version of Scorza–Dragoni theorem for upper semicontinuous multifunctions:

THEOREM 2.4. *Let X and Y be two Polish topological spaces. Let $C\colon T \to 2^X \setminus \{\emptyset\}$ be a multifunction with measurable graph G in $T \times X$; that is $G \in \widehat{\mathcal{B}}(T) \otimes \mathcal{B}(X)$. Let $F\colon G \to \mathcal{K}(Y)$ be a multifunction such that:*

(i) *For every $t \in T$, $\mathrm{graph}(F_t) = \{(x,y) \in X \times Y : x \in C(t)$ and $y \in F(t,x)\}$ is closed in $X \times Y$.*

(ii) *For every $\sigma \in L^0_C$, the multifunction $t \mapsto F(t,\sigma(t))$ admits at least a μ-measurable selection.*

(iii) *For every $\varepsilon > 0$, there exists a compact set $J_\varepsilon \subset T$ with $\mu(T \setminus J_\varepsilon) < \varepsilon$ such that for every compact subset $\mathcal{M}$ of the set $(J_\varepsilon \times X) \cap G$, endowed by the topology inherited from $T_\tau \times X$, the image set $F(\mathcal{M})$ is relatively compact in Y.*

Then, there exists a multifunction $F_0: G \to \mathcal{K}(X) \cup \{\emptyset\}$ such that

(a) *The graph of F_0 belongs to $\widehat{\mathcal{B}}(T) \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$.*

(b) *There exists a μ-negligible set N such that*

$$\emptyset \subsetneqq F_0(t,x) \subseteq F(t,x) \quad \textit{for all } t \in T \setminus N \textit{ and all } x \in C(t).$$

(c) *If $u: T \to X$ and $v: T \to Y$ are two μ-measurable functions such that $(t,u(t)) \in G$ and $v(t) \in F(t,u(t))$ μ-a.e. then $v(t) \in F_0(t,u(t))$ μ-a.e.*

(d) *For every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset T$ with $\mu(T \setminus K_\varepsilon) < \varepsilon$ such that the restriction $F_{0|G_\varepsilon}$ of F_0 to the set $G_\varepsilon = (K_\varepsilon \times X) \cap G$, equipped with the topology inherited from the product space $T_\tau \times X$, is upper semicontinuous.*

Proof. For every $t \in T$ put $\Phi(t) := \mathrm{graph}(F_t)$. Then, by condition (i), Φ is a multifunction from T to the closed subsets of $X \times Y$. By Proposition 2.2, there exists a largest weakly measurable multifunction $\Phi_0: T \to \mathrm{cl}(X \times Y) \cup \{\emptyset\}$ such that $\Phi_0(t) \subset \Phi(t)$, for all $t \in T \setminus N_0$, for some μ-negligible set N_0 of T. By [15, Proposition III.13], the multifunction Φ_0 is also graph-measurable. Hence, putting

$$F_0(t,x) := \{y \in Y : (x,y) \in \Phi_0(t)\}, \quad (t,x) \in G$$

we get a multifunction from G to the closed subsets of Y such that $\mathrm{graph}(F_0) \in \widehat{\mathcal{B}}(T) \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$. Moreover, we have

(A1) for all $t \in T \setminus N_0$ and for all $x \in C(t)$, $F_0(t,x) \subset F(t,x)$.

(A2) If $u: T \to X$ and $v: T \to Y$ are two μ-measurable functions such that $(t,u(t)) \in G$, for all $t \in T$ and $v(t) \in F(t,u(t))$ μ-a.e. then $v(t) \in F_0(t,u(t))$ μ-a.e.

Let $\{\sigma_n\}_{n \geq 0}$ be a sequence of μ-measurable selections of the multifunction C such that $\{\sigma_n(t) : n \geq 0\}$ is dense in $C(t)$ for every $t \in T$ ([15, Theorem III.22]). By condition (ii), for every $n \geq 0$ there exists a μ-measurable function $v_n: T \to Y$ such that $v_n(t) \in F(t,\sigma_n(t))$ μ-a.e. By (A2), we have also $v_n(t) \in F_0(t,\sigma_n(t))$ μ-a.e. for every n. By modifying the μ-negligible set N_0, we may suppose that

$$v_n(t) \in F_0(t,\sigma_n(t)) \quad \text{for all } n \in \mathbb{N} \text{ and all } t \in T \setminus N_0. \tag{2.2}$$

In particular, it follows that $\Phi_0(t) \neq \emptyset$ for all $t \in T \setminus N_0$. By virtue of the choice of Φ_0, we may suppose without loss of generality that $\Phi_0(t) \neq \emptyset$ for all

$t \in T$. Let $\varepsilon > 0$. By hypothesis, there exists a compact set $J_\varepsilon \subset T$, with $\mu(T \setminus J_\varepsilon) < \varepsilon$, such that compactness condition (iii) holds. Now we define a function $h_0 \colon T \times (X \times Y) \to [0, \infty[$ by setting $h_0(t, z) := d(z, \Phi_0(t))$ for all $(t, z) \in T \times (X \times Y)$. It is clear that function h_0 is separately μ-measurable in t and separately continuous in z. By Proposition 2.3, there exists a compact set $K_\varepsilon \subset T$ such that $\lambda(T \setminus K_\varepsilon) < \varepsilon$ and the restricted function $h_{0|K_\varepsilon \times (X \times Y)}$ is continuous. We may suppose without loss of generality that $K_\varepsilon \subset J_\varepsilon \setminus N_0$. Let us put $G_\varepsilon = (K_\varepsilon \times X) \cap G$. Then, from the continuity of $h_{0|K_\varepsilon \times (X \times Y)}$, it follows easily that the graph of the multifunction $F_{0|G_\varepsilon}$ is closed in $G_\varepsilon \times Y$. Hence it is also closed in $G_\varepsilon^\tau \times Y$, where G_ε^τ denotes the set G_ε equipped with the topology induced by the product space $T_\tau \times X$. By condition (iii), $F_0(\mathcal{M})$ is relatively compact in Y for every compact set $\mathcal{M}$ of G_ε^τ. Hence by Lemma 2.1, we deduce that $F_{0|G_\varepsilon^\tau}$ is upper semicontinuous[2].

To finish the proof of the theorem, we will prove that

$$F_0(t, x) \neq \emptyset \quad \text{for all } (t, x) \in G_\varepsilon. \tag{2.3}$$

Let $(t, x) \in G_\varepsilon$. There exists a subsequence $(\sigma_{n_j}(t))_j$ of $(\sigma_n(t))_n$ converging to x in X (this follows from the density of $(\sigma_n(t))$ in $C(t)$). Let us consider the compact set $B' = \{x\} \cup \{\sigma_{n_j}(t) : j \geq 0\}$. By the upper semicontinuity of $F_0(t, \cdot\,)$ on $C(t)$, the set $F_0(t, B')$ is compact in Y. Furthermore, by (2.2), $v_{n_j}(t) \in F_0(t, B')$, for all $j \geq 0$. So, we can suppose (along a subsequence) that there exists $y \in Y$ such that $v_{n_j}(t) \to y$ in Y. Since the multifunction $F_{0|\Gamma_\varepsilon}$ is closed, we deduce that $y \in F_0(t, x)$. Hence claim (2.3) is proved. □

Remarks. (1) Assertion (b) in Theorem 2.4 is crucial. It relies strongly on the essential supremum property given in Proposition 2.2.

(2) We can find in the literature several works on multivalued versions of the Scorza–Dragoni theorem (see for instance [11], [30], [36], [13]). Our result is an extension of the result given in [13].

(3) Let us choose a sequence (J_n) of compact subsets of T with $\mu(T \setminus J_n) < 2^{-n}$ such that compactness condition (iii) holds on $G_n = (J_n \times X) \cap G$ for every n. Take t in $\bigcup_n J_n$ and M an arbitrary compact subset of $C(t)$. Then, by (iii) the image set $F_t(M) = F(\{t\} \times M)$ is relatively compact in Y. By Lemma 2.1, it follows that F_t is u.s.c. on $C(t)$. Hence it is equivalent to replace condition (i) of the theorem by the following stronger condition:

(i_1) For almost every $t \in T$, the multifunction $F(t, \cdot\,)$ is upper semicontinuous on $C(t)$.

[2] Note that condition (A1) imply that F_0 is compact valued in Y.

Now we will give an interesting application of Theorem 2.4. We suppose that $X = Y$ is a separable Banach space. We denote by α_X the *Kuratowski measure of noncompactness* defined on the set of all bounded subsets of X.

Let $T = I = [0, a]$, with $a > 0$, be a compact interval of $\mathbb{R}$ and λ the Lebesgue measure on I. We denote by $\mathcal{L}(I)$ the tribe of Lebesgue measurable subsets of I. The *left topology* τ_ℓ (resp. *right topology* τ_r) on I is defined as the topology generated by the basis of open sets $\{]s,t] \cap I : s \le t\}$ (resp. $\{[s,t[\cap I : s \le t\}$). We denote by I_ℓ (resp. I_r) the set I equipped with the left topology (resp. right topology). Let us denote by τ_u the usual topology on I. For $t \in I$, $\delta > 0$ and $\tau \in \{\tau_u, \tau_\ell, \tau_r\}$, we set

$$I^\tau_{t,\delta} := \begin{cases} [t-\delta, t+\delta] \cap I & \text{if } \tau = \tau_u, \\ [t-\delta, t] \cap I & \text{if } \tau = \tau_\ell, \\ [t, t+\delta] \cap I & \text{if } \tau = \tau_r. \end{cases}$$

We have the following result:

COROLLARY 2.5. *Let $\tau \in \{\tau_u, \tau_\ell, \tau_r\}$ and I_τ be the interval I equipped with topology τ. Let X be a separable Banach space. Let $C: I \to 2^X \setminus \{\emptyset\}$ be a multifunction with measurable graph G in $I \times X$. Let $F: G \to \mathcal{K}(X)$ be a multifunction such that:*

(1) *For every $t \in I$,* $\mathrm{graph}(F_t) = \{(x,y) \in X \times X : x \in C(t)$ *and* $y \in F(t,x)\}$ *is closed in $X \times X$.*
(2) *For every $\sigma \in L^0_C$, the multifunction $t \mapsto F(t, \sigma(t))$ admits at least a λ-measurable selection.*
(3) *For every $\varepsilon > 0$, there exists a compact set $J_\varepsilon \subset I$ with $\lambda(I \setminus J_\varepsilon) < \varepsilon$ such that for every $t \in J_\varepsilon$ and every compact set B of X, we have*

$$\inf_{\delta > 0} \alpha_X[F(((I^\tau_{t,\delta} \cap J_\varepsilon) \times B) \cap G)] = 0.$$

Then, there exists a multifunction $F_0: G \to \mathcal{K}(X) \cup \{\emptyset\}$ such that

(a) *The graph of F_0 belongs to $\mathcal{L}(I) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$.*
(b) *There exists a λ-negligible set N such that*

$$\emptyset \subsetneqq F_0(t,x) \subseteq F(t,x) \quad \text{for all } t \in I \setminus N \text{ and all } x \in C(t).$$

(c) *If $u: I \to X$ and $v: I \to X$ are two λ-measurable functions such that $(t, u(t)) \in G$ and $v(t) \in F(t, u(t))$ λ-a.e. then $v(t) \in F_0(t, u(t))$ λ-a.e.*
(d) *For every $\varepsilon > 0$, there exists a compact set $I_\varepsilon \subset I$ with $\lambda(I \setminus I_\varepsilon) < \varepsilon$ such that the restriction $F_{0|G_\varepsilon}$ of F_0 to the set $G_\varepsilon = (I_\varepsilon \times X) \cap G$, equipped with the topology inherited from the product space $I_\tau \times X$, is upper semicontinuous.*

Moreover, if the multifunction F is convex compact valued in X then so is F_0.

Proof. Taking $T = I$ and $Y = X$, we remark that all conditions of Theorem 2.4 are satisfied except for the compactness condition (iii). Let us check it. It is enough to give the proof in the case where $\tau = \tau_\ell$ (other cases are similar). Let $\varepsilon > 0$ and J_ε be the compact subset of I given by condition (3). Take $\mathcal{M}$ an arbitrary compact subset of G^ℓ_ε (where G^ℓ_ε denotes the set $(J_\varepsilon \times X) \cap G$ equipped with the induced topology from $I_\ell \times X$). Let $(y_n)_{n\geq 1}$ be a sequence in $F(\mathcal{M})$. Then $y_n \in F(t_n, x_n)$ for some $(t_n, x_n) \in \mathcal{M}$, $n = 1, 2, \ldots$ Since $\mathcal{M}$ is compact, we can suppose (along a subsequence) that $(t_n, x_n) \to (t, x)$ in G^ℓ_ε for some $(t, x) \in G_\varepsilon$. Consider the compact subset $B := \{x\} \cup \{x_n : n \geq 1\}$ of $C(t)$ and apply the compactness condition (3) to t. Then for every $\eta > 0$, there exists $\delta > 0$ such that

$$\alpha_X[F((([t-\delta, t] \cap J_\varepsilon) \times B) \cap G)] < \eta.$$

Since $t_n \to t$ in I_ℓ, there exists $n_1 = n(\eta)$ such that $t_n \in [t-\delta, t] \cap J_\varepsilon$ for all $n \geq n_1$. It follows that y_n belongs to $F(([t-\delta, t] \cap J_\varepsilon) \times B \cap G)$ for every $n \geq n_1$. Hence

$$\alpha_X[\{y_n : n \geq n_1\}] < \eta.$$

Since $\eta > 0$ is arbitrary, we deduce that $\alpha_X[\{y_n : n \geq 1\}] = 0$. This proves that the set $F(\mathcal{M})$ is relatively compact in X.

For the last assertion of the theorem, let us preserve notations of the proof of Theorem 2.4. By construction, $\Phi_0(t) = \mathrm{graph} F_0(t, \cdot\,)$ is the largest weakly measurable multifunction such that $\Phi_0 \preccurlyeq \Phi$. On the other hand, the multifunction $\Phi_1(t) := \mathrm{graph}\overline{\mathrm{co}}F_0(t, \cdot\,)$ is also weakly measurable and satisfies $\Phi_1 \preccurlyeq \Phi$. Hence $\Phi_1(t) = \Phi_0(t)$ a.e. Modifying F_0 on a λ-negligible set we can infer that $F_0 = \overline{\mathrm{co}\, F_0}$. This finish the proof of corollary. □

Remarks. Let us give some explicit examples providing compactness condition (3) of Corollary 2.5.

(1) Suppose X is a finite dimensional space, $F: G \to \mathcal{K}(X)$ a multifunction such that $|F(t, x)| \leq g(t)(1+\|x\|)$, for all $(t, x) \in G$ for some measurable function $g: I \to \mathbb{R}^+$. Then compactness condition (3) holds (apply Lusin theorem to g).

(2) Suppose X infinite dimensional and $F: G \to \mathcal{K}(X)$ a multifunction such that for some $k \in L^1_{\mathbb{R}^+}(I)$ we have

$$\alpha_X[F(t, B)] \leq k(t)\alpha_X(B)$$

for every bounded subset B of X. Then compactness condition (3) is satisfied.

We shall need in what follows the following lemmas:

Lemma 2.6. *Let $c \in L^1_{\mathbb{R}^+}(I)$ and let N be a λ-negligible set of I. Then there exists a lower semicontinuous function $\overline{c}: I \to\,]0, \infty]$ such that $0 \leq c(t) < \overline{c}(t)$, for all $t \in I$, $\overline{c}(t) = \infty$, for all $t \in N$, and $\int_0^a \overline{c}(s)\, ds < \infty$.*

PROOF. Let us set as in Bothe (see [10]):

$$A_n := \{t \in I : n-1 \le c(t) < n\} \cup N, \quad n \in \mathbb{N}^*.$$

Let us choose, for every $n \in \mathbb{N}^*$, an open set O_n of I such that $A_n \subset O_n$ and $\lambda(O_n \setminus A_n) < 2^{-n}$ and set $\overline{c} := \sum_{n=1}^{\infty} n\chi_{O_n}$. It is clear that $0 \le c < \overline{c}$ on I and that $\overline{c}$ is lower semicontinuous on I. Moreover, if $t \in N$, we have $t \in O_n$ for every $n \ge 1$, so $\overline{c}(t) = \infty$. Now, from the choice of A_n and O_n, we have $n\chi_{O_n} = n\chi_{O_n \setminus A_n} + n\chi_{A_n} \le n\chi_{O_n\setminus A_n} + \chi_{A_n} c + \chi_{A_n}$. Hence, λ-a.e. $\overline{c} \le g + c + 1$, where $g = \sum_{n=1}^{\infty} n\chi_{O_n \setminus A_n}$. Since $\int_0^a g \le \sum_{n=1}^{\infty} n2^{-n} < \infty$, it follows that $\overline{c}$ is λ-integrable. □

Let E be a normed space (with norm $\|\cdot\|$), K a subset of E and $x \in \overline{K}$. We recall that the *Bouligand cone* of K at x, denoted by $T_K(x)$, is defined by:

$$T_K(x) := \left\{u \in E : \liminf_{h \to 0^+} \frac{1}{h} d(x + hu, K) = 0\right\}.$$

We have the following lemma, providing a measurability property of the Bouligand cone.

LEMMA 2.7. *Let E be a separable Banach space and K a nonempty subset of E. Let us endow K with the topology induced by E. Then, there exists a decreasing sequence (Φ_n) of weakly measurable multifunctions $\Phi_n : K \to \mathrm{cl}(E)$ such that $T_K(\xi) = \bigcap_n \Phi_n(\xi)$ for each $\xi \in K$.*

PROOF. By definition, we have $T_K(\xi) = \{v \in E : f(\xi, v) = 0\}$, where $f(\xi, v) := \liminf_{h\to 0^+} d(\xi + hv; K)/h$. Let (ε_n) be a sequence of positive numbers such that $\varepsilon_n \downarrow 0$. Let us set

$$f_n(\xi, v) := \inf_{0<h<\varepsilon_n} \frac{1}{h} d(\xi + hv; K)$$

for $n \in \mathbb{N}$, $\xi \in K$ and $v \in E$. It is clear that every function f_n is upper semicontinuous on $K \times E$ and that $f_n(\xi, \cdot)$ is continuous on E for every $\xi \in K$. It follows by Theorem 6.2 of Himmelberg [29] (more exactly see the remark following this theorem) that the multifunction $\Phi_n : K \to \mathrm{cl}(E)$ defined by

$$\Phi_n(\xi) := \mathrm{cl}\{v \in E : f_n(\xi, v) < \varepsilon_n\}, \quad \xi \in K$$

is weakly measurable. Now we let the reader check that $T_\Gamma(\xi) = \bigcap_n \Phi_n(\xi)$ for every $\xi \in K$. □

REMARK. Weak measurability in the statement of Lemma 2.7 is relative to the measure space $(K, \mathcal{B}(K))$, where the set K is equipped with the separable metric topology inherited from E.

3. The main result

Let us introduce some notations and definitions. Let $I = [0, a]$ $(a > 0)$ be an interval of $\mathbb{R}$ and X a separable Banach space. If G is a subset of $I \times X$, we set $G^* := \{(t, x) \in G : t < a\}$. A *Kamke function* on $I \times \mathbb{R}^+$ is a Carathéodory mapping $\omega: I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t, 0) = 0$ for all $t \in I$ and that the unique absolutely continuous function $r: I \to \mathbb{R}^+$ such that $r(0) = 0$ and $r'(t) \le \omega(t, r(t))$ λ-a.e. is the function identically equal to zero. Examples of Kamke functions on $I \times \mathbb{R}^+$ are the functions of type $\omega(t, x) = k(t)x$ with $k \in L^1_{\mathbb{R}^+}(I)$.

We state now the main result of this paper:

Theorem 3.1. *Let $I = [0, a]$ (with $a > 0$) be an interval of $\mathbb{R}$ and X a separable Banach space. Let $C: I \to \mathrm{cl}(X)$ be a multifunction with closed graph G in $I_\ell \times X$. Let $F: G \to \mathrm{ck}(X)$ be a multifunction such that:*

(i) *There exists $c \in L^1_{\mathbb{R}^+}(I)$ such that*
$$|F(t, x)| \le c(t)(1 + \|x\|) \quad \text{for all } (t, x) \in G.$$
(ii) *For every $t \in I$, the multifunction $F(t, \cdot\,)$ is u.s.c. on $C(t)$.*

(iii) *For every $(t, x) \in G^*$,*
$$\liminf_{h \to 0^+} \frac{1}{h}\, d(x, C(t + h)) < \infty.$$
(iv) *For every $\sigma \in L^0_C$, the multifunction*
$$\Lambda_\sigma: t \mapsto (\{1\} \times F(t, \sigma(t))) \cap T_G(t, \sigma(t))$$
is λ-a.e. nonempty valued on I and admits at least a measurable selection.

(v) *There exists a Kamke-function ω on $I \times \mathbb{R}^+$ such that for every $\varepsilon > 0$, there exists a compact set $J_\varepsilon \subset I$, with $\lambda(I \setminus J_\varepsilon) < \varepsilon$, such that for every $t \in J_\varepsilon$ we have:*
$$\inf_{\delta > 0} \alpha_X[F((([t - \delta, t] \cap J_\varepsilon) \times B) \cap G)] \le \omega(t, \alpha_X(B))$$
for every bounded set B of X.

Then, given $x_0 \in C(0)$, there exists an absolutely continuous function $x: I \to X$ and a function $x' \in L^1_X(I)$ such that $x(t) = x_0 + \int_0^t x'(s)\, ds$ for all $t \in I$ and
$$\text{(P)} \qquad \begin{cases} x(t) \in C(t) & \text{on } I, \\ x'(t) \in F(t, x(t)) & \text{a.e. on } I. \end{cases}$$

Proof. By virtue of Corollary 2.5, there exists a multifunction $F_0: G \to \mathrm{ck}(X)$ and a λ-negligible set N_0 such that:

(B1) $\mathrm{graph}(F_0) \in \mathcal{L}(I) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$.

(B2) For every $t \in I \setminus N_0$ and every $x \in C(t)$ we have $\emptyset \ne F_0(t, x) \subset F(t, x)$.

(B3) If $u: I \to X$ and $v: I \to X$ are two measurable functions such that $u \in L^0_C$ and $v(t) \in F(t, u(t))$ a.e. then $v(t) \in F_0(t, u(t))$ a.e.

(B4) For every $\varepsilon > 0$, there exists a compact set $J_\varepsilon \subset I$ with $\lambda(I \setminus J_\varepsilon) < \varepsilon$, such that if we denote by G^ℓ_ε the set $G_\varepsilon = (J_\varepsilon \times X) \cap G$ equipped with the topology induced by $I_\ell \times X$, then the restriction $F_{0|G_\varepsilon}$ of the multifunction F_0 to G_ε is u.s.c. on G^ℓ_ε.

Before going on with the proof of Theorem 3.1 we will need some preliminaries lemmas. The first one is the following lemma.

LEMMA 3.2. *We preserve the preceding assumptions and notations. Then, there exists a λ-negligible set N of I such that for every $t \in I \setminus N$ and every $x \in C(t)$, $\emptyset \neq F_0(t,x) \subset F(t,x)$ and $(\{1\} \times F_0(t,x)) \cap T_G(t,x) \neq \emptyset$.*

PROOF. By virtue of condition (B4), there exists a sequence of compact sets $(J_n)_{n \geq 1}$ in I such that for every n, $\lambda(I \setminus J_n) < 2^{-n}$ and the restriction $F_{0|G_n}$ of F_0 to the set $G_n = (J_n \times X) \cap G$ (equipped with the topology induced by $I_\ell \times X$) is u.s.c. Let us put $J := \bigcup_{n \geq 1} J_n$ and consider the multifunction $\mathcal{T}$ defined by

$$\mathcal{T}(t) := \{x \in C(t) : (\{1\} \times F_0(t,x)) \cap T_G(t,x) \neq \emptyset\}, \quad t \in J.$$

We consider on J the complete tribe $\mathcal{L}(J) := \mathcal{L}(I) \cap J$ (which is actually the λ-completion tribe of the Borel tribe $\mathcal{B}(J)$). We shall prove that $\operatorname{graph}(\mathcal{T}) \in \mathcal{L}(J) \otimes \mathcal{B}(X)$. Let G' denote the Borel subset $(J \times X) \cap G$ of $I \times X$ equipped with the usual topology. We define on G' a multifunction Φ by setting: $\Phi(t,x) := \{1\} \times F_0(t,x)$ for $(t,x) \in G'$. Then Φ is a weakly measurable multifunction from G' to $\operatorname{ck}(\mathbb{R} \times X)$. Indeed, this follows from the fact that for each $n \geq 1$, $F_{0|(J_n \times X) \cap G}$ is u.s.c. Furthermore, by Lemma 2.7, there exists a sequence of weakly measurable multifunctions $\Phi_n: G' \to \operatorname{cl}(\mathbb{R} \times X)$ such that $T_G(t,x) = \bigcap_n \Phi_n(t,x)$ for every $(t,x) \in G'$. Hence, we have

$$\operatorname{graph}(\mathcal{T}) = \left\{(t,x) \in G' : \Phi(t,x) \cap \left(\bigcap_n \Phi_n(t,x)\right) \neq \emptyset\right\}.$$

In virtue of Theorem 4.1 of Himmelberg [29], the multifunction $H: G' \rightrightarrows \mathbb{R} \times X$ defined by $H(t,x) := \Phi(t,x) \cap (\bigcap_n \Phi_n(t,x))$ is weakly measurable (note that Φ is compact valued). Since $\operatorname{graph}(\mathcal{T}) = \{(t,x) \in G' : H(t,x) \neq \emptyset\}$, we deduce that $\operatorname{graph}(\mathcal{T}) \in \mathcal{B}(G')$. Now, by remarking that $\mathcal{B}(G') \subset \mathcal{L}(J) \otimes \mathcal{B}(X)$, we see that our claim is proved.

To finish the proof of the lemma, we will prove that $\mathcal{T}(t) = C(t)$ a.e. on J. Indeed, suppose by contradiction that the contrary holds. Then, there exists a set $A \in \mathcal{L}(J)$ with $\lambda(A) > 0$ such that for every $t \in A$, $D(t) := C(t) \setminus \mathcal{T}(t) \neq \emptyset$. Since the graph of the multifunction $D(\,\cdot\,)$ belongs to $\mathcal{L}(J) \otimes \mathcal{B}(X)$, selection theorem of Aumann ([15, Theorem III.22]) implies that there exists a measurable function

$\sigma_0: A \to X$ such that $\sigma_0(t) \in D(t)$ for all $t \in A$. It is clear that we can extend the function σ_0 to all the set I in such a way that $\sigma_0 \in L^0_C$. By assumption (iv) of the Theorem 3.1, there exists a measurable function $v_0: I \to X$ such that $(1, v_0(t)) \in \Lambda_{\sigma_0}(t)$ a.e. on I. In particular, $v_0(t) \in F(t, \sigma_0(t))$ a.e. So, by (B3), we have also $v_0(t) \in F_0(t, \sigma_0(t))$ a.e. on I. Hence,

$$(1, v_0(t)) \in (\{1\} \times F_0(t, \sigma_0(t))) \cap T_G(t, \sigma_0(t)) \quad \text{a.e. on } I.$$

This contradicts the fact that $\sigma_0(t) \in C(t) \setminus \mathcal{T}(t)$, for all $t \in A$. So $\mathcal{T}(t) = C(t)$ a.e. on J and the lemma is proved. □

Let us choose, by Lemma 2.6, an integrable and lower semicontinuous function $\overline{c}: I \to]0, \infty]$ such that $c(t) < \overline{c}(t)$ for every $t \in I$ and $\overline{c}(t) = \infty$, for every $t \in N$, where N is the negligible set given by Lemma 3.2. Let $m := (1 + \|x_0\| + a) \exp(\int_0^a \overline{c}(s)\, ds)$ and consider the integrable function g on I defined by $g(t) := m\overline{c}(t) + 1$, $t \in I$. For each $\tau \in I$ and each function $u \in L^1_X([0, \tau])$, we shall associate the absolutely continuous function $\widetilde{u}: [0, \tau] \to X$ defined by

$$\widetilde{u}(t) := x_0 + \int_0^t u(s)\, ds, \quad t \in [0, \tau].$$

Let us prove now the following lemma.

LEMMA 3.3. *For every $\varepsilon \in]0, 1]$, there exist a compact set $J_\varepsilon \subset I \setminus N$ with $\lambda(I \setminus J_\varepsilon) < \varepsilon$, and a function $w_\varepsilon \in L^1_X(I)$ such that*

(a) *The compactness condition* (v) *of Theorem* 3.1 *is satisfied by J_ε and with $G_\varepsilon = (J_\varepsilon \times X) \cap G$, the restriction $F_{0|G^\ell_\varepsilon}$ of the multifunction F_0 to G^ℓ_ε is u.s.c.*
(b) *$\chi_{I \setminus J_\varepsilon} w_\varepsilon = 0$ and $\|w_\varepsilon(t)\| \le g(t)$ λ-a.e. on I.*
(c) *$(t, \widetilde{w_\varepsilon}(t)) \in G$ for all $t \in I$ and $w_\varepsilon(t) \in F_0(t, \widetilde{w_\varepsilon}(t))$ λ-a.e. on J_ε.*

PROOF. Let $\varepsilon \in]0, 1]$ and choose a compact set $J_\varepsilon \subset I \setminus N$ satisfying the compactness assumption (v) of Theorem 3.1 and the Scorza–Dragoni condition (B4). Let us introduce some notations and definitions that we shall need bellow. For $\eta \in]0, 1]$ and $\tau \in I$, we denote by $\mathcal{J}^-_\eta([0, \tau])$ the set of all nondecreasing right continuous functions $\theta: [0, \tau] \to [0, \tau]$ such that $\theta(0) = 0$, $\theta(\tau) = \tau$ and $\theta(t) \in [t - \eta, t]$, $\theta(\theta(t)) = \theta(t)$ for all $t \in [0, \tau]$. We denote by $\mathcal{P}_{\varepsilon,\eta}([0, \tau])$ the set of all pairs (θ, u) of functions $\theta \in \mathcal{J}^-_\eta([0, \tau])$ and $u \in L^1_X([0, \tau])$ satisfying the following five conditions:

(C1) $(\theta(t), \widetilde{u}(t)) \in G$ for all $t \in [0, \tau]$.
(C2) $u(t) \in F_0(\theta(t), \widetilde{u}(\theta(t))) + \eta B_X$ for a.e. $t \in [0, \tau] \cap J_\varepsilon$.
(C3) $u(t) = 0$ on $[0, \tau] \setminus J_\varepsilon$.
(C4) For all t if $t \in [0, \tau[\cap J_\varepsilon$ then $\theta(t) \in J_\varepsilon$.
(C5) $\|u(t)\| \le \overline{c}(t)(1 + \|\widetilde{u}(\theta(t))\|) + 1$ for all $t \in [0, \tau]$.

We will consider, in what follows, the set $\mathcal{P}_{\varepsilon,\eta} := \bigcup\{\mathcal{P}_{\varepsilon,\eta}([0,\tau]) : \tau \in I\}$. Note that $\mathcal{P}_{\varepsilon,\eta}$ is nonempty. Indeed, at least the set $\mathcal{P}_{\varepsilon,\eta}([0,0])$ is nonempty since it contains the pair (θ_0, u_0) where θ_0 and u_0 are the identically null functions $0 \mapsto 0$.

Let us begin by stating the following property of the elements of $\mathcal{P}_{\varepsilon,\eta}$:

(C6) For every $\tau \in I$ and every $(\theta, u) \in \mathcal{P}_{\varepsilon,\eta}([0,\tau])$, we have $\|u(t)\| \le g(t)$, for all $t \in [0,\tau]$.

Indeed, let us set $\rho(t) := 1 + \|\widetilde{u}(\theta(t))\|$ for $t \in [0,\tau]$. We have

$$0 \le \rho(t) \le 1 + \|x_0\| + \int_0^t \|u(s)\|\,ds, \quad t \in [0,\tau]. \tag{3.1}$$

It follows that $\rho \in L_X^\infty([0,\tau])$. On the other hand, by (C5), condition (3.1) implies

$$\begin{aligned}\rho(t) &\le 1 + \|x_0\| + \int_0^t [\overline{c}(s)(1 + \|\widetilde{u}(\theta(s))\|) + 1]\,ds \\ &\le 1 + \|x_0\| + a + \int_0^t \overline{c}(s)\rho(s)\,ds.\end{aligned}$$

Using the Gronwall lemma ([5]), we deduce that

$$\rho(t) \le (1 + \|x_0\| + a)\exp\bigg(\int_0^a \overline{c}(s)\,ds\bigg) = m, \quad \text{for all } t \in [0,\tau].$$

Hence, again by (C5), we obtain that $\|u(t)\| \le m\overline{c}(t) + 1 = g(t)$, for all $t \in [0,\tau]$.

Now, we introduce a preorder in $\mathcal{P}_{\varepsilon,\eta}$ by setting $(\theta_1, u_1) \preceq (\theta_2, u_2)$ (with $(\theta_i, u_i) \in \mathcal{P}_{\varepsilon,\eta}([0,\tau_i])$ for $i = 1, 2$) if and ond only if $\tau_1 \le \tau_2$, $\theta_{2|[0,\tau_1]} = \theta_1$ and $u_{2|[0,\tau_1]} = u_1$ a.e.

Let us prove that the set $\mathcal{P}_{\varepsilon,\eta}$ satisfies the conditions of Zorn lemma for the preorder $\preceq$ (we refer to [24, Chapter I], for related notions). Indeed, let $\mathcal{C} = \{(\theta_\alpha, u_\alpha) : \alpha \in A\}$ be a totally ordered subset of $\mathcal{P}_{\varepsilon,\eta}$ with $(\theta_\alpha, u_\alpha) \in \mathcal{P}_{\varepsilon,\eta}([0,\tau_\alpha])$, for all $\alpha \in A$. Let us set $\tau := \sup_{\alpha\in A} \tau_\alpha \in I$. If there exists $\alpha_0 \in A$ such that $\tau_{\alpha_0} = \tau$, then we have $(\theta_\alpha, u_\alpha) \preceq (\theta_{\alpha_0}, u_{\alpha_0})$ for every $\alpha \in A$. Suppose now that $\tau_\alpha < \tau$ for every $\alpha \in A$. Then there exists a sequence $(\alpha_n)_{n\ge1} \subset A$ such that $\tau_{\alpha_n} < \tau_{\alpha_{n+1}}$, for all n and $\tau = \sup_{n\ge1} \tau_{\alpha_n}$. Let us define a function $\theta\colon [0,\tau] \to [0,\tau]$ by setting $\theta_{|[0,\tau_\alpha]} = \theta_\alpha$ for every $\alpha \in A$ and $\theta(\tau) := \tau$. It is clear that we have $\theta \in \mathcal{J}_\eta^-([0,\tau])$. Now, we have to define a suitable function u: $[0,\tau] \to X$. By hypothesis, for each $n \ge 1$ there is a negligible set $L_n \subset [0,\tau_{\alpha_n}]$ such that $u_{\alpha_{n+1}}(t) = u_{\alpha_n}(t)$, for all $t \in [0,\tau_{\alpha_n}] \setminus L_n$. Let us set $L := \bigcup_{n\ge1} L_n$. We define the function u on $[0,\tau]$ by setting $u(t) := u_{\alpha_n}(t)$ if $t \in [0,\tau_{\alpha_n}] \setminus L$ with $n \ge 1$, and $u(t) := 0$ if $t \in L \cup \{\tau\}$. Then u is a measurable function from $[0,\tau]$ to X such that $u_{|[0,\tau_{\alpha_n}]} = u_{\alpha_n}$ a.e. for every n. On the other hand, condition (C6), applied to each u_{α_n}, implies that $\|u(t)\| \le g(t)$ a.e. on $[0,\tau]$.

So, we can state that $u \in L^1_X([0,\tau])$. Moreover, one can easily check that all conditions (C1)–(C5) are satisfied by the pair (θ, u). Hence $(\theta, u) \in \mathcal{P}_{\varepsilon,\eta}([0,\tau])$ and by construction (θ, u) is an upper bound for the set $\mathcal{C}$.

By Zorn lemma, the set $\mathcal{P}_{\varepsilon,\eta}$ admits a maximal element $(\theta^*, u^*) \in \mathcal{P}_{\varepsilon,\eta}([0,\tau^*])$. Let us prove that $\tau^* = a$. Suppose by contradiction that the contrary holds. Then, we will construct a positive number $h^* > 0$ with $\tau^* + h^* \in I$, a vector $x^* \in C(\tau^* + h^*)$ and a vector $y^* \in X$ such that the following four conditions are satisfied:

(C7) $\|(x^* - \widetilde{u^*}(\tau^*))/h^* - y^*\| < \eta$.
(C8) $\|y^*\| \le \overline{c}(t)(1 + \|\widetilde{u^*}(\tau^*)\|)$ for every $t \in [\tau^*, \tau^* + h^*]$.
(C9) If $\tau^* \in I \setminus N$ then $y^* \in F_0(\tau^*, \widetilde{u^*}(\tau^*))$.
(C10) If $\tau^* \in I \setminus J_\varepsilon$ then $J_\varepsilon \cap [\tau^*, \tau^* + h^*] = \emptyset$.

Indeed, let us distinguish two cases:

(a) If $\tau^* \in I \setminus N$ then by Lemma 3.2, there exists a vector $y^* \in F_0(\tau^*, \widetilde{u^*}(\tau^*))$ such that $(1, y^*) \in T_G(\tau^*, \widetilde{u^*}(\tau^*))$. Hence, there exists a sequence $h_n \to 0^+$ with $\tau^* + h_n \in I$ for every n, and a sequence of vectors $x_n \in C(\tau^* + h_n)$ such that $\lim_{n\to\infty} \|1/h_n(x_n - \widetilde{u^*}(\tau^*)) - y^*\| = 0$. We have

$$\|y^*\| \le |F_0(\tau^*, \widetilde{u^*}(\tau^*))| \le c(\tau^*)(1 + \|\widetilde{u^*}(\tau^*)\|) < \overline{c}(\tau^*)(1 + \|\widetilde{u^*}(\tau^*)\|).$$

Since function $\overline{c}$ is lower semicontinuous, there exists an integer n_0 such that

$$\|y^*\| < \overline{c}(t)(1 + \|\widetilde{u^*}(\tau^*)\|) \quad \text{for all } t \in [\tau^*, \tau^* + h_{n_0}]. \tag{3.2}$$

We may also suppose that n_0 satisfies $\|1/h_{n_0}(x_{n_0} - \widetilde{u^*}(\tau^*)) - y^*\| < \eta$.

(b) If $\tau^* \in N$, by virtue of condition (iii) applied to $(\tau^*, \widetilde{u^*}(\tau^*))$, there exists a constant $M > 0$, a sequence $h_n \to 0^+$ with $\tau^* + h_n \in I$, for all n and a sequence $x_n \in C(\tau^* + h_n)$ such that $\|1/h_n(x_n - \widetilde{u^*}(\tau^*))\| \le M$ for all n. We have $M < \overline{c}(\tau^*)(1 + \|\widetilde{u^*}(\tau^*)\|) = \infty$ and the function $\overline{c}$ is lower semicontinuous. Hence there exists an integer n_0 such that

$$M < \overline{c}(t)(1 + \|\widetilde{u^*}(\tau^*)\|) \quad \text{for all } t \in [\tau^*, \tau^* + h_{n_0}].$$

We suppose also that the integer n_0 is such that $J_\varepsilon \cap [\tau^*, \tau^* + h_{n_0}] = \emptyset$ if $\tau^* \notin J_\varepsilon$. In this case we set $y^* := 1/h_{n_0}(x_{n_0} - \widetilde{u^*}(\tau^*))$.

Now with the integer n_0 and the vectors x_{n_0} and y^* constructed as in the cases (a) or (b), we see that the conditions (C7)–(C10) are satisfied by $h^* := h_{n_0}$, $x^* := x_{n_0}$ and y^*.

Let us put $\sigma := \tau^* + h^* \in I$ and define two functions ϑ and v on $[0,\sigma]$ as follows. We set $\vartheta(t) := \theta^*(t)$ if $t \in [0, \tau^*[$, $\vartheta(t) := \tau^*$ if $t \in [\tau^*, \sigma[$ and $\vartheta(\sigma) := \sigma$. We define the function v by $v(t) := u^*(t)$ if $t \in [0, \tau^*[$, $v(t) := 1/h^*(x^* - \widetilde{u^*}(\tau^*))$ if $t \in [\tau^*, \sigma] \cap J_\varepsilon$ and $v(t) := 0$ if $t \in [\tau^*, \sigma] \setminus J_\varepsilon$. Then, it is clear that $\vartheta \in \mathcal{J}^-_\eta([0,\sigma])$

and $v \in L^1_X([0,\sigma])$. Moreover, by construction of ϑ and v and conditions (C7)–(C10) above, it is easy to check that $(\vartheta, v) \in \mathcal{P}_{\varepsilon,\eta}([0,\sigma])$. Now, we remark that $(\theta^*, u^*) \preceq (\vartheta, v)$ and $(\vartheta, v) \not\preceq (\theta^*, u^*)$. So we get a contradiction with the maximal assumption on (θ^*, u^*). Hence, we have proved the following property

(C11) For every $\eta \in \,]0,1]$, the set $\mathcal{P}_{\varepsilon,\eta}([0,a])$ is nonempty.

Let $(\eta_n)_{n\ge1}$ be a sequence in $]0,1]$ such that $\eta_n \downarrow 0$. By (C11), for every $n \ge 1$ there exists $\theta_n \in \mathcal{J}^-_{\eta_n}([0,a])$ and $u_n \in L^1_X(I)$ such that with $v_n := \widetilde{u_n}$, we have:

(C12) For every $t \in I$, $(\theta_n(t), v_n(\theta_n(t))) \in G$.
(C13) For almost all $t \in J_\varepsilon$, $u_n(t) \in F_0(\theta_n(t), v_n(\theta_n(t))) + \eta_n B_X$.
(C14) For every $t \in I \setminus J_\varepsilon$, $u_n(t) = 0$.
(C15) For every $t \in J_\varepsilon$, $\theta_n(t) \in J_\varepsilon$.
(C16) For every $t \in I$, $\|u_n(t)\| \le g(t)$.

By virtue of (C16), the set of continuous functions $\mathcal{F} := \{v_n : n \ge 1\}$ is bounded and equicontinuous in $\mathcal{C}_X(I)$. For $t \in I$, let us put $A(t) := \{v_n(t) : n \ge 1\}$ and $r(t) := \alpha_X(A(t))$. In virtue of Ascoli theorem, to prove that $\mathcal{F}$ is relatively compact in $\mathcal{C}_X(I)$ it is sufficient to prove that $r(t) = 0$ for every $t \in I$. For $t_1 \le t_2$ in I, we have

$$A(t_2) \subset A(t_1) + \{v_n(t_2) - v_n(t_1) : n \ge 1\} \subset A(t_1) + \left(\int_{t_1}^{t_2} g(s)\,ds\right) B_X.$$

So $\alpha_X(A(t_2)) \le \alpha_X(A(t_1)) + 2\int_{t_1}^{t_2} g(s)\,ds$. By symmetry, we deduce that

$$|r(t_2) - r(t_1)| \le 2\int_{t_1}^{t_2} g(s)\,ds \quad \text{for } 0 \le t_1 \le t_2 \le a.$$

It follows that function r is absolutely continuous on I. Let $\dot r := dr/d\lambda \in L^1_{\mathbb{R}}(I)$ be the derivative of r with respect to λ. Let us choose N' a λ-negligible set of I such that the condition (C13) is satisfied *everywhere* on $J_\varepsilon \setminus N'$ for all n and that:

$$\dot r(t) = \lim_{\gamma\to0^+} \frac{1}{\gamma}[r(t) - r(t-\gamma)] \qquad \text{for every } t \in I \setminus N', \tag{3.3}$$

$$\lim_{\gamma\to0^+} \frac{1}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J_\varepsilon)} g(s)\,ds = 0 \quad \text{for every } t \in J_\varepsilon \setminus N'. \tag{3.4}$$

We shall prove that

(C17) For every $t \in I \setminus N'$ with $t \ne 0$, we have $\dot r(t) \le \omega(t, r(t))$.

Indeed, let $t \in I \setminus N'$ with $t \ne 0$. For $\gamma > 0$ so small that $[t-\gamma, t] \subset I$, we put

$$U^\gamma_n(t) := \left\{\frac{1}{\gamma}(v_k(t) - v_k(t-\gamma)) : k \ge n\right\}, \quad n \in \mathbb{N}^*.$$

Then, we have $A(t) \subset A(t-\gamma) + \gamma . U_1^\gamma(t)$ and for every $n_0 \in \mathbb{N}^*$,

$$\alpha_X(A(t)) \le \alpha_X(A(t-\gamma)) + \gamma \alpha_X(U_1^\gamma(t)) = \alpha_X(A(t-\gamma)) + \gamma \alpha_X(U_{n_0}^\gamma(t)).$$

Hence $r(t) \le r(t-\gamma) + \gamma \alpha_X(U_{n_0}^\gamma(t))$ and so

$$\frac{1}{\gamma}[r(t) - r(t-\gamma)] \le \alpha_X(U_{n_0}^\gamma(t)).$$

From (3.3), given $\delta > 0$, there exists $\gamma' = \gamma(\delta) > 0$ such that for every $\gamma \in\,]0, \gamma']$ we have

$$\dot{r}(t) \le \frac{1}{\gamma}[r(t) - r(t-\gamma)] + \delta.$$

It follows that

$$\dot{r}(t) \le \alpha_X(U_{n_0}^\gamma(t)) + \delta \tag{3.5}$$

for all $n_0 \in \mathbb{N}^*$ and all $\gamma \in\,]0, \gamma']$. Let us suppose first that $t \notin J_\varepsilon$. Then, we may assume that $[t-\gamma', t] \subset I \setminus J_\varepsilon$. So, by (C14), we have

$$\frac{1}{\gamma}(v_k(t) - v_k(t-\gamma)) = \frac{1}{\gamma}\int_{[t-\gamma,t]} u_k(s)\,ds = 0 \quad \text{for all } k \in \mathbb{N}^*.$$

Hence $U_{n_0}^\gamma(t) = \{0\}$ and $\dot{r}(t) \le \delta$ by (3.5). Since δ is arbitrary, we get $\dot{r}(t) \le 0 \le \omega(t, r(t))$ and (C17) is proved in this case.

Suppose now that $t \in J_\varepsilon$. Let $h > 0$ be too small such that $[t-h, t] \subset I$, and let us consider the bounded set

$$B_{t,h} := \bigcup_{s \in [t-h,t]} A(s).$$

By virtue of compactness condition (v) (applied to t and $B_{t,h}$) and (3.4), there exists $\gamma'' \in\,]0, \inf(\gamma', h/2)]$ such that for every $\gamma \in\,]0, \gamma'']$ we have

$$\alpha_X[F((([t-2\gamma, t] \cap J_\varepsilon) \times B_{t,h}) \cap G)] \le \omega(t, \alpha_X(B_{t,h})) + \delta, \tag{3.6}$$

$$\frac{1}{\gamma}\int_{[t-\gamma,t]\cap(I\setminus J_\varepsilon)} g(s)\,ds \le \delta. \tag{3.7}$$

Let us take $\gamma \in\,]0, \gamma'']$ and choose $n_0 \ge 1$ (n_0 depending on γ) such that $\eta_n \le \gamma$ for all $n \ge n_0$. Then, by the mean value theorem, we have the following inclusion

$$U_{n_0}^\gamma(t) \subset \Big(\bigcup_{n \ge n_0} \overline{\mathrm{co}}[\{0\} \cup u_n([t-\gamma, t] \cap (J_\varepsilon \setminus N'))]\Big) + \Big\{\frac{1}{\gamma}\int_{[t-\gamma,t]\cap(I\setminus J_\varepsilon)} u_n(s)\,ds : n \ge n_0\Big\}.$$

It follows by (C16) and (3.7) that

$$U_{n_0}^\gamma(t) \subset \Big(\bigcup_{n \ge n_0} \overline{\mathrm{co}}[\{0\} \cup u_n([t-\gamma, t] \cap (J_\varepsilon \setminus N'))]\Big) + \delta B_X.$$

By condition (C13), for every $n \geq n_0$, we have

$$u_n([t-\gamma, t] \cap (J_\varepsilon \setminus N')) \subset \bigcup_{s \in [t-\gamma, t] \cap J_\varepsilon} F_0(\theta_n(s), v_n(\theta_n(s))) + \eta_n B_X.$$

Now, recall condition (C15) and note that for every $n \geq n_0$ and every $s \in [t-\gamma, t] \cap J_\varepsilon$, we have $\eta_n \leq \gamma$ and $t - 2\gamma \leq t - \gamma - \varepsilon_n \leq \theta_n(t-\gamma) \leq \theta_n(s) \leq \theta_n(t)$, so $\theta_n(s) \in [t-2\gamma, t] \cap J_\varepsilon \subset [t-h, t] \cap J_\varepsilon$ and $v_n(\theta_n(s)) \in B_{t,h}$. Hence for all $n \geq n_0$,

$$u_n([t-\gamma, t] \cap (J_\varepsilon \setminus N')) \subset F_0((([t-2\gamma, t] \cap J_\varepsilon) \times B_{t,h}) \cap G) + \gamma B_X.$$

It follows that

$$U_{n_0}^\gamma(t) \subset (\overline{\mathrm{co}}\{\{0\} \cup [F_0((([t-2\gamma, t] \cap J_\varepsilon) \times B_{t,h}) \cap G) + \gamma B_X]\}) + \delta B_X.$$

From the properties of the measure of noncompactness α_X and (3.6), we deduce that

$$\begin{aligned} \alpha_X(U_{n_0}^\gamma(t)) &\leq \alpha_X(F_0((([t-2\gamma, t] \cap J_\varepsilon) \times B_{t,h}) \cap G)) + 2\gamma + 2\delta \\ &\leq \omega(t, \alpha_X(B_{t,h})) + h + 3\delta. \end{aligned}$$

It follows, by (3.5), that

$$\dot{r}(t) \leq \omega(t, \alpha_X(B_{t,h})) + h + 4\delta. \tag{3.8}$$

Now it is easy to check that

$$A(t) \subset B_{t,h} \subset A(t) + \left(\int_{t-h}^{t} g(s)\, ds\right) B_X.$$

Applying the measure of noncompactness α_X, we get

$$r(t) \leq \alpha_X(B_{t,h}) \leq r(t) + 2\int_{t-h}^{t} g(s)\, ds$$

and hence $\lim_{h \to 0^+} \alpha_X(B_{t,h}) = r(t)$. Taking $h \to 0^+$ and $\delta \to 0^+$ in the inequality (3.8), we get $\dot{r}(t) \leq \omega(t, r(t))$. Hence condition (C17) is proved.

Now since $r(0) = 0$ and ω is a Kamke function, we deduce from (C17) that $r(t) = 0$ for all $t \in I$. This completes the proof of the relative compactness of the set $\mathcal{F} = \{v_n : n \geq 1\}$ in $\mathcal{C}_X(I)$.

Without loss of generality, we can suppose that there exists $v \in \mathcal{C}_X(I)$ such that $\|v_n - v\|_\infty \to 0$ as $n \to \infty$. Now we remark from (C16) that

$$\|v_n(t) - v_n(\theta_n(t))\| \leq \int_{\theta_n(t)}^{t} g(s)\, ds \overset{n\to\infty}{\longrightarrow} 0$$

for every $t \in I$. It follows that $v_n(\theta_n(t)) \to v(t)$ as $n \to \infty$ in norm for each $t \in I$. Since G is closed in $I_\ell \times X$, condition (C12) implies that $(t, v(t)) \in G$ for all $t \in I$. Consider the set $\mathcal{M}(t) := \{(\theta_n(t), v_n(\theta_n(t))) : n \geq 1\} \cup \{(t, v(t))\}$ which is compact in $I_\ell \times X$. By condition (C15) for each $t \in J_\varepsilon$, $\mathcal{M}(t) \subset G_\varepsilon = (J_\varepsilon \times X) \cap G$; hence $\mathcal{M}(t)$ is also a compact set in G_ε^ℓ. Since the multifunction $F_{0|G_\varepsilon^\ell}$ is u.s.c., it follows that for each $t \in J_\varepsilon$ the set $F_0(\mathcal{M}(t))$ is compact in X. Let us put

$$\mathcal{H}(t) := \begin{cases} F_0(\mathcal{M}(t)) & \text{if } t \in J_\varepsilon, \\ \{0\} & \text{if } t \in I \setminus J_\varepsilon. \end{cases}$$

Then $\mathcal{H}$ is a measurable multifunction from I to $\mathcal{K}(X)$. By virtue of conditions (C13) and (C14), we have

$$u_n(t) \in \mathcal{H}(t) + \eta_n B_X \quad \lambda\text{-a.e. on } I.$$

Furthermore, by (C16), the sequence (u_n) is uniformly integrable in $L^1_X(I)$. It follows, by standard arguments (see [1]), that $(u_n)_n$ is relatively weakly compact in $L^1_X(I)$. By Eberlein–Smulian theorem we may suppose without loss of generality that there exists $w \in L^1_X(I)$ such that the sequence $(u_n)_n$ converges weakly in $L^1_X(I)$ to w. It is clear that then w satisfies also $(\chi_{I \setminus J_\varepsilon} w)(t) = 0$ and $\|w(t)\| \le g(t)$ a.e. on I. On the other hand, by condition (C13), for each $n \ge 1$ there exists a measurable function $w_n\colon J_\varepsilon \to X$ such that

$$\text{(3.9)} \quad \|u_n(t) - w_n(t)\| \le \eta_n \quad \text{and} \quad w_n(t) \in F_0(\theta_n(t), v_n(\theta_n(t))) \quad \lambda\text{-a.e. on } J_\varepsilon.$$

Let us extend w_n to all the set I by setting $w_n(t) = 0$ on $I \setminus J_\varepsilon$. Then we get $w_n \in L^1_X(I)$ and $w_n \to w$ weakly in $L^1_X(I)$. By Mazur lemma, we have

$$w(t) \in \bigcap_{n \ge 1} \overline{\text{co}}\{w_k(t) : k \ge n\} \quad \lambda\text{-a.e on } I. \tag{3.10}$$

By (C15), $(\theta_n(t), v_n(\theta_n(t))) \in G_\varepsilon$, for all $t \in J_\varepsilon$, and $F_{0|G_\varepsilon^\ell}$ is u.s.c. hence

$$\bigcap_{n \ge 1} \overline{\text{co}} \bigcup_{k \ge n} F_0(\theta_k(t), v_k(\theta_k(t))) \subset F_0(t, v(t)), \quad \text{for all } t \in J_\varepsilon. \tag{3.11}$$

From (3.9)–(3.11), we deduce finally that

$$w(t) \in F_0(t, v(t)) \quad \lambda\text{-a.e. on } J_\varepsilon. \tag{3.12}$$

Remark now that for each $t \in I$, the sequence $v_n(t) = x_0 + \int_0^t u_n(s)\,ds$ converges weakly in X to $\widetilde{w}(t) := x_0 + \int_0^t w(s)\,ds$. We deduce that $v(t) = \widetilde{w}(t)$ for all $t \in I$. Putting $w_\varepsilon := w$, we see now that Lemma 3.3 is completely proved. □

END OF PROOF OF THEOREM 3.1. Let us now finish the proof of Theorem 3.1. Let $(\varepsilon_n)_{n \ge 1}$ be a sequence in $]0,1]$ such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. For every $n \ge 1$, let $J_n := J_{\varepsilon_n}$ and $w_n := w_{\varepsilon_n}$ be given as in Lemma 3.3 corresponding to $\varepsilon = \varepsilon_n$. Let us put $v_n := \widetilde{w_n}$ for $n \ge 1$. From the condition (b) of Lemma 3.3, it follows that the sequence of continuous functions $(v_n)_n$ is bounded and equicontinuous in $\mathcal{C}_X(I)$. Let us prove that $(v_n)_n$ is relatively compact in $\mathcal{C}_X(I)$. By Ascoli theorem it suffice to prove that for every $t \in I$ the set $V(t) := \{v_n(t) : n \geq 1\}$ is relatively compact in X.

Let us set $\varrho(t) := \alpha_X(V(t))$ for $t \in I$. As in the proof of Lemma 3.3, we can easily check that the non-negative function ϱ is absolutely continuous on I. Let then $\dot{\rho} := d\rho/d\lambda$ be its derivative with respect to λ. For every $n \geq 1$, let us choose a λ-negligible subset N_n of I such that

(D1) For every $t \in J_n \setminus N_n$, $u_n(t) \in F_0(t, v_n(t))$.
(D2) For every $t \in I \setminus N_n$, $\dot{\varrho}(t) = \lim_{\gamma \to 0^+} [\varrho(t) - \varrho(t-\gamma)]/\gamma$.

Let us set $N_* := \bigcup_n N_n$ and $J := \liminf_n J_n$. Since $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, we have $\lambda(I \setminus J) = 0$. Let us prove that

(D3) For every $t \in J \setminus N_*$, with $t \neq 0$, we have $\dot{\varrho}(t) \leq \omega(t, \varrho(t))$.

By the properties of the Kamke function ω, this will implies that $\varrho(t) = 0$ for all $t \in I$ (notice that $\varrho(0) = 0$) and hence that the sets $V(t)$ are relatively compact for every $t \in I$. Let then $t \in J \setminus N_*$, with $t \neq 0$, be fixed. Notice that $J = \bigcup_n \uparrow J_n'$, where $J_n' := \bigcap_{k \geq n} J_k$ for $n \geq 1$. Hence, there exists an integer n_t (depending on t) such that $t \in J_{n_t}'$.

Let $h > 0$ be too small such that $[t-h, t] \subset I$ and consider the bounded set

$$D_{t,h} := \bigcup_{s \in [t-h,t]} V(s).$$

By condition (a) of Lemma 3.3 (applied to $\varepsilon = \varepsilon_n$ with $n = n_t$) and condition (D2), given $\varepsilon > 0$, there exists $\gamma_\varepsilon^t \in]0, h]$ such that for every $\gamma \in]0, \gamma_\varepsilon^t]$, we have

$$\alpha_X[F((([t-\gamma, t] \cap J_{n_t}) \times D_{t,h}) \cap G)] \leq \omega(t, \alpha_X(D_{t,h})) + \varepsilon, \tag{3.13}$$

$$\dot{\varrho}(t) \leq \frac{1}{\gamma}[\varrho(t) - \varrho(t-\gamma)] + \varepsilon. \tag{3.14}$$

Let us take $\gamma \in]0, \gamma_\varepsilon^t]$ and put for $n \geq 1$,

$$W_n^\gamma(t) := \left\{ \frac{1}{\gamma}(v_k(t) - v_k(t-\gamma)) : k \geq n \right\}.$$

As in the proof of Lemma 3.3, we can easily check that for every $n \geq 1$,

$$\dot{\varrho}(t) \leq \alpha_X(W_n^\gamma(t)) + \varepsilon. \tag{3.15}$$

Moreover, we have $W_n^\gamma(t) \subset A_n^\gamma(t) + B_n^\gamma(t)$, where

$$A_n^\gamma(t) := \left\{ \frac{1}{\gamma} \int_{[t-\gamma,t] \cap (J_n' \setminus N_*)} u_k(s)\, ds : k \geq n \right\}$$

and

$$B_n^\gamma(t) := \left\{ \frac{1}{\gamma} \int_{[t-\gamma,t] \cap (I \setminus J_n')} u_k(s)\, ds : k \geq n \right\}.$$

Hence

$$\alpha_X(W_n^\gamma(t)) \le \alpha_X(A_n^\gamma(t)) + \alpha_X(B_n^\gamma(t)) = \alpha_X(A_{n_t}^\gamma(t)) + \alpha_X(B_n^\gamma(t)). \tag{3.16}$$

By the mean value theorem, we have

$$A_{n_t}^\gamma(t) \subset \bigcup_{k\ge n_t} \overline{\mathrm{co}}[\{0\} \cup u_k([t-\gamma,t] \cap (J'_{n_t} \setminus N_*))].$$

By (D1) (notice that $J'_{n_t} \setminus N_* \subset J_k \setminus N_k$ for $k \ge n_t$), we have

$$u_k([t-\gamma,t] \cap (J'_{n_t} \setminus N_*)) \subset F_0[(([t-\gamma,t] \cap J'_{n_t}) \times D_{t,h}) \cap G],$$

for all $k \ge n_t$. Hence

$$A_{n_t}^\gamma(t) \subset \overline{\mathrm{co}}[\{0\} \cup F_0[(([t-\gamma,t] \cap J'_{n_t}) \times D_{t,h}) \cap G]].$$

It follows that

$$\begin{aligned} \alpha_X(A_{n_t}^\gamma(t)) &\le \alpha_X(F_0[(([t-\gamma,t] \cap J'_{n_t}) \times D_{t,h}) \cap G]) \\ &\le \omega(t, \alpha_X(D_{t,h})) + \varepsilon \end{aligned} \tag{3.17}$$

(the last inequality follows from (3.13)). On the other hand, for every n and $k \ge 1$, we have

$$\left\| \frac{1}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J'_n)} u_k(s)\,ds \right\| \le \frac{1}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J'_n)} g(s)\,ds.$$

Hence

$$\alpha_X(B_n^\gamma(t)) \le \frac{2}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J'_n)} g(s)\,ds. \tag{3.18}$$

We deduce now from (3.16)–(3.18) that

$$\alpha_X(W_n^\gamma(t)) \le \omega(t, \alpha_X(D_{t,h})) + \frac{2}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J'_n)} g(s)\,ds + \varepsilon,$$

and hence by (3.15), that

$$\dot{\varrho}(t) \le \omega(t, \alpha_X(D_{t,h})) + \frac{2}{\gamma} \int_{[t-\gamma,t]\cap(I\setminus J'_n)} g(s)\,ds + 2\varepsilon, \tag{3.19}$$

for every $n \ge 1$ and $\gamma \in \,]0, \gamma_\varepsilon^t]$. Passing to the limit in the inequality (3.19) as $n \to \infty$, with h, ε and γ fixed, we get

$$\dot{\varrho}(t) \le \omega(t, \alpha_X(D_{t,h})) + 2\varepsilon. \tag{3.20}$$

Moreover, $\lim_{h\to 0^+} \alpha_X(D_{t,h}) = \varrho(t)$ (see the proof of Lemma 3.3 for $B_{t,h}$). Hence passing to the limit in the inequality (3.20) as $h \to 0^+$ and $\varepsilon \to 0^+$, we obtain $\dot{\varrho}(t) \le \omega(t, \varrho(t))$. This finish the proof of assertion (D3).

Applying Ascoli theorem, we can suppose (by passing to a subsequence) that there exists a function $v \in \mathcal{C}_X(I)$ such that $\|v_n - v\|_\infty \to 0$ as $n \to \infty$. As the multifunction C is closed valued, we conclude from condition (c) of Lemma 3.3 that $(t, v(t)) \in G$ for all $t \in I$. Let us put

$$K(t) := \{v_n(t) : n \geq 1\} \cup \{v(t)\} \quad \text{and} \quad \Phi(t) := F_0(t, K(t)), \quad t \in I.$$

Remark that for almost every t in I, the multifunction $F_0(t, \cdot\,)$ is compact valued and u.s.c. on $C(t)$. It follows that for almost every t in I, the set $\Phi(t)$ is compact in X. Moreover, the multifunction $\Phi: I \rightrightarrows X$ is measurable. By conditions (b) and (c) of Lemma 3.3, we have for all n, $\chi_{I \setminus J_n} w_n = 0$ and $w_n(t) \in \Phi(t)$ λ-a.e. on J_n. Hence

$$w_n(t) \in \Phi(t) \cup \{0\} \quad \lambda\text{-a.e. on } I. \tag{3.21}$$

As on the other hand, $\|w_n(t)\| \leq g(t)$ a.e. we conclude that the sequence $(w_n)_n$ is relatively weakly compact in $L^1_X(I)$. We may suppose (by passing to a subsequence) that there exists $u \in L^1_X(I)$ such that $(w_n)_n$ converges weakly to u in $L^1_X(I)$. By standard arguments, it can be easily shown that $v(t) = x_0 + \int_0^t u(s)\, ds$ for $t \in I$.

It remains to prove that

$$u(t) \in F_0(t, v(t)) \quad \lambda\text{-a.e. on } I. \tag{3.22}$$

By virtue of the inclusion (3.21), a classical result ([1]) implies that

$$u(t) \in \overline{\text{co}}\, \text{Ls}_n\{w_n(t)\} \quad \lambda\text{-a.e. on } I. \tag{3.23}$$

Let us choose N^* a λ-negligible set of I such that $N_* \subset N^*$ and the condition (3.23) is satisfied everywhere on $I \setminus N^*$. Let us take $t \in J \setminus N^*$. Then there exists an integer $n_t \geq 1$ such that for every $n \geq n_t$, $t \in J_n$. Hence $u_n(t) \in F_0(t, v_n(t))$ for all $n \geq n_t$ (recall here the condition (D1)). It follows that

$$\text{Ls}_n\{u_n(t)\} \subset \text{Ls}_n F_0(t, v_n(t)) \subset F_0(t, v(t))$$

where the last inclusion is due to the upper semicontinuity of the multifunction $F_0(t, \cdot\,)$. By (3.23) and since $F_0(\,\cdot\,)$ convex closed valued, we deduce that $u(t) \in F_0(t, v(t))$. This completes the proof of the theorem. □

REMARKS. (1) Assertion (iv) of the theorem is both a tangential condition and a measurability hypothesis for the multifunction F. It is weaker than conditions we find usually in the literature ([10], [21], [25]–[27]). Most authors suppose the multifunction F globally measurable with respect the product tribe $\mathcal{L}(I) \otimes \mathcal{B}(X)$. Let us illustrate a simple case where this measurability assumption holds.We suppose that $C(t) \equiv X$ is constant on I. Let $F: I \times X \to \text{ck}(X)$ be such that for almost every t, $F(t, \cdot\,)$ is u.c.s. on X and for every x, $F(\,\cdot\,, x)$ admits at least a measurable selection. Then a routine argument shows that for every measurable function σ on I the multifunction $t \mapsto F(t, \sigma(t))$ admits at least a measurable selection.

(2) In [10] Bothe introduced the condition

(B) For every $t \in N$ and every $x \in C(t)$, $(\{1\} \times X) \cap T_G(t, x) \neq \emptyset$.

The author didn't give a real interpretation of such property. In fact (B) hides in the background a topological/analytical property of the constraint C. Condition (iii) in the statement of our theorem is the correct extension of (B) to infinite dimensional spaces. A real meaning of (iii) find its base in a generalization of the notions of derivability and absolute continuity for multifunctions. Particularly Lipschitzian and absolutely continuous multifunctions (after a change of variable for the last) satisfy condition (iii). Moreover, condition (iii) is illustrated by Example 3.1, p. 29 of [9] which is not covered by (B).

We give an immediate consequence of Theorem 3.1.

Corollary 3.4. *Let I and C as in Theorem* 3.1. *We suppose that $F: G \to \mathrm{ck}(X)$ is globally measurable and satisfies the conditions:*

(a) *There exists $c \in L^1_{\mathbb{R}^+}(I)$ such that*

$$|F(t, x)| \leq c(t)(1 + \|x\|) \quad \text{for all } (t, x) \in G.$$

(b) *For every $t \in I$, the multifunction $F(t, \cdot\,)$ is u.s.c. on $C(t)$.*

(c) *For every $(t, x) \in G^*$,*

$$\liminf_{h \to 0^+} \frac{1}{h}\, d(x, C(t + h)) < \infty.$$

(d) *There exists a negligible set N of I such that*

$$(\{1\} \times F(t, x)) \cap T_G(t, x) \neq \emptyset \quad \text{for all } t \in I \setminus N,\ x \in C(t).$$

(e) *There exists a Kamke-function ω on $I \times \mathbb{R}^+$ such that for every $\varepsilon > 0$, there exists a compact set $J_\varepsilon \subset I$, with $\lambda(I \setminus J_\varepsilon) < \varepsilon$, such that for every $t \in J_\varepsilon$ we have:*

$$\inf_{\delta > 0} \alpha_X [F((([t - \delta, t] \cap J_\varepsilon) \times B) \cap G)] \leq \omega(t, \alpha_X(B))$$

for every bounded set B of X.

Then, given $x_0 \in C(0)$, there exists an absolutely continuous function $x: I \to X$ and a function $x' \in L^1_X(I)$ such that $x(t) = x_0 + \int_0^t x'(s)\, ds$ for all $t \in I$ and

$$\text{(P)} \qquad \begin{cases} x(t) \in C(t) & \text{on } I, \\ x'(t) \in F(t, x(t)) & \text{a.e. on } I. \end{cases}$$

Proof. It is enough to take $F_0 = F$ in the proof of Theorem 3.1. Moreover, as shown in the proof of Lemma 3.2, the multifunction H from G to $\mathcal{K}(\mathbb{R} \times X)$

defined by:

$$H\colon (t,x) \mapsto (\{1\} \times F(t,x)) \cap T_G(t,x)$$

is weakly measurable. So by the measurable selection theorem, condition (d) implies condition (iv) of Theorem 3.1. $\square$

4. Application to known results. The non convex sweeping process

We will suppose in all what follows that X is a *finite dimensional space*. Let K be a nonempty closed subset of X and $x \in K$. We put

$$\mathrm{proj}_K(x) := \{y \in K : d(x,K) = \|x-y\|\}$$

the set of projections of x onto K. The *proximal normal cone* to K at x is defined by

$$N_K^P(x) := \{v \in X : \text{there exists } \delta > 0 \text{ such that } d(x+\delta v, K) = \delta\|v\|\}.$$

The *limiting proximal normal cone* to K at x is defined by

$$\widehat{N}_K(x) := \left\{ \lim_{n\to\infty} v_n : \text{for all } n, v_n \in N_K^P(x_n),\ x_n \in K \text{ and } \lim_{n\to\infty} x_n = x \right\}.$$

For these notions and related topics we refer to [16] and [17] (see also [6]).

Let us recall some facts about absolutely continuous functions. Let $[a,b]$ (with $a<b$) be a compact interval of $\mathbb{R}$. Denote by λ the Lebesgue measure on $[a,b]$. A function $f\colon [a,b] \to X$ is called *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite family of disjoint sub-intervals $]s_i,t_i[$ $(i=1,\dots,n)$ of $[a,b]$, we have

$$\sum_{i=1}^{n}(t_i - s_i) \le \delta \Rightarrow \sum_{i=1}^{n} \|f(t_i) - f(s_i)\| \le \varepsilon.$$

It is known that any absolutely continuous function $f\colon [a,b] \to X$ is of *bounded variation* on I and that if df denotes the differential measure[3] of f, then $|df| \ll \lambda$ (cf. Moreau–Valadier [34, Section 3, Lemma 1]). Moreover, f is λ-a.e. derivable on I and

$$f'(t) = \frac{df}{d\lambda}(t) \quad \lambda\text{-a.e.}$$

for any Radon–Nikodym density $df/d\lambda$ of df with respect to λ (cf. [34, Section 3, Proposition 2]).

We will need next the following lemma:

[3] In [24, III.5, p. 142] it is called Borel–Stieltjes measure determined by the function f.

Lemma 4.1. *Let* $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ *be two compact intervals of* $\mathbb{R}$ *with* $a_1 < b_1$ *and* $a_2 < b_2$. *For* $i \in \{1, 2\}$ *let* λ_i *denotes the Lebesgue measure on* I_i. *Let* $\varphi: I_1 \to I_2$ *be an absolutely continuous function on* I_1 *such that* $\varphi(a_1) = a_2$ *and* $\varphi(b_1) = b_2$. *We suppose that* $d\varphi$ *admits a Radon–Nikodym density* $\dot\varphi$ *with respect to* λ_1 *such that* $\dot\varphi(t) > 0$ *for all* $t \in I_1$. *Then we have the following properties:*

(a) *For every Borel subset* A *of* I_1, $\lambda_2(\varphi(A)) = d\varphi(A)$.
(b) *If* N *is a* λ_1*-negligible subset of* I_1 *then* $\varphi(N)$ *is a* λ_2*-negligible subset of* I_2.
(c) *The inverse function* $\phi = \varphi^{-1}$ *of* φ *is strictly increasing and absolutely continuous on* I_2. *Moreover, if* $\dot\phi$ *is a Radon–Nikodym density of* $d\phi$ *with respect to* λ_2 *on* I_2, *then for* λ_2*-a.e.* $\tau \in I_2$, $\dot\phi(\tau) = 1/\dot\varphi(\phi(\tau))$.
(d) *Let* $f_2: I_2 \to X$ *be a given absolutely continuous function. Then the composite function* $f_1 = f_2 \circ \varphi: I_1 \to X$ *is absolutely continuous on* I_1. *Moreover, if* $\dot f_1$ *(resp.* $\dot f_2$*) denotes a Radon–Nikodym density of* df_1 *(resp.* df_2*) with respect to* λ_1 *(resp.* λ_2*), then* $\dot f_1(t) = \dot f_2(\varphi(t))\dot\varphi(t)$ λ_1*-a.e.*

Proof. Notice first that the hypothesis implies that φ is a non decreasing homeomorphism from the interval I_1 to the interval I_2.

(a) Consider the positive measure μ defined on $\mathcal{B}(I_1)$ by $\mu(A) := \lambda_2(\varphi(A))$. Since φ is a homeomorphism from I_1 to I_2 and since the Lebesgue measure λ_2 is regular on $\mathcal{B}(I_2)$, it is easy to check that the measure μ is regular on $\mathcal{B}(I_1)$. For $t \le t'$ in I_1, denote by $J = (t, t')$ any sub-interval of I_1 of extremities t and t'. We have $\varphi(J) = (\varphi(t), \varphi(t'))$. Hence $\mu(J) = \varphi(t') - \varphi(t) = d\varphi(J)$. Let $\mathcal{A}$ be the field of all finite unions

$$A = J_1 \cup \ldots \cup J_n \tag{4.1}$$

of sub-intervals $J_i = (t_i, t'_i)$, $i = 1, \ldots, n$, of I_1. If the intervals J_i, $i = 1, \ldots, n$ in (4.1) are disjoints, then

$$\varphi(A) = \varphi(J_1) \cup \ldots \cup \varphi(J_n)$$

where $\varphi(J_i) = (\varphi(t_i), \varphi(t'_i))$, $i = 1, \ldots, n$ are also disjoints sub-intervals of I_2. It follows that

$$\mu(A) = \sum_{i=1}^{n} \lambda_2(\varphi(J_i)) = \sum_{i=1}^{n} d\varphi(J_i) = d\varphi(A).$$

Hence the regular measures μ and $d\varphi$ coincide on the field $\mathcal{A}$. Since $\mathcal{B}(I_1)$ is the σ-field generated by $\mathcal{A}$, it follows by [24, Theorem III.5.14], that $\mu = d\varphi$ on $\mathcal{B}(I_1)$.

(b) Let $\varepsilon > 0$. There exists $\eta_\varepsilon > 0$ such that for every $A \in \mathcal{B}(I_1)$, the condition $\lambda_1(A) \le \eta_\varepsilon$ implies $d\varphi(A) \le \varepsilon$ (this result from the fact that $d\varphi \ll \lambda_1$). Since N is λ_1-negligible, there exists a Borel subset U of I_1 containing N such that $\lambda_1(U) \le \eta_\varepsilon$. Hence $\lambda_2(\varphi(U)) = d\varphi(U) \le \varepsilon$. Moreover, $\varphi(U)$ is a Borel subset containing $\varphi(N)$. Since ε is arbitrary it follows that $\lambda_2^*(\varphi(N)) = 0$.

(c) Let $A \in \mathcal{B}(I_1)$ such that $\mu(A) = 0$. Then $\int_A \dot\varphi(s)\,ds = 0$. Since $\dot\varphi(s) > 0$ on I_1, we deduce that $\lambda_1(A) = 0$. Hence $\lambda_1 \ll \mu = d\varphi$. Let f be a Radon–Nikodym density of λ_1 with respect to μ. We have $f \in L^1_{\mathbb{R}^+}(I_1, \mu)$ and

$$\int_E f(s)\mu(ds) = \lambda_1(E), \quad E \in \mathcal{B}(I_1). \tag{4.2}$$

Consider the notations $S_1 := I_2$, $\Sigma_1 := \mathcal{B}(I_2)$, $\mu_1 := \lambda_2$, $S_2 := I_1$, $\Sigma_2 := \mathcal{B}(I_1)$, $\mu_2 := \mu$. By virtue of (a) we have $\mu_1(\phi^{-1}(A)) = \mu_2(A)$ for all $A \in \Sigma_2$. Moreover, since ϕ is a homeomorphism we have $\Sigma_1 = \{\phi^{-1}(A) : A \in \Sigma_2\}$. Hence we can apply the "generalized change of variable lemma" [24, Lemma III.10.8]. We get that $f(\phi(\cdot))$ is λ_2-integrable and

$$\int_E f(s)\mu(ds) = \int_{\phi^{-1}(E)} f(\phi(t))\,dt \quad \text{for all } E \in \mathcal{B}(I_1). \tag{4.3}$$

Let $\tau_1 < \tau_2$ in I_2 and put $E = [\phi(\tau_1), \phi(\tau_2)] = \phi([\tau_1, \tau_1])$. Then, applying (4.2) and (4.3), we get

$$\int_{\tau_1}^{\tau_2} f(\phi(t))\,dt = \lambda_1(E) = \phi(\tau_2) - \phi(\tau_1).$$

This proves that ϕ is absolutely continuous on I_2 and that $\dot\phi := f(\phi(\cdot))$ is a Radon–Nikodym density of $d\phi$ with respect to λ_2. Since $f = d\lambda_1/d\mu$, by virtue of Jeffery theorem (cf. e.g. [34, §4, Théorème 3]) for μ-almost every (hence also λ_1-almost every) $s \in I_1$, we have

$$f(s) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\mu([s, s+\varepsilon])} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\varphi(s+\varepsilon) - \varphi(s)}.$$

It follows easily that $f(s) = 1/\varphi'(s) = 1/\dot\varphi(s)$ λ_1-a.e. on I_1. This completes the proof of assertion (c).

(d) The fact that $f_1 = f_2 \circ \varphi$ is absolutely continuous follows easily from application of the $\varepsilon - \delta$-definition of absolute continuity to f_2 and φ. For the second assertion, let N_1 be a λ_1-negligible subset of I_1 and N_2 be a λ_2-negligible subset of I_2 such that

$$\dot\varphi(s) = \lim_{\varepsilon \to 0^+} \frac{\varphi(s+\varepsilon) - \varphi(s)}{\varepsilon} \quad \text{and} \quad \dot f_2(t) = \lim_{\delta \to 0^+} \frac{f_2(t+\delta) - f_2(t)}{\delta}$$

for all $s \in I_1 \setminus N_1$ and all $t \in I_2 \setminus N_2$. Let N_1' another λ_1-negligible subset of I_1 such that

$$\dot f_1(s) = \lim_{\varepsilon \to 0^+} \frac{f_1(s+\varepsilon) - f_1(s)}{\varepsilon} \quad \text{for all } s \in I_1 \setminus N_1'.$$

Consider the λ_1-negligible subset $N := N_1 \cup N_1' \cup \phi(N_2)$ of I_1. Take $s \in I_1 \setminus N$. Then for $\varepsilon > 0$ enough small, we have

$$\frac{f_1(s+\varepsilon) - f_1(s)}{\varepsilon} = \frac{f_2(\varphi(s+\varepsilon)) - f_1(\varphi(s))}{\varphi(s+\varepsilon) - \varphi(s)} . \frac{\varphi(s+\varepsilon) - \varphi(s)}{\varepsilon}. \tag{4.4}$$

We have $\varphi(s) \notin N_2$ and $\varphi(s+\varepsilon) \searrow \varphi(s)$ as $\varepsilon \to 0^+$. Hence by passing to the limit in (4.4) as $\varepsilon \to 0^+$ we get $\dot{f}_1(s) = \dot{f}_2(\varphi(s))\dot{\varphi}(s)$. $\square$

The following theorem is a variant of our main result:

Theorem 4.2. *Let $I = [0,a]$ be a compact interval of $\mathbb{R}$ and $C: I \to \mathrm{cl}(X)$ be a multifunction with closed graph G in $I_\ell \times X$. Let $F: G \to \mathrm{ck}(X)$ be a globally measurable multifunction such that:*

(a) *There exists $c \in L^1_{\mathbb{R}^+}(I)$ such that*

$$|F(t,x)| \le c(t)(1 + \|x\|) \quad \textit{for all } (t,x) \in G.$$

(b) *For every $t \in I$, the multifunction $F(t,\,\cdot\,)$ is u.s.c. on $C(t)$.*

(c) *There exists a function $r: I \to \mathbb{R}^+$ strictly increasing and absolutely continuous such that,*

$$\liminf_{h \to 0^+} \frac{d(x, C(t+h))}{r(t+h) - r(t)} < \infty \quad \textit{for all } (t,x) \in G^*.$$

(d) *There exists a negligible set N of I such that*

$$(\{1\} \times F(t,x)) \cap T_G(t,x) \ne \emptyset \quad \textit{for all } t \in I \setminus N,\ x \in C(t).$$

Then, given $x_0 \in C(0)$, problem (P) *has an absolutely continuous solution $x: I \to X$ such that $x(0) = x_0$.*

Proof. Denote by r' the derivative of the function r. For $t \in I$, put $\gamma(t) := \max\{1, c(t), r'(t)\}$ and consider the function

$$\varphi(t) := \int_0^t \gamma(s)\, ds, \quad t \in I.$$

Then φ is a strictly increasing homeomorphism from the interval I to the interval $I_1 := [0, a_1]$ where $a_1 := \varphi(a)$. Denote by λ (resp. λ_1) the Lebesgue measure on I (resp. on I_1). By definition φ is absolutely continuous on I with $d\varphi/d\lambda = \gamma$. Consider $\phi := \varphi^{-1}$ the inverse function of φ. For $t \in I_1$, put $C_1(t) := C(\phi(t))$ and

$$F_1(t,x) := \frac{1}{\gamma(\phi(t))} F(\phi(t), x) \quad \text{for } t \in I_1,\ x \in C_1(t).$$

Then F_1 is a globally measurable multifunction from $G_1 := \mathrm{graph}(C_1)$ to $\mathrm{ck}(X)$ such that $F_1(t,\,\cdot\,)$ is u.s.c. on $C_1(t)$ for all $t \in I_1$ and $|F_1(t,x)| \le 1 + \|x\|$ for all $(t,x) \in G_1$. Moreover, we have $r(s+h) - r(s) \le \varphi(s+h) - \varphi(s)$ for $s \in I$, $h > 0$

and φ is a strictly increasing homeomorphism from I to I_1. Hence condition (c) of the theorem implies

$$\liminf_{h\to 0^+} \frac{1}{h}\, d(x, C_1(t+h)) < \infty \quad \text{for all } (t,x) \in G_1^*. \tag{4.5}$$

By Lemma 4.1 the function ϕ is absolutely continuous on I_1 with $d\phi/d\lambda_1(t) = 1/\gamma(\phi(t))$. Choose a λ_1-negligible subset N_1 of I_1 containing $\varphi(N)$ such that for all $t \in I_1 \setminus N_1$ the derivative $\phi'(t)$ exists and is equal to $1/\gamma(\phi(t))$. We shall prove that

$$(\{1\} \times F_1(t,x)) \cap T_{G_1}(t,x) \neq \emptyset \quad \text{for all } t \in I_1 \setminus N_1 \text{ and all } x \in C_1(t). \tag{4.6}$$

Let $t \in I_1 \setminus N_1$ and $x \in C_1(t)$ be fixed. Since $\phi(t) \in I \setminus N$, by virtue of condition (d) there exists $y \in F(\phi(t), x)$, a sequence $h_n \to 0^+$ with $\phi(t) + h_n \in I$ and a sequence $x_n \in C(\phi(t) + h_n)$ such that

$$\left\| \frac{x_n - x}{h_n} - y \right\| \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

Set $k_n := \varphi(\phi(t)+h_n) - t > 0$. Then $k_n \to 0^+$, $t + k_n \in I_1$ and $\phi(t+k_n) - \phi(t) = h_n$. Moreover, (4.7) is equivalent to

$$\left\| \frac{x_n - x}{k_n} \cdot \frac{k_n}{\phi(t+k_n) - \phi(t)} - y \right\| \to 0 \quad \text{as } n \to \infty$$

and

$$\frac{k_n}{\phi(t+k_n) - \phi(t)} \to \gamma(\phi(t)) \quad \text{as } n \to \infty.$$

It follows easily that

$$\left\| \frac{x_n - x}{k_n} - \frac{1}{\gamma(\phi(t))} \cdot y \right\| \to 0 \quad \text{as } n \to \infty.$$

Hence the point $y_1 := (1/\gamma(\phi(t))) \cdot y$ belongs to $(\{1\} \times F_1(t,x)) \cap T_{G_1}(t,x)$.

We can now apply Corollary 3.4 to F_1 and C_1. There exists an absolutely continuous function $x_1\colon I_1 \to X$ such that $x_1(0) = x_0$, $x_1(t) \in C_1(t)$, for all $t \in I_1$ and $x_1'(t) \in F_1(t, x_1(t))$ a.e. on I_1. Set $x(s) := x_1(\varphi(s))$ for $s \in I$. By Lemma 4.1(d) the function x is absolutely continuous on I and for almost every s in I, $x'(s) = x_1'(\varphi(s)) \cdot \gamma(s)$. On the other hand λ-a.e.

$$\gamma(s) \cdot x_1'(\varphi(s)) \in \gamma(s) F_1(\varphi(s), x_1(\varphi(s))) = F(s, x(s)).$$

Hence x is a solution of the problem (P). $\square$

Now we provide a measurable characterization of the tangential condition (d) in the light of results given in [2, Proposition 7.1] and [17, Theorem 2.10, p. 193].

Proposition 4.3. *Let $C: I \to \mathrm{cl}(X)$ be a multifunction with closed graph G in $I \times X$. Let $F: G \to \mathrm{ck}(X)$ be a globally measurable multifunction such that $F(t, \cdot)$ is upper semicontinuous on $C(t)$ for every $t \in I$. We suppose that*

$$|F(t,x)| \le c(t)(1 + \|x\|) \quad \textit{for all } (t,x) \in G$$

for some measurable function $c: I \to \mathbb{R}^+$. Then the following assertions are equivalent:

(T1) *For almost every $t \in I$,*

$$(\{1\} \times F(t,x)) \cap T_G(t,x) \neq \emptyset \quad \textit{for all } x \in C(t).$$

(T2) *For almost every $t \in I$,*

$$(\{1\} \times F(t,x)) \cap \overline{\mathrm{co}}\, T_G(t,x) \neq \emptyset \quad \textit{for all } x \in C(t).$$

(T3) *For almost every $t \in I$,*

$$-\alpha + \delta^*(-p, F(t,x)) \ge 0 \quad \textit{for all } x \in C(t) \textit{ and all } (\alpha, p) \in (T_G(t,x))^-.$$

(T4) *For almost every $t \in I$,*

$$-\alpha + \delta^*(-p, F(t,x)) \ge 0 \quad \textit{for all } x \in C(t) \textit{ and all } (\alpha, p) \in N_G^P(t,x).$$

Proof. The proof of the implications (T1)⇒(T2)⇒(T3)⇒(T4) follows from arguments similar to those given in [2, Proposition 7.1]. It remains to prove (T4)⇒(T1).

Assume that (T4) is satisfied. Let $\varepsilon > 0$ and find by Corollary 2.5(d) a compact set $J_\varepsilon \subset I$ with $\lambda(I \setminus J_\varepsilon) < \varepsilon$ such that $F_{|G_\varepsilon}$ is upper semicontinuous where $G_\varepsilon := (J_\varepsilon \times X) \cap G$. We may suppose also that condition (T4) is satisfied by each point t of J_ε. Now using a special multivalued version of Dugundji theorem, we shall extend $F_{|G_\varepsilon}$ to an upper semicontinuous multifunction $\widetilde{F}$ defined on all the space $E := \mathbb{R} \times X$. Indeed, by [7, Théorème 2.2], there exists a locally finite open cover $(U_k)_{k \in K}$ of $E \setminus G_\varepsilon$, a partition of unity $(\psi_k)_{k \in K}$ subordinate to $(U_k)_{k \in K}$ and a family $(t_k, x_k)_{k \in K}$ of points of G_ε such that the multifunction $\widetilde{F}$ defined by

$$\widetilde{F}(t,x) := \begin{cases} F(t,x) & \text{if } (t,x) \in G_\varepsilon, \\ \sum_k \psi_k(t,x) F(t_k, x_k) & \text{if } (t,x) \in E \setminus G_\varepsilon, \end{cases}$$

is upper semicontinuous on E and has nonempty convex compact values. We shall prove that

$$-\alpha + \delta^*(-p, \widetilde{F}(t,x)) \ge 0 \quad \text{for all } (t,x) \in G \text{ and all } (\alpha, p) \in N_G^P(t,x). \tag{4.8}$$

Let $(t,x) \in G$. If $(t,x) \in G_\varepsilon$ then (4.8) holds since $\widetilde{F}_{|G_\varepsilon} = F_{|G_\varepsilon}$ and (T4) is satisfied by each point of J_ε.

Now suppose that $t \in \mathbb{R} \setminus J_\varepsilon$ and take (α, p) in $N_G^P(t,x)$. The set $\{k \in K : \psi_k(t,x) \neq 0\}$ is finite, say equal to $\{k_1, \dots, k_n\}$. For each $i = 1, \dots, n$ choose y_i in $F(t_{k_i}, x_{k_i})$ such that $\delta^*(-p, F(t_{k_i}, x_{k_i})) = \langle -p, y_i\rangle$. By virtue of (T4) we have

$$-\alpha + \langle -p, y_i\rangle \geq 0 \quad \text{for } i = 1, \dots, n.$$

Consider the vector $y := \sum_{i=1}^n \psi_{k_i}(t,x) y_i$ which belongs to $\widetilde{F}(t,x)$. We have

$$-\alpha + \langle -p, y\rangle = \sum_{i=1}^n \psi_{k_i}(t,x)(-\alpha + \langle -p, y_i\rangle) \geq 0.$$

It follows that $-\alpha + \delta^*(-p, \widetilde{F}(t,x)) \geq 0$. Hence (4.8) is proved.

We conclude from (4.8) that the upper semicontinuous multifunction $(t,x) \mapsto \{1\} \times \widetilde{F}(t,x)$ satisfies condition (iv) of [2, Proposition 7.1], with respect to the closed set G. Applying implication (iv)⇒(i) of that proposition, we get

$$(\{1\} \times \widetilde{F}(t,x)) \cap T_G(t,x) \neq \emptyset \quad \text{for all } (t,x) \in G.$$

In particular, we have

$$(\{1\} \times F(t,x)) \cap T_G(t,x) \neq \emptyset \quad \text{for all } t \in J_\varepsilon \text{ and all } x \in C(t). \tag{4.9}$$

Now let $\varepsilon = 2^{-n}$ and find a sequence of compact sets $J_n \subset I$ with $\lambda(I \setminus J_n) < 2^{-n}$ such that (4.9) holds true for each $J_\varepsilon = J_n$. Since $\bigcup_n J_n$ is a set of full measure, it is obvious that condition (T1) is satisfied. □

Remark. The equivalences (T1)⇔(T2)⇔(T3) can be found in the works [26], [25] in a different context, namely, the constraint C there is assumed to be absolutely continuous with respect to ρ-Hausdorff distances. Here we don't assume any analytic property on the multifunction C except that it's graph is closed.

To end this paper, we give an application to non convex sweeping process which arises from Mechanics [31]–[33] and Mathematical Economics [19], [20], [28].

Let $I = [0,a]$ with $a > 0$. Let $C: I \to \mathrm{cl}(X)$ be a given multifunction and $x_0 \in C(0)$. The *sweeping process* by the moving set $C(t)$ consists on finding absolutely continuous solutions $x : I \to X$ of the differential inclusion

$$\text{(Sw)} \qquad \begin{cases} x(0) = x_0, & \\ x(t) \in C(t) & \text{for } t \in I, \\ x'(t) \in -N_{C(t)}(x(t)) & \text{a.e. on } I, \end{cases}$$

where $N_{C(t)}(x(t))$ denotes the Clarke cone of $C(t)$ at $x(t)$.

Intuitively, the problem (Sw) can be described as follows: at the initial time a point belongs to $C(0)$; during the time this point is possibly caught up by the boundary of $C(t)$ so that it can only proceed in an inward normal direction of $C(t)$ as if pushed by this boundary. For an exclusive study of the sweeping process in the convex case we refer to [33], [31] and [41]. Recently problem (Sw) was extended to the case where the moving sets $C(t)$ are not necessarily convex ([4], [18], [38]).

We will assume the following hypothesis on the multifunction C:

(H1) C is a multifunction from I to $\mathrm{cl}(X)$ with closed graph G in $I_\ell \times X$.

(H2) There exists a nondecreasing absolutely continuous function $r: I \to \mathbb{R}^+$ such that

$$e(C(t), C(t')) \leq r(t') - r(t) \quad \text{for all } t \leq t' \text{ in } I.$$

Without loss of generality we may suppose that the derivative $\dot{r}$ of r satisfies $\dot{r}(t) > 0$ almost everywhere on I. Set

$$F(t,x) := \overline{\mathrm{co}}[-\widehat{N}_{C(t)}(x) \cap \dot{r}(t) B_X] \quad \text{for } (t,x) \in G \tag{4.10}$$

where $\widehat{N}_{C(t)}(x)$ denotes the limiting proximal normal cone to $C(t)$ at x. Then F is a multifunction from G to $\mathrm{ck}(X)$ such that

$$|F(t,x)| \leq \dot{r}(t) \quad \text{for all } (t,x) \in G.$$

For $t \in I$ fixed, the multifunction $x \mapsto \widehat{N}_{C(t)}(x)$ has a closed graph in $C(t) \times X$, hence obviously the multifunction $F(t, \cdot\,)$ is upper semicontinuous on $C(t)$. Moreover, in virtue of Lemma 2.2 in [4] and Theorem III.40 in [15], the multifunction F has a measurable graph.

Theorem 4.4. *Assume that C satisfies the hypothesis* (H1), (H2) *and F is the multifunction defined by* (4.10). *Then for every $x_0 \in C(0)$ there exists an absolutely continuous function $x(\cdot\,)$ such that $x(0) = x_0$, $x(t) \in C(t)$ for all $t \in I$ and $\dot{x}(t) \in F(t, x(t))$ a.e. on I. Consequently the sweeping process problem* (Sw) *admits at least an absolutely continuous solution $x(\cdot\,)$ such that $\|\dot{x}(t)\| \leq \dot{r}(t)$ almost everywhere.*

Proof. By virtue of the preceding considerations, conditions (a)–(c) of Theorem 4.2 are satisfied by the multifunction F and the constraint C. We shall prove that condition (d) is also satisfied. Let $\varepsilon = 2^{-n}$ and choose by Corollary 2.5 a compact set $I_n \subset I$ with $\lambda(I \setminus I_n) < 2^{-n}$ such that the restriction of F to $G_n := (I_n \times X) \cap G$ is upper semicontinuous. We suppose also that r is derivable on each point t of I_n with derivative equal to $\dot{r}(t)$. For each n choose a compact set $J_n \subset I_n$ with $\lambda(I \setminus J_n) < 2^{-n}$ such that each point t of J_n is a density point of I_n. Consider the negligible subset $N := I \setminus \bigcup_n J_n$ of I. Take $t \in I \setminus N$ and $x \in C(t)$. There exists an integer n_0 such that $t \in J_{n_0}$. Since t is a density point for I_{n_0} there exists a sequence (h_k) of strictly positive numbers such that $t + h_k \in I_{n_0}$ and $h_k \to 0^+$ as $k \to \infty$. For each k choose $x_k \in \mathrm{proj}_{C(t+h_k)}(x)$. By virtue of (H2) we have

$$\left\| \frac{x - x_k}{h_k} \right\| = \frac{d(x, C(t+h_k))}{h_k} \le \frac{r(t+h_k) - r(t)}{h_k}.$$

Since r is derivable on t, we deduce that the sequence $y_k := (x_k - x)/h_k$ is bounded in X. Hence a subsequence (again denoted by) y_k converges to some point $y \in X$. Now by construction of x_k we have $\delta(x_k - x) \in -N^P_{C(t+h_k)}(x_k)$ for all $\delta \ge 0$. In particular, putting

$$z_k := \frac{\dot r(t)}{r(t+h_k) - r(t)} \cdot (x_k - x)$$

we get

$$\|z_k\| \le \dot r(t) \quad \text{and} \quad z_k \in -N^P_{C(t+h_k)}(x_k).$$

Hence

$$z_k \in F(t + h_k, x_k) \quad \text{for all } k. \tag{4.11}$$

Furthermore, since $\dot r(t)$ is the derivative of r at t, it is easy to check that $z_k \to y$ as $k \to \infty$. Now remark that $(t+h_k, x_k) \in G_{n_0}$ and $F_{|G_{n_0}}$ is upper semicontinuous. Hence condition (4.11) implies that $y \in F(t,x)$. We have thus proved that the point $(1, y)$ lies in $(\{1\} \times F(t,x)) \cap T_G(t,x)$. Now we complete the proof by applying the conclusion of Theorem 4.2. $\square$

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