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PERIODIC SOLUTIONS OF LAGRANGE EQUATIONS

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Dedicated to Andrzej Granas

Abstract. Nontrivial periodic solutions of Lagrange Equations are investigated. Sublinear and superlinear nonlinearity are included. Convexity assumptions are significiently relaxed. The method used is the duality developed by the authors.

1. Introduction

We investigate the nonlinear problem:

$$\begin{aligned} \frac{d}{dt}L_{x'}(t,x'(t))+V_x(t,x(t)) &= 0, \quad \text{a.e. in } \mathbb{R} \\ x(t+T) &= x(t), \end{aligned} \tag{1.1}$$

where

(H) $T>0$ is arbitrary, $L,V:\mathbb{R}\times\mathbb{R}^n\to R$ are Gateaux differentiable in the second variable, T-periodic and measurable in t functions and L is convex in the second variable.

We are looking for solutions of (1.1) being a pair (x,p) of periodic absolutely continuous functions $x,p:[0,T]\to\mathbb{R}^n$, $x(0)=x(T)$, $p(0)=p(T)$ such that

$$\begin{aligned} \frac{d}{dt}p(t)+V_x(t,x(t))=0,\\ p(t)=L_{x'}(t,x'(t)). \end{aligned}$$

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Of course, if $L(t,x') = |x'|^2/2$ or $t \to L_p^*(t,p(t))$ (L^* denotes Fenchel conjugate of $L(t,\,\cdot\,)$) is an absolutely continuous function, then our solution of (1.1) belongs to $C^{1,+}([0,T],\mathbb{R}^n)$ of continuously differentiable functions x whose derivatives x' are absolutely continuous. It is clear that (1.1) is the Euler-Lagrange equation to the functional

$$J(x) = \int_0^T (-V(t,x(t)) + L(t,x'(t)))\,dt \tag{1.2}$$

considered on the space A_p of absolutely continuous T-periodic functions x: $\mathbb{R} \to \mathbb{R}^n$. The dual functional to it is

$$J_D(p) = -\int_0^T L^*(t,p(t))\,dt + \int_0^T V^*(t,-p'(t))\,dt$$

considered also on the space A_p, where L^* is a Fenchel conjugate to L and where V^* is a Fenchel conjugate to V.

Periodic problem (1.1) was studied in eighties by many authors as well in sublinear case as in superlinear one (see e.g. [7]), sublinear cases also in [1], [3], [7], [8], some cases of (1.1) for superlinear V_x in [5], [6], [2], [9]. It is interesting that the method developed in [5] is based on the dual variational method for the problem, according to the idea discovered by Clarke. However, here we develop our duality method from [9]. We relax the convexity assumption on V and significantly improve the construction of the set X.

Our aim is to find a nonlinear subspace X of A_p defined by the type of nonlinearity of V (and in fact also L). In this paper we develop absolutely new construction of the set X in comparison to the paper [9]. First we define $\overline{X}$ *to be a given, closed with respect to the norm "maximum", and convex subset of the set* $\{v \in A_0 : v' \in A\}$, where A is the space of absolutely continuous functions with derivatives in L^2, A_0 is the subspace of A of all functions $v\colon [0,T] \to \mathbb{R}^n$ satisfying $v(0) = 0$.

Let B be a convex set in $\mathbb{R}^n$ such that for each $v \in \overline{X}$, $v(t) \in B$, $t \in [0,T]$. Let us set the basic hypothesis we need:

(H1) There exist $0 < \alpha_1 \le \alpha_2$ and $d_1, d_2 \in \mathbb{R}$ such that for $x' \in L^2$

$$d_1 + \frac{\alpha_1}{2}\|x'\|_{L^2}^2 \le \int_0^T L(t,x'(t))\,dt \le \frac{\alpha_2}{2}\|x'\|_{L^2}^2 + d_2, \tag{1.3}$$

where $L(t,\,\cdot\,)$ is strictly convex, there exist $0 < \beta_1$, $q_1 > 1$, $k \in \mathbb{R}$ such that for each $v \in \overline{X}$ and for all $x \in \mathbb{R}^n$

$$k + \frac{\beta_1}{q}\,|x|^{q_1} \le \int_0^T V(t,v(t)+x)\,dt.$$

$V_x(t, \cdot)$ is continuous, $t \in [0,T]$, and there exist $k_1, k_2 \in L^2$ such that

$$(1.4)\qquad \begin{aligned} &(t \to \sup\{V(t,x) : x \in B\}) \le k_1(t), \quad t \in [0,T], \\ &(t \to \sup\{V_{x_i}(t,x) : x \in B\}) \le k_2(t), \quad t \in [0,T],\ i = 1, \dots, n. \end{aligned}$$

Having the basic hypothesis we are able to define a nonlinear subspace X as follows. We reduce the space $\overline{X}$ to the set $X \subset \overline{X}$ with the property:

- For each $v \in X$ and $c_v \in \mathbb{R}^n$, where c_v is a minimizer for the functional $\mathbb{R}^n \ni c \to \int_0^T V(t, v(t) + c)\,dt$ (such a minimizer, by (H) and (H1) certainly exists), there exists (possible another) $\widetilde{v} \in X$ such that

$$(1.5)\qquad \int_0^T \{\langle v(t) + c_v, -\widetilde{p}'(t) > -V^*(t, -\widetilde{p}'(t))\}\,dt = \int_0^T V(t, v(t) + c_v)\,dt,$$

where $\widetilde{p}(t) = L_{x'}(t, \widetilde{v}'(t)) - d_p$ for a.e. $t \in [0,T]$, $d_p \in \mathbb{R}^n$and V^* is the Fenchel conjugate of V in the second variable.

We assume that

(HX) The set X contains at least one element x with $x(0) = x(T) = 0$.

It is clear that the set X is much smaller than $\widetilde{X}$ and that it depends strongly on the type of nonlinearity V and L. We easily see that X is not in general a closed set in A and that in X the subdifferential $\partial_v V(t, v(t)) \neq \emptyset$ and $V(t, v(t)) = V^{**}(t, v(t))$ for a.e. $t \in [0,T]$. As the dual set to X we shall consider the following set

$$X^d = \{p \in A_T : \text{there exist } v \in X \text{ such that}$$
$$p(t) = L_{x'}(t, v'(t)) - d_p,\ t \in [0,T] \text{ a.e. } d_p \in \mathbb{R}^n\},$$

where A_T denotes the subspace of A of all functions $w\colon [0,T] \to \mathbb{R}^n$ with $w(T) = 0$. Therefore, by (1.5), we derive that for $v \in X$ there exists c_v and $p \in X^d$ such that $-p'(t) = V_x(t, v(t) + c_v)$, $t \in [0,T]$ a.e.

Taking into account the structure of the space X we shall study the functional

$$J(x,c) = \int_0^T (-V(t, x(t) + c) + L(t, x'(t)))\,dt + l(x(T))$$

on the space $X \oplus \mathbb{R}^n$ instead of (1.2) on the space A_p, where

$$l(a) = \begin{cases} 0 & \text{if } a = 0, \\ \infty & \text{if } a \neq 0, \end{cases}$$

and the functional

$$J_D(p,d) = -\int_0^T L^*(t, p(t) + d)\,dt + \int_0^T V^*(t, -p'(t))\,dt + l(p(0)).$$

We shall look for a "min" of J over the set X i.e. actually

$$\min_{x \in X} \max_{c \in \mathbb{R}^n} J(x,c).$$

To show that element $\overline{x} \in X$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D, described in the next section. Just because of the duality theory we are able to avoid in our proof of an existence of critical points the deformation lemmas, the Ekeland variational principle or PS type conditions. We would like to stress that we do not assume explicitly convexity of $V(t,\cdot)$.

The main result of our paper is the following:

THEOREM 1.1 (Main Theorem). *Under hypotheses* (H) *and* (H1) *there exists a pair* $\overline{x}+c_{\overline{x}}, \overline{p}+d_{\overline{p}})$, $\overline{x} \in X$, $\overline{p} \in X^d$, $c_{\overline{x}} \in \mathbb{R}^n$, $d_{\overline{p}} \in \mathbb{R}^n$ *being a solution of* (1.1) *and such that*

$$J(\overline{x}, c_{\overline{x}}) = \min_{x\in X}\max_{c\in\mathbb{R}^n} J(x,c) = \min_{p\in X^d}\max_{d\in\mathbb{R}^n} J_D(p,d) = J_D(\overline{p}, d_{\overline{p}}).$$

We see that our hypotheses on L and V concern only convexity of $L(t,\cdot)$ and at most local convexity of $V(t,\cdot)$ (see Example). We do not assume that $V(t,x) \geq 0$. However we require that the above set X is nonempty, which we must check in each concrete type of equation. Some routine how to do that we show at the end of the paper for the equation

$$k(t)x'' + V_x(t,x) = 0.$$

2. Duality results

To obtain a duality principle we need a kind of perturbation of J. Thus define for each $x \in X$ the perturbation of J as

$$J_x(a,y) = \int_0^T (V(t, x(t) + c_x + y(t)) - L(t, x'(t)))\,dt - l(x(T) + a) \tag{2.1}$$

for $y \in L^2, a \in \mathbb{R}^n$. Of course, $J_x(0,0) = -J(x,c_x)$. For $x \in X$ and $p \in X^d$, $d \in \mathbb{R}^n$ we define a type of conjugate of J by

$$\begin{aligned} J_x^{\#}(p,d) = \sup_{y\in L^2}\sup_{c}&\bigg\{\int_0^T \langle y(t), p'(t)\rangle\,dt - \int_0^T V(t, x(t)+c+y(t))\,dt\bigg\} \\ &+ \int_0^T L(t,x'(t))\,dt + \inf_{a\in\mathbb{R}^n}\{\langle a,d\rangle + l(x(T)+a)\}. \end{aligned}$$

By a direct calculation we obtain

$$\begin{aligned} J_x^{\#}(p,d) = \sup_{c}\bigg\{ &- \langle x(T), d\rangle - \int_0^T \langle x(t)+c, p'(t)\rangle\,dt \\ &+ \int_0^T L(t,x'(t))\,dt + \int_0^T V^*(t, -p'(t))\,dt\bigg\} \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&= \sup_c\{-\langle c, p(0)\rangle\} + \int_0^T \langle x'(t), p(t)+d\rangle\, dt \\
&\quad + \int_0^T L(t, x'(t))\, dt + \int_0^T V^*(t, -p'(t))\, dt \\
&= \int_0^T \langle x'(t), p(t)+d\rangle\, dt + \int_0^T L(t, x'(t))\, dt \\
&\quad + \int_0^T V^*(t, -p'(t))\, dt + l(p(0)).
\end{aligned}
$$

Now we take "min" from $J_x^{\#}(p,d)$ with respect to $x \in X$ and calculate it. Because X is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this effect we use the special structure of the set X^d. First we observe that for each $p \in X^d$ there exists $x_p \in X$ such that

$$p(t) + d_p = L_{x'}(t, x_p'(t))$$

and, by classical convex analysis argument,

$$x_p'(t) = L_p^*(t, p(t) + d_p),$$

where L^* is a Fenchel conjugate to L. Therefore

$$\int_0^T \langle x_p'(t), p(t)+d_p\rangle\, dt - \int_0^T L(t, x_p'(t))\, dt = \int_0^T L^*(t, p(t)+d_p)\, dt.$$

Next let us note that, on the other hand,

$$
\begin{aligned}
&\int_0^T \langle x_p'(t), p(t)+d_p\rangle\, dt - \int_0^T L(t, x_p'(t))\, dt \\
&\qquad \le \sup_{x\in X}\left\{\int_0^T \langle x'(t), p(t)+d_p\rangle\, dt - \int_0^T L(t, x'(t))\, dt\right\} \\
&\qquad \le \sup_{x'\in L^2}\left\{\int_0^T \langle x'(t), p(t)+d_p\rangle\, dt - \int_0^T L(t, x'(t))\, dt\right\} \\
&\qquad = \int_0^T L^*(t, p(t)+d_p)\, dt
\end{aligned}
$$

and actually all inequalities above are equalities. Therefore we can calculate for $p \in X^d$ and appropriate d_p

$$
\begin{aligned}
\sup_{x\in X} -J_x^{\#}(-p, -d_p) &= \sup_{x\in X}\left\{\int_0^T \langle x'(t), p(t)+d_p\rangle\, dt - \int_0^T L(t, x'(t))\, dt\right\} \\
&\quad - \int_0^T V^*(t, -p'(t))\, dt - l(p(0)) \\
&= \int_0^T L^*(t, p(t)+d_p)\, dt - \int_0^T V^*(t, -p'(t))\, dt - l(p(0)).
\end{aligned}
$$

Let us put, for $p \in X^d$

$$J_D(p,d) = -\int_0^T L^*(t,p(t)+d)\,dt + \int_0^T V^*(t,-p'(t))\,dt + l(p(0)).$$

From (2.3) we infer, for $p \in X^d$, that

$$\sup_{x\in X} -J_x^{\#}(-p,-d_p) = -J_D(p,d_p). \tag{2.4}$$

We can also define a type of the second conjugate of J: for $y \in L^2$, $a \in \mathbb{R}^n$, $x \in X$, $p \in X^d$, put

$$\begin{aligned} J_x^{\#\#}(y,a) = \sup_{p\in X^d} \bigg\{ &\int_0^T \langle y(t), -p'(t)\rangle\,dt + \int_0^T \langle x(t)+c_x, -p'(t)\rangle\,dt \\ &- \int_0^T L(t,x'(t))\,dt - \int_0^T V^*(t,-p'(t))\,dt \bigg\} \\ &+ \inf_{d\in\mathbb{R}^n} \{\langle a,d\rangle + \langle x(T),d\rangle\}. \end{aligned}$$

We assert that $J_x^{\#\#}(0,0) = -J(x,c_x)$. To prove that, we use the special structure of X. First we observe that for each $x \in X$ there exists $\overline{p} \in X^d$ such that $\overline{p}'(\,\cdot\,) = -V_x(\,\cdot\,,x(\,\cdot\,)+c_x)$ and

$$\int_0^T \langle -\overline{p}'(t), x(t)+c_x\rangle\,dt - \int_0^T V^*(t,-\overline{p}'(t))\,dt = \int_0^T \breve{V}(t,x(t)+c_x)\,dt.$$

Next let us note that

$$\begin{aligned} \int_0^T \langle -\overline{p}'(t),&x(t)+c_x\rangle\,dt - \int_0^T V^*(t,-\overline{p}'(t))\,dt \\ &\le \sup_{p\in X^d} \bigg\{ \int_0^T \langle -p'(t), x(t)+c_x\rangle\,dt - \int_0^T V^*(t,-p'(t))\,dt \bigg\} \\ &= \sup_{p'\in L^2} \{ \int_0^T \langle -p'(t), x(t)+c_x\rangle\,dt - \int_0^T V^*(t,-p'(t))\,dt \bigg\} \\ &= \int_0^T \breve{V}(t,x(t)+c_x)\,dt. \end{aligned}$$

Hence we see that, for $x \in X$,

$$J_x^{\#\#}(0,0) = -\int_0^T (-V(t,x(t)+c_x)+L(t,x'(t)))\,dt - l(x(T)) = -J(x,c_x). \tag{2.5}$$

We easily compute (see (2.4))

$$\begin{aligned} \sup_{x\in X} J_x^{\#\#}(0,0) &= \sup_{x\in X}\sup_{p\in X^d} -J_x^{\#}(-p,-d_p) = \sup_{p\in X^d}\sup_{x\in X} -J_x^{\#}(-p,-d_p) \\ &= \sup_{p\in X^d} -J_D(p,d_p) = \sup_{p\in X^d}\inf_d -J_D(p,d) \end{aligned} \tag{2.6}$$

where the last equality is a consequence of the following lemma.

LEMMA 2.1. *For any $p \in X^d$ which corresponds to $x \in X$ with $x(0) = x(T) = 0$ the constant d_p from the specification of X^d is a minimizer of the functional*

$$d \to \int_0^T L^*(t, p(t) + d)\, dt.$$

PROOF. From the definition of X^d we have $p(t) + d_p = L_x{}'(t, x'(t))$ a.e. in $[0,T]$ for some $x \in X$. This means, that $x'(t) = L_p^*(t, p(t) + d_p)$ a.e. in $[0,T]$. Integrating this equality yields, since x is periodic and L^* convex, the assertion of the lemma. □

We shall need one more

LEMMA 2.2. *For any $p \in A$ such that $p'(t) = -V_x(t, x(t) + c_x)$, for $x \in X$, and c_x being a minimizer of the functional $c \to \int_0^T V(t, x(t) + c)\, dt$ we have $p(0) = p(T)$.*

PROOF. The assumption of the lemma yields that $\int_0^T V_x(s, x(s) + c_x)\, ds = 0$ and the proof is completed. □

Hence, from above and (2.6) we obtain the following duality principle.

THEOREM 2.3. *For functionals J and J_D we have the duality relation*

$$\inf_{x \in X} \sup_c J(x, c) = \inf_{p \in X^d} \sup_d J_D(p, d). \tag{2.7}$$

Denote by $\partial J_x(y, a)$ the subdifferential of J_x with respect to the *first* variable. In particular, if $1/q + 1/q' = 1$ then

$$\partial J_{\overline{x}}(0,0) = \left\{ q \in L^{q'} : \int_0^T V^*(t, q(t))\, dt + \int_0^T V(t, \overline{x}(t) + c_{\overline{x}})\, dt \right.$$
$$\left. = \int_0^T \langle q(t), \overline{x}(t) + c_{\overline{x}} \rangle\, dt \right\}.$$

The next result formulates a variational principle for "minmax" arguments.

THEOREM 2.4. *Let $\overline{x} \in X$ be such that*

$$\infty > J(\overline{x}, c_{\overline{x}}) = \inf_{x \in X} \sup_c J(x, c) > -\infty$$

and let the set $\partial J_{\overline{x}}(0,0)$ be nonempty. Then there exist $-\overline{p}' \in \partial J_{\overline{x}}(0,0)$ with $\overline{p}(t) = -\int_t^T \overline{p}'(s)\, ds$ belonging to X^d, such that $\overline{p}$ together with $d_{\overline{p}}$ satisfies

$$J_D(\overline{p}, d_{\overline{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d).$$

Furthermore

$$J_{\overline{x}}(0,0) + J_{\overline{x}}^{\#}(-\overline{p}, -d_{\overline{p}}) = 0, \tag{2.8}$$

$$J_D(\overline{p}, d_{\overline{p}}) - J_{\overline{x}}^{\#}(-\overline{p}, -d_{\overline{p}}) = 0. \tag{2.9}$$

Proof. By Theorem 2.3 to prove the first assertion it suffices to show that $J(\overline{x}, c_{\overline{x}}) \geq J_D(\overline{p}, d_{\overline{p}})$. We note, from the form of $J(x)$ and the finiteness of $J(\overline{x}, c_{\overline{x}})$ that $\overline{x}(T) = \overline{x}(0) = 0$. Let us observe that $-\overline{p}' \in \partial J_{\overline{x}}(0,0)$ means, in fact, that $-\overline{p}'(t) = V_x(t, \overline{x}(t) + c_{\overline{x}})$ a.e. $t \in [0,T]$. By Lemma 2.2 each primitive of $\overline{p}'$ is a periodic function. Since $\overline{x} \in X$ hence there exists an $\widetilde{x} \in X$ such that

$$\begin{aligned}\overline{p}(t) &= \int_t^T V_x(s, \overline{x}(s) + c_{\overline{x}})\,ds \\ &= \int_t^T -\frac{d}{ds} L_{x'}(s, \widetilde{x}'(s))\,ds = L_{x'}(t, \widetilde{x}'(t)) - L_{x'}(T, \widetilde{x}'(T)).\end{aligned}$$

Putting $d_{\overline{p}} = L_{x'}(T, \widetilde{x}'(T))$ we have that $\overline{p}(t) = L_{x'}(t, \widetilde{x}'(t)) - d_{\overline{p}}$ belongs to X^d. Hence, we have

$$\begin{aligned}-J(\overline{x}, c_{\overline{x}}) &= \int_0^T (V(t, \overline{x}(t) + c_{\overline{x}}) - L(t, \overline{x}'(t)))\,dt \\ &= \int_0^T (-V^*(t, -\overline{p}'(t)) - L(t, \overline{x}'(t)))\,dt + \int_0^T \langle \overline{x}(t) + c_{\overline{x}}, -\overline{p}'(t)\rangle\,dt \\ &\leq \int_0^T (-V^*(t, -\overline{p}'(t)) + L^*(t, \overline{p}(t) + d_{\overline{p}}))\,dt = -J_D(\overline{p}, d_{\overline{p}}).\end{aligned}$$

Therefore, $J(\overline{x}, c_{\overline{x}}) \geq J_D(\overline{p}, d_{\overline{p}})$, and since, by Lemma 2.1 $d_{\overline{p}}$ is a minimizer of the functional $d \to \int_0^T L^*(t, \overline{p}(t) + d)\,dt$ we also have

$$J(\overline{x}, c_{\overline{x}}) = J_D(\overline{p}, d_{\overline{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d).$$

Thus the first assertion is proved.

The second assertion is a simple consequence of two facts: $J_{\overline{x}}(0,0) = -J(\overline{x}, c_{\overline{x}})$ so $J_{\overline{x}}(0,0) + J(\overline{x}, c_{\overline{x}}) = 0$ and $-\overline{p}' \in \partial J_{\overline{x}}(0,0)$ i.e. $J_{\overline{x}}(0,0) + J_{\overline{x}}^{\#}(-\overline{p}, -d_{\overline{p}}) = 0$ and so equality (2.8). Then equality (2.9) we get joining the last equality and the equality $J(\overline{x}, c_{\overline{x}}) = J_D(\overline{p}, d_{\overline{p}})$.

From equations (2.8), (2.9) we are able to derive a dual to (1.1) Euler–Lagrange equations. □

Corollary 2.5. *Let $\overline{x} \in X$ be such that*

$$\infty > J(\overline{x}, c_{\overline{x}}) = \inf_{x \in X} \sup_c J(x, c) > -\infty.$$

Then there exists $\overline{p} \in X^d$ suchthatthepair $(\overline{x}, \overline{p})$ satisfies the relations

$$-\overline{p}'(t) = V_x(t, \overline{x}(t) + c_{\overline{x}}), \tag{2.10}$$

$$\overline{p}(t) + d_{\overline{p}} = L_{x'}(t, \overline{x}'(t)), \tag{2.11}$$

$$J_D(\overline{p}, d_{\overline{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d) = \inf_{x \in X} \sup_c J(x, c) = J(\overline{x}, c_{\overline{x}}). \tag{2.12}$$

PROOF. Since $\overline{x} \in X$ therefore $\partial J_{\overline{x}}(0,0)$ is nonempty and so the existence of $\overline{p}'$ in Theorem 2.4 is now obvious. Equations (2.8) and (2.9) imply

$$\begin{aligned} \int_0^T V(t,\overline{x}(t)+c_{\overline{x}})\,dt + \int_0^T V^*(t,-\overline{p}'(t))\,dt - \int_0^T \langle \overline{x}(t)+c_{\overline{x}}, -\overline{p}'(t)\rangle\,dt &= 0, \\ \int_0^T L^*(t,\overline{p}(t)+d_{\overline{p}})\,dt + \int_0^T L(t,\overline{x}'(t))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t)+d_{\overline{p}}\rangle\,dt &= 0, \end{aligned}$$

and then (2.10), (2.11). Relations (2.12) are a direct consequence of Theorems 2.3 and 2.4. □

As a direct consequence of the above corollary and definition of X^d we have

COROLLARY 2.6. *By the same assumptions as in Corollary* 2.5 *there exists a pair* $(\overline{x},\overline{p}) \in X \times X^d$ *satisfying, together with* $(c_{\overline{x}}, d_{\overline{p}})$ *relations* (2.12)*, and the pair* $(\overline{x}+c_{\overline{x}}, \overline{p}+d_{\overline{p}})$ *is a solution of* (1.1)*. Conversely, each pair* $(\overline{x},\overline{p})$ *satisfying, together with* $(c_{\overline{x}}, d_{\overline{p}})$*, relations* (2.12) *satisfies also equations* (2.10), (2.11).

3. Variational principles and a duality gap for minimizing sequences

In this section we show that a statement similar to Theorem 2.4 is true for a minimizing sequence of J.

THEOREM 3.1. *Let* $\{(x_j, c_{x_j})\}$, $x_j \in X$, $j = 1,2,\ldots$, *be a minimizing sequence for* J *and let*

$$\infty > J(x_j, c_{x_j}) > -\infty \quad \text{for } j = 1,2,\ldots$$

Then there exist $-p_j' \in \partial J_{x_j}(0,0)$ *with* $p_j \in X^d$*, such that* $\{(p_j, d_{p_j})\}$ *is a minimizing sequence for* J_D *i.e.*

$$\inf_{x_j\in X} J(x_j, c_{x_j}) = \inf_{x_j \in X} \sup_{c\in\mathbb{R}^n} J(x_j, c) = \inf_{p_j\in X^d} \sup_{d\in\mathbb{R}^n} J_D(p_j, d) = \inf_{p_j \in X^d} J_D(p_j, d_{p_j}).$$

Furthermore

$$\begin{aligned} J_{x_j}(0,0) + J^{\#}_{x_j}(-p_j, -d_{p_j}) &= 0, \\ J_D(p_j, d_{p_j}) - J^{\#}_{x_j}(-p_j, -d_{p_j}) &\le \varepsilon, \\ 0 \le J(x_j, c_{x_j}) - J_D(p_j, d_{p_j}) &\le \varepsilon, \end{aligned}$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

PROOF. We have that $\infty > \inf_{x_j\in X} J(x_j, c_{x_j}) = a > -\infty$, and therefore we may assume $x_j(0) = x_j(T)$. Thus, for a given $\varepsilon > 0$ there exists j_0 such that $J(x_j, c_{x_j}) - a < \varepsilon$, for all $j \ge j_0$. Further, the proof is similar to that of Theorem 2.4, so we only sketch it. First we observe that $\partial J_{x_j}(0,0)$ is nonempty for $j \ge j_0$ and $-p_j' \in \partial J_{x_j}(0,0)$ implies that $\int_0^T p_j'(t)\,dt = 0$. Accordingly to the definition of X^d let us take as a primitive of p_j' such p_j that $p_j(T) = 0$ and in fact $p_j(0) = 0$. Therefore, for all $d \in \mathbb{R}^n$, we also have

$$\begin{aligned}
-J(x_j, c_{x_j}) &= \int_0^T (V(t, x_j(t) + c_{x_j}) - L(t, x_j'(t)))\, dt \\
&= \int_0^T (-V^*(t, -p_j'(t)) - L(t, x_j'(t)))\, dt + \int_0^T \langle x_j(t) + c_{x_j}, -p_j'(t)\rangle\, dt \\
&\leq \int_0^T (-V^*(t, -p_j'(t)) + L^*(t, p_j(t) + d))\, dt = -J_D(p_j, d).
\end{aligned}$$

Hence, due to Theorem 2.3,

$$a + \varepsilon \geq \sup_{d \in \mathbb{R}^n} J_D(p_j, d) = J_D(p_j, d_{p_j}) \geq a \quad \text{for } j \geq j_0.$$

The second assertion is a simple consequence of two facts:

$$J_{x_j}(0, 0) = -J(x_j, c_{x_j})$$

so

$$J_{x_j}(0, 0) + J(x_j, c_{x_j}) = 0 \quad \text{and} \quad -p_j' \in \partial J_{x_j}(0, 0)$$

i.e. $J_{x_j}(0, 0) + J^{\#}_{x_j}(-p_j, -d_{p_j}) = 0$. □

A direct consequence of this theorem is the following corollary.

Corollary 3.2. *Let $\{(x_j, c_{x_j})\}$, $x_j \in X$, $j = 1, 2, \ldots$, be a minimizing sequence for J and let $\infty > J(x_j, c_{x_j}) > -\infty$, for $j = 1, 2, \ldots$ If*

$$-p_j'(t) = V_x(t, x_j(t) + c_{x_j})$$

then $p_j(t) = -\int_t^T p_j'(s)\, ds$ belongs to X^d and $\{(p_j, d_{p_j})\}$ is a minimizing sequence for J_D i.e.

$$\inf_{x_j \in X} J(x_j, c_{x_j}) = \inf_{x_j \in X} \sup_{c \in \mathbb{R}^n} J(x_j, c) = \inf_{p_j \in X^d} \sup_{d \in \mathbb{R}^n} J_D(p_j, d) = \inf_{p_j \in X^d} J_D(p_j, d_{p_j}).$$

Furthermore

$$\begin{aligned}
&J_D(p_j, d_{p_j}) - J^{\#}_{x_j}(-p_j, -d_{p_j}) \leq \varepsilon, \\
&0 \leq J(x_j, c_{x_j}) - J_D(p_j, d_{p_j}) \leq \varepsilon,
\end{aligned} \tag{3.1}$$

for a given $\varepsilon > 0$ and sufficiently large j.

4. The existence of "maxmin"

The last problem which we have to solve is to prove the existence of $\overline{x} \in X$ such that

$$J(\overline{x}, c_{\overline{x}}) = \min_{x \in X} \max_{c \in \mathbb{R}^n} J(x, c).$$

To obtain this it is enough to use hypothesis (H1), the results of the former section and known compactness theorems.

Theorem 4.1. *Under hypothesis* (H1) *there exists* $\overline{x} \in X$ *such that*

$$J(\overline{x}, c_{\overline{x}}) = \min_{x\in X} \max_{c\in\mathbb{R}^n} J(x,c).$$

Proof. Let us observe that by (H1) for each $x \in X$ there exists c_x such that $\max_{c\in\mathbb{R}^n} J(x,c) = J(x,c_x)$. Next, we observe, that by the definition of $\overline{X}$, $J(x,c_x)$ is bounded below on X as well as that the sets $S_b = \{(x,c_x) : x \in X, c_x \in \mathbb{R}^n,\ J(x,c_x) \le b\}$, $b \in \mathbb{R}$ are nonempty for sufficiently large b and bounded with respect to the norm $|c_x| + \|x'\|_{L^2}$. The last means that S_b, $b \in \mathbb{R}$ are relatively weakly compact in $A_0 \oplus \mathbb{R}^n$. It is a well known fact that the functional J is weakly lower semicontinuous in $A_0 \oplus \mathbb{R}^n$ and thus also in $X \oplus \mathbb{R}^n$. Therefore there exists a sequence $\{x_n\}$, $x_n \in X$, such that $x_n \rightharpoonup \overline{x}$ weakly in A_0 with $\overline{x} \in A_0$, together with $c_{x_n} \to c_{\overline{x}} \in \mathbb{R}^n$, and $\liminf_{n\to\infty} J(x_n, c_{x_n}) \ge J(\overline{x}, c_{\overline{x}})$. Moreover, we know that $\{(x_n, c_{x_n})\}$ is uniformly convergent to $(\overline{x}, c_{\overline{x}})$. In order to finish the proof we must only show that $\overline{x} \in X$.

To prove that we apply the duality results of Section 3. To this effect let us recall from Corollary 3.2 that for

$$p_n'(t) = -V_x(t, x_n(t) + c_{x_n}) \tag{4.2}$$

$p_n(t) = -\int_t^T p_n'(s)\,ds$ belongs to X^d and take d_{p_n} such that $\max_{d\in\mathbb{R}^n} J_D(p_n,d) = J_D(p_n, d_{p_n})$. Then $\{(p_n, d_{p_n})\}$ is a minimizing sequence for J_D. We easily check that $\{d_{p_n}\}$ is a bounded sequence and therefore we may assume (up to a subsequence) that it is convergent. From the fact that $\{p_n'\} \subset \overline{X}^d$ and (4.2) we infer that $\{p_n'\}$ is a bounded sequence in $L^{q'}$ norm and that it is pointwise convergent to

$$\overline{p}'(t) = -V_x(t, \overline{x}(t) + c_{\overline{x}}) \tag{4.3}$$

and so $\{p_n\}$ is uniformly convergent to $\overline{p}$ where $\overline{p}(t) = -\int_t^T \overline{p}'(s)\,ds$. We can choose $d_{\overline{p}}$ satisfying equality: $\max_{d\in\mathbb{R}^n} J_D(\overline{p}, d) = J_D(\overline{p}, d_{\overline{p}})$.

By Corollary 3.2 (see (3.1)) we also have (taking into account (4.2)) that for $\varepsilon_n \to 0$ ($n \to \infty$)

$$0 \le \int_0^T (L^*(t, p_n(t) + d_{p_n}) + L(t, x_n'(t)))\,dt - \int_0^T \langle x_n'(t), p_n(t) + d_{p_n}\rangle\,dt \le \varepsilon_n$$

and so, taking a limit

$$0 = \int_0^T L^*(t, \overline{p}(t) + d_{\overline{p}})\,dt + \lim_{n\to\infty} \int_0^T L(t, x_n'(t))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t) + d_{\overline{p}}\rangle\,dt$$

and next, in view of the property of Fenchel inequality,

$$0 = \int_0^T L^*(t, \overline{p}(t) + d_{\overline{p}})\,dt + \int_0^T L(t, \overline{x}'(t)))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t) + d_{\overline{p}}\rangle\,dt. \tag{4.4}$$

Applying now Ekeland's variational principle (see [4]) to the ε-subdifferential of $\int_0^T L(t, x_n'(t))\,dt$ at $x_n'(\cdot)$ we deduce that $\{x_n'\}$ is strongly convergent in L^2 to $\overline{x}'$. Therefore, as $x_n \in \overline{X}$, $\overline{x}$ belongs to $\overline{X}$. From (4.4) we also have

$$\overline{p}(t) + d_{\overline{p}} = L_{x'}(t, \overline{x}'(t)). \tag{4.5}$$

Joining (4.3) and (4.5) we get

$$\frac{d}{dt} L_{x'}(t, \overline{x}'(t)) = -V_x(t, \overline{x}(t) + c_{\overline{x}}).$$

The last means that $\overline{x} \in X$ and so the proof is completed. □

A direct consequence of Theorem 4.1 and Corollary 2.6 is the following main theorem.

Theorem 4.2. *Assume hypotheses* (H), (H1) *and* (HX). *Then there exists a pair* $(\overline{x} + c_{\overline{x}}, \overline{p} + d_{\overline{p}})$ *being a solution of* (1.1) *and such that*

$$J(\overline{x}, c_{\overline{x}}) = \min_{x\in X} \max_{c\in\mathbb{R}^n} J(x, c) = \min_{p\in X^d} \max_{d\in\mathbb{R}^n} J_D(p, d) = J_D(\overline{p}, d_{\overline{p}}).$$

5. Example

Let us denote by P the positive cone in $\mathbb{R}^n$ i.e. $P = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$ and by $\overline{P} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$. We say that $x \geq y$ for $x, y \in \mathbb{R}^n$ if $x - y \in \overline{P}$.

Consider the problem

$$\begin{gathered} k(t)x''(t) + V_x(t, x(t)) = 0, \quad \text{a.e. in } \mathbb{R}, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{gathered} \tag{5.1}$$

where $V(\cdot, x)$ is a T-periodic, measurable function in R, $V(t, \cdot)$ is Gateaux differentiable function. In the notation of the paper we have $L(t, x') = k(t)|x'|^2/2$. If $b, c \in \mathbb{R}^n$ by bc we always mean a vector $[b_i c_i]_{i=1,\dots,n}$. We set the basic hypotheses we need:

- (H1') the function k is absolutely continuous, positive and $V_x(t, \cdot)$ is continuous and nonnegative in P, (i.e. $V_x(t, v) \in \overline{P}$) for $t \in [0, T]$, and $\int_0^T (1/k(t)) V_x(t, 0)\,dt \neq 0$,
- (H2') for a given $\theta \in P$, there exists $v \in P$ such that

$$\int_0^T V_x(t, \beta v)\,dt \leq \theta v, \tag{5.2}$$

  where $\theta v = [\theta_i v_i]_{i=1,\dots,n}$, $\beta v = [\beta_i v_i]_{i=1,\dots,n}$, and $\beta = \theta \int_0^T (1/k(r))\,dr$,
- (H3') there exist $l, l_1 \in L^2([0, T], R)$ such that

$$\begin{aligned} &(t \to \sup\{V(t, x) : x \in \overline{P},\ x \leq \beta v\}) \leq l(t), \quad t \in [0, T], \\ &(t \to \sup\{V_{x_i}(t, x) : x \in \overline{P},\ x \leq \beta v\}) \leq l_1(t), \quad t \in [0, T],\ i = 1, \dots, n, \end{aligned}$$

moreover, V satisfies in $\overline{X} = \{x \in A_0 : x' \in A,\ x(t) \in \overline{P}, x(t) \le \beta v,\ t \in [0,T]\}$ the growth conditions: there exist $0 < \beta_1,\ q_1 > 1,\ k_1 \in \mathbb{R}$ such that for each $w \in \overline{X}$ and for all $c \in \mathbb{R}^n$

$$k_1 + \beta_1 |c|^{q_1} \le \int_0^T V(t, w(t) + c)\, dt. \tag{5.2}$$

Note that (5.2) implies that if c_x is a minimizer of the functional $c \to \int_0^T V(t, x(t) + c)\, dt$ for $x \in \overline{X}$ then

$$|c_x| \le \left(\frac{1}{\beta_1}\left(\int_0^T l(t)\, dt - k_1\right)\right)^{1/q_1} = b.$$

(H4') $V(t, \cdot)$ is convex in the set $B + D$ for $t \in [0,T]$, where $B = \{c : |c| \le b\}$, $D = \{x \in \mathbb{R}^n : x \in \overline{P},\ x \le \beta v\}$.

We would like to stress that because of (H1') and (H4') each function

$$x_j \to V_{x_i}(t, (x_1, \dots, x_j, \dots x_n)), \quad i = 1, \dots, n,\ j = 1, \dots, n,\ t \in [0,T]$$

is increasing if $(x_1, \dots, x_j, \dots x_n)$ lies in $B + D$. Moreover, if c_x is a minimizer of the functional $c \to \int_0^T V(t, x(t) + c)\, dt$ for $x \in \overline{X}$ then $\int_0^T V_x(s, x(s) + c_x)\, ds = 0$. Hence and because of (H1') $c_x \in -\overline{P}$.

It is easily seen that assumptions (H) and (H1) are satisfied. Therefore, what we have to do is to construct a nonempty set X. We prove that $\overline{X}$ is our set X. To this effect let us define in $\overline{X}$ an operator

$$\mathrm{A}\, x(t) = \int_0^t \frac{1}{k(r)}\left(\int_r^T V_x(s, x(s) + c_x)\, ds - a_x\right) dr, \tag{5.4}$$

where $a_x = (a_x^1, \dots, a_x^n)$ and

$$a_x^i = \max\left\{0, \min\left\{\int_r^T V_x(s, x(s) + c_x)\, ds : r \in [0,T]\right\}\right\}.$$

Then $\mathrm{A}\, x(t) \ge 0$ (i.e. $\mathrm{A}\, x(t) \in \overline{P}$), $t \in [0,T]$ and

$$\begin{aligned}\mathrm{A}\, x(t) &\le \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, x(s) + c_x)\, ds\, dr \le \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, x(s))\, ds\, dr \\ &\le \int_0^T \frac{1}{k(r)} \int_0^T V_x(s, \beta v)\, ds\, dr \le \int_0^T \frac{1}{k(r)}\, dr \theta v = \beta v.\end{aligned}$$

Hence $\mathrm{A}\, x \in \overline{X}$. We observe that if we take $\widetilde{p}(t) = k(t)(\mathrm{A}\, x(t))' + a_x$ then, by (5.3), $-\widetilde{p}'(t) = V_x(t, x(t) + c_x)$. It is clear that $\overline{X}$ contains at least one element w such that $w(0) = w(T) = 0$. What we still have to check is the relation (1.5). By (H4') $V(t, \cdot)$ is convex and by (H1') it is continuously differentiable. However subdifferential is a global notion thus we need to extend convexity of $V(t,\,\cdot\,)$ to the whole space. To this effect let us define

$$\check{V}(t,x) = \begin{cases} V(t,x) & \text{if } x \in B + D,\ t \in [0,T], \\ \infty & \text{if } x \notin B + D,\ t \in [0,T]. \end{cases}$$

As our all investigation reduce to the set $B + D$, therefore $\check{V} = V$ in it. We need this notation only for the purpose of duality in Section 2. Of course (1.5) is satisfied for $\check{V}$ in $\overline{X}$. Therefore $\overline{X}$ is our set X and *problem* (5.1) *has at least one nonzero* (because of (H1')) *periodic solution*.

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