Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 131–145

THE RELATIVE REIDEMEISTER NUMBERS OF FIBER MAP PAIRS

Fernanda S. P. Cardona — Peter N.-S. Wong

Dedicated to the memory of Helga Schirmer

Abstract. The relative Reidemeister number, denoted by $\mathrm{R}(f;X,A)$, is an upper bound for the relative Nielsen number, denoted by $\mathrm{N}(f;X,A)$). If (f,f_A) is a pair of fibre-preserving maps of a pair of Hurewicz fibrations, then under certain conditions, the relative Reidemeister number can be calculated in terms of those on the base and on the fiber. In this paper, we give addition formulas for $\mathrm{R}(f;X,A)$ and for the relative Reidemeister number on the complement $\mathrm{R}(f;X-A)$. As an application, we give estimation of the asymptotic Nielsen type number $\mathrm{NI}^\infty(f)$.

1. Introduction

For a selfmap $f\colon X\to X$ of a compact connected polyhedron, the Nielsen number $\mathrm{N}(f)$ provides a lower bound for the minimal number of fixed points in the homotopy class of f. If X is a manifold of dimension at least 3, this lower bound is sharp. The computation of $\mathrm{N}(f)$ is in general difficult and therefore is one of the central problems in Nielsen fixed point theory.

Let $p\colon E\to B$ be a Hurewicz fibration and $f\colon E\to E$ a fiber-preserving map with induced map $\overline{f}\colon B\to B$. R. Brown in [1] initiated the study of the Nielsen fixed point theory for fiber-preserving maps and gave conditions for which $\mathrm{N}(f)=$

2000 *Mathematics Subject Classification*. Primary 55M20; Secondary 57T15.

Key words and phrases. Relative Nielsen number, relative Reidemeister number, fixed points of pairs, fiber-preserving maps, asymptotic Nielsen type number.

This work was conducted in part during the first author's visit to Bates College, supported by a grant from FAPESP of Brasil.

$\mathrm{N}(f_b)\mathrm{N}(\overline{f})$, where f_b denotes the restriction of f on the fiber over $b \in \mathrm{Fix}(\overline{f})$. Such a product formula was further studied by Fadell [4]; necessary and sufficient conditions for the validity of this naïve product formula were given by You [15].

One of the standing assumptions in these works is that the fibration p be orientable in the sense of [15]. This turns out to be rather restrictive. By relaxing the orientability of p, a naïve addition formula, rather than a product formula has been obtained by Heath, Keppelmann and Wong [7]. The main objective of these formulas is to compute the Nielsen number of f in terms of possibly simpler Nielsen type invariants of $\overline{f}$ and of f_b.

In 1986, H. Schirmer (see [12]) introduced a Nielsen type theory for maps of pairs. More precisely, for any map $f\colon (X,A) \to (X,A)$ of a compact polyhedral pair (X,A), a Nielsen type number $\mathrm{N}(f;X,A)$ was introduced and a Wecken type theorem was proven. This relative Nielsen theory has since prompted many subsequent works by Schirmer and others. By definition, $\mathrm{N}(f;X,A)$ is defined in terms of f and of $f_A\colon A \to A$ so the computation of $\mathrm{N}(f;X,A)$ is more difficult than that of the (absolute) Nielsen number. In his thesis [13], A. Schusteff established the naïve product formula for the relative Nielsen number of a fiber preserving map of pairs. More precisely, given a commutative diagram

$$\begin{array}{ccc} (E,E_0) & \xrightarrow{f,f_0} & (E,E_0) \\ {\scriptstyle p}\big\downarrow{\scriptstyle p_0} & & {\scriptstyle p}\big\downarrow{\scriptstyle p_0} \\ (B,B_0) & \xrightarrow[\overline{f},\overline{f_0}]{} & (B,B_0) \end{array}$$

conditions were given to ensure that the naïve product formula holds, i.e.

$$\mathrm{N}(f;E,E_0) = \mathrm{N}(f_b;F_b,F_{0b})\mathrm{N}(\overline{f};B,B_0),$$

where $E_0 \xrightarrow{p_0} B_0$ is a sub-fibration of a Hurewicz fibration $E \xrightarrow{p} B$ and F_b, F_{0b} are the fibers of p and p_0, respectively, over $b \in \mathrm{Fix}(\overline{f})|_{B_0}$. Furthermore, a relative Reidemeister number $\mathrm{R}(f;E,E_0)$ was introduced in [13] to give computational results when the spaces are Jiang spaces. Subsequently in [2], the basic properties of the relative Reidemeister number were established and a relative Reidemeister number on the complement, analogous to the Nielsen type invariant of Zhao (see [16]), was introduced.

The main objective of this paper is to compute the relative Reidemeister number $\mathrm{R}(f;E,E_0)$ and the relative Reidemeister number on the complement $\mathrm{R}(f;E-E_0)$ of fiber-preserving maps of pairs. We obtain addition formulas for these homotopy invariants. Another motivation for computing $\mathrm{R}(f;E-E_0)$ is its connection with an asymptotic Nielsen type invariant introduced by Jiang (see [9] and [14]). As an application, we estimate the asymptotic Nielsen type number $\mathrm{NI}^\infty(f)$ when f is a fiber-preserving map on a compact polyhedron.

This paper is organized as follows. We present an addition formula for the absolute mod K Reidemeister number in Section 2. The main results of the paper are contained in Section 3 in which $\mathrm{R}(f; E, E_0)$ and $\mathrm{R}(f; E - E_0)$ are computed or estimated in terms of the relative Reidemeister numbers of $\overline{f}$ and of f_b, generalizing some results of Schusteff. In the final section, we make use of the relative Reidemeister number on the complement and equivariant fixed point theory to give an estimate for $\mathrm{NI}^\infty(f)$.

The authors wish to thank Robert F. Brown for his careful reading of an earlier version of the manuscript. In particular, the first author wishes to thank Robert F. Brown for his help and comments during the first months of her visit to the U.S., as well as the staff of the Math Dept. at UCLA. Also, she could not let this Introduction finish without thanking the invitation to Bates College and all the people in the College that made the rest of her trip a very productive one.

2. Addition formulas for the mod K Reidemeister number of fiber-preserving maps

In [6], an addition formula for the Reidemeister number for coincidences was given via an algebraic approach. We will adapt what was done there for our fixed-point settings. Throughout, we will let π_X denote the group of deck transformations of the universal cover of X and thus π_X is also identified with $\pi_1(X)$ with an appropriate basepoint. Given a group homomorphism $\varphi\colon \pi_X \to \pi_X$, we have the so-called *Reidemeister action* of π_X on π_X, given by $\sigma \cdot \alpha = \sigma\alpha\varphi(\sigma^{-1})$. The Reidemeister classes are the orbits of this action, and the set of Reidemeister classes is denoted by $\mathcal{R}(\varphi, \pi_X)$. The *Reidemeister number* of f is given by $\mathrm{R}(f) = \#\mathcal{R}(\varphi, \pi_X)$, where $\#$ denotes the cardinality.

Let (E, p, B) be a Hurewicz fibration with the typical fiber $F = p^{-1}(b)$ for some fixed $b \in B$, with all spaces being 0-connected. Let $f\colon E \to E$ be a fiber preserving map. Suppose $K = \ker i_\#$, where $i_\#\colon \pi_F \to \pi_E$ is induced by $i\colon F \hookrightarrow E$. We will denote by $i_{\#_K}$ the induced map on the quotient, $i_{\#_K}\colon \pi_F/K \to \pi_E$. Also, the set of orbits of the Reidemeister action of φ' in π_F/K will be denoted by $\mathcal{R}_K(\varphi', \pi_F)$. Moreover, we can suppose, without loss of generality, that $i_{\#_K}$ is the inclusion map (notice that $\pi_F/K \cong \operatorname{Im} i_{\#_K} < \pi_E$).

Now, we give a general addition formula for the Reidemeister number for a fiber-preserving map. In what follows we will indicate the conjugation map by $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$, without being explicit where it is defined, since the context will make it clear.

Theorem 2.1. *Let (E, p, B) be a fibration as defined as above; let f be a fiber-preserving map. Then, there is a one-to-one correspondence between the* *sets*

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}\mathcal{R}_K(\tau_\alpha\varphi',\pi_F),$$

where $\widehat{i_{\alpha K}}$ *is induced by* $i_{\#K}$ *defined above, for any* $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$. *If the cardinalities of the sets involved are finite, we have*

$$\mathrm{R}(f) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]}$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ *and* $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$.

Proof. Consider the following commutative diagram of homomorphisms and groups, whose rows are short exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & \pi_F/K \arrow[r, "i_{\#K}"] \arrow[d, "\varphi'"'] & \pi_E \arrow[r, "p_\#"] \arrow[d, "\varphi"'] & \pi_B \arrow[r] \arrow[d, "\overline{\varphi}"'] & 1 \\
1 \arrow[r] & \pi_F/K \arrow[r, "i_{\#K}"'] & \pi_E \arrow[r, "p_\#"'] & \pi_B \arrow[r] & 1
\end{tikzcd}$$

This induces the following sequence, which is exact at $\mathcal{R}(\varphi,\pi_E)$ and $\mathcal{R}(\overline{\varphi},\pi_B)$,

$$\mathcal{R}_K(\varphi',\pi_F) \xrightarrow{\widehat{i_K}} \mathcal{R}(\varphi,\pi_E) \xrightarrow{\widehat{p}} \mathcal{R}(\overline{\varphi},\pi_B) \longrightarrow 1 \tag{2.1}$$

i.e. $\widehat{p}$ is surjective and $\mathrm{Im}\,\widehat{i_K} = \widehat{p}^{-1}[\overline{e}]$, where $\overline{e}$ is the identity element of π_B.

Similarly, we have the following commutative diagram of homomorphisms and groups, whose rows are short exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & \pi_F/K \arrow[r, "i_{\#K}"] \arrow[d, "\tau_\alpha\varphi'"'] & \pi_E \arrow[r, "p_\#"] \arrow[d, "\tau_\alpha\varphi"'] & \pi_B \arrow[r] \arrow[d, "\tau_{\overline{\alpha}}\overline{\varphi}"] & 1 \\
1 \arrow[r] & \pi_F/K \arrow[r, "i_{\#K}"'] & \pi_E \arrow[r, "p_\#"'] & \pi_B \arrow[r] & 1
\end{tikzcd}$$

A sequence similar to (2.1) is given by

$$\mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \xrightarrow{\widehat{i_{\alpha K}}} \mathcal{R}(\varphi,\pi_E) \xrightarrow{\widehat{p_\alpha}} \mathcal{R}(\tau_{\overline{\alpha}}\overline{\varphi},\pi_B) \longrightarrow 1 \tag{2.2}$$

such that $\widehat{p_\alpha}$ is surjective and $\mathrm{Im}\,\widehat{i_{\alpha K}} = \widehat{p_\alpha}^{-1}[\overline{e}]$. Note that there is a natural bijection between $\mathcal{R}(\varphi,\pi_E)$ and $\mathcal{R}(\tau_\alpha\varphi,\pi_E)$ by sending the class of β in $\mathcal{R}(\varphi,\pi_E)$ to the class of $\beta\tau^{-1}$ in $\mathcal{R}(\tau_\alpha\varphi,\pi_E)$. This bijection in turn gives a one-to-one correspondence between $\widehat{p}^{\,-1}[\overline{\alpha}]$ and $\widehat{p_\alpha}^{\,-1}[\overline{e}]$. (For coincidences, see [6].)

It follows that we have the 1-1 bijections

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p}^{\,-1}([\overline{\alpha}]) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}(\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)),$$

for any $[\alpha] \in \widehat{p}^{\,-1}[\overline{\alpha}]$. Therefore,

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi, \pi_E) = \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \#\widehat{p}^{\,-1}([\overline{\alpha}]).$$

Consider in $\mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ the equivalence relation given by

$$\beta_1 \sim \beta_2 \Rightarrow \widehat{i_{\alpha K}}(\beta_1) = \widehat{i_{\alpha K}}(\beta_2).$$

Notice that $\#\widehat{p}^{\,-1}([\overline{\alpha}]) = \#\mathcal{R}_K(\tau_\alpha \varphi', \pi_F)/_\sim$, for any $[\alpha] \in \widehat{p}^{\,-1}[\overline{\alpha}]$. Given $\overline{\theta} \in \mathrm{Fix}(\tau_{\overline{\alpha}} \overline{\varphi})$ and $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ define $\overline{\theta}[\beta] = [\theta\beta\tau_\alpha\varphi(\theta^{-1})]$, where $\beta \in [\beta]$ and $p_\#(\theta) = \overline{\theta}$. This is a well-defined action

$$\begin{aligned} \mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) \times \mathcal{R}_K(\tau_\alpha \varphi', \pi_F) &\to \mathcal{R}_K(\tau_\alpha \varphi', \pi_F), \\ (\overline{\theta}, [\beta]) &\mapsto \overline{\theta}[\beta], \end{aligned}$$

of a group on a set. (Notice that $\theta\beta\tau_\alpha\varphi(\theta^{-1}) \in \ker p_\# = \mathrm{Im}\, i_\#$). It is straightforward to verify that two classes in $\mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ are in the same class of $\mathcal{R}(\varphi, \pi_E)$ if, and only if, they are in the same orbit by the action just defined (see also [5, Propositions 1.5–1.7]).

For $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ and $p_\#(\beta\alpha) = \overline{\alpha}$, the isotropy subgroups of this action are $p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))$, therefore the cardinality of the orbits is given by the local index $[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]$. Therefore

$$\#\widehat{p}^{\,-1}([\overline{\alpha}]) = \sum_{[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]},$$

for any $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$. □

Remark 2.2. The existence of the natural map $p_\#^{-1}(\overline{\alpha}) \to \widehat{p}^{\,-1}([\overline{\alpha}])$ allows us to take $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ as in the above formula.

Remark 2.3. Note that if $[\alpha_1] = [\alpha_2]$ as elements of $\mathcal{R}(\varphi, \pi_E)$ then

$$[\mathrm{Fix}(\tau_{\overline{\alpha_1}}\overline{\varphi}) : p(\mathrm{Fix}(\tau_{\alpha_1}\varphi))] = [\mathrm{Fix}(\tau_{\overline{\alpha_2}}\overline{\varphi}) : p(\mathrm{Fix}(\tau_{\alpha_2}\varphi))].$$

Observe that there is $\gamma \in \pi_E$ such that $\alpha_1 = \gamma\alpha_2\varphi(\gamma^{-1})$ and the map

$$\mathrm{Fix}(\tau_{\overline{\alpha_1}}\overline{\varphi}) \to \mathrm{Fix}(\tau_{\overline{\alpha_2}}\overline{\varphi}), \quad b \mapsto \tau_{p(\gamma^{-1})} b$$

is an isomorphism on the respective cosets.

Remark 2.4. To see that the sum above yields an integer, remember that within the orbit of $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ the index $[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \ (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]$ (which counts the "size" of the orbit), for every $[\gamma] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ in the orbit of $[\beta]$, remains the same; and after summing up these fractions over all of $\mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$, we will get in return exactly 1 for each orbit. Thus we end up obtaining the number of orbits.

The following gives the naïve addition as well as the naïve product formulas for the Reidemeister numbers.

Corollary 2.5. *Let* (E,p,B) *be a fibration as described above. Suppose* $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \{\overline{e}\}$ *for all* $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$. *There is a one-to-one correspondence between the sets*

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \mathcal{R}_K(\tau_\alpha\varphi',\pi_F).$$

Also

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi,\pi_E) < \infty \Leftrightarrow \#\mathcal{R}(\overline{\varphi},\pi_B) < \infty \text{ and } \#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F) < \infty$$

for all $[\alpha] \in \mathcal{R}(\varphi,\pi_E)$. *In this case, we have*

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi,\pi_E) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F).$$

If, in addition, $\#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ *is* mod K *fiber uniform, i.e. it is independent of the choice of* $[\alpha] \in \mathcal{R}(\varphi,\pi_E)$, *then we have a product formula*

$$\#\mathcal{R}(\varphi,\pi_E) = \#\mathcal{R}_K(\varphi',\pi_F)\cdot\#\mathcal{R}(\overline{\varphi},\pi_B).$$

Proof. Since $\widehat{i_{\alpha K}}$ is injective (due to the hypothesis $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \{\overline{e}\}$ for all $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$), the sequence (2.2) becomes a short exact sequence.

$$1 \longrightarrow \mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \xrightarrow{\widehat{i_{\alpha K}}} \mathcal{R}(\tau_\alpha\varphi,\pi_E) \xrightarrow{\widehat{p_\alpha}} \mathcal{R}(\tau_{\overline{\alpha}}\overline{\varphi},\pi_B) \longrightarrow 1.$$

Under the mod K fiber uniform condition, the assertions now follow from Theorem 2.1. $\square$

Let $\alpha \in \mathcal{R}(\varphi,\pi_E)$. The *index of* α is simply $\mathrm{index}(f,\eta_E\mathrm{Fix}(\alpha\widetilde{f}))$, the usual fixed point index where $\eta_E\colon\widehat{E} \to E$ denotes the universal covering. So, if the index of α is nonzero then α is said to be *essential*. Denote by $\mathcal{N}(\varphi,\pi_E)$ the set of essential $\alpha \in \mathcal{R}(\varphi,\pi_E)$. Therefore, $\mathrm{N}(f) = \#\mathcal{N}(\varphi,\pi_E)$. Similarly, we denote by $\mathcal{N}_K(\varphi',\pi_F)$ the set of essential $\alpha' \in \mathcal{R}_K(\varphi',\pi_F)$, and $\mathrm{N}_K(f') = \#\mathcal{N}_K(\varphi',\pi_F)$.

Definition 2.6. Let f be a fiber-preserving map, let φ and $\overline{\varphi}$ be the homomorphisms induced by f and $\overline{f}$. We say that f is *locally* (resp. *essentially locally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#\ (\mathrm{Fix}(\tau_\alpha\varphi))]$$

does not depend on $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ (resp. $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}]) \cap \mathcal{N}(\varphi,\pi_E)$). Similarly, we say that f is *globally* (resp. *essentially globally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#\ (\mathrm{Fix}(\tau_\alpha\varphi))]$$

does not depend on $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$ (resp. $\mathcal{N}(\overline{\varphi},\pi_B)$).

3. Addition formulas for Reidemeister numbers of pairs of fiber preserving maps

Given a selfmap f of a polyhedral pair (X, A), the inclusion $i: A \hookrightarrow X$ induces a well-defined function

$$\widehat{i}: \mathcal{R}(\varphi_A, \pi_A) \to \mathcal{R}(\varphi, \pi), \quad [\alpha] \to [i_\# \alpha].$$

The set of *weakly common classes* is given by

$$\mathcal{R}(\varphi, \varphi_A) = \{[\alpha] \in \mathcal{R}(\varphi, \pi) \mid [\alpha] = \widehat{i}([\beta]) \text{ for some } [\beta] \in \mathcal{R}(\varphi_A, \pi_A)\}.$$

The Reidemeister number of f on the complement $X - A$ is defined as

$$\mathrm{R}(f; X - A) = \#(\mathcal{R}(\varphi, \pi) - \mathcal{R}(\varphi, \varphi_A))$$

and, if $\mathrm{R}(f) < \infty$, then

$$\mathrm{R}(f; X - A) = \mathrm{R}(f) - \mathrm{R}(f, f_A) \quad \text{where } \mathrm{R}(f, f_A) = \#\mathcal{R}(\varphi, \varphi_A).$$

The relative Reidemeister number of f on the pair (X, A) is defined as

$$\mathrm{R}(f; X, A) = \mathrm{R}(f_A) + \mathrm{R}(f; X - A).$$

REMARK 3.1. Note that if one of $\mathrm{R}(f), \mathrm{R}(f_A)$ or $\mathrm{R}(f, f_A)$ is infinite then $\mathrm{R}(f; X, A)$ is infinite. For more details on the basic properties of these homotopy invariants, we refere the reader to [2] and [3].

For the rest of this paper, by a *fiber map of the pair*, we mean a pair of fiber preserving maps $(f, f_0): (E, E_0) \to (E, E_0)$ with $f_0 = f|_{E_0}$ where (E_0, p_0, B_0) is a Hurewicz sub-fibration of a Hurewicz fibration (E, p, B). Furthermore, we assume that E, E_0, B, B_0 and the typical fibers are all 0-connected spaces.

Suppose $K_0 = \ker i_{0\#}$, where $i_{0\#}: \pi_{F_0} \to \pi_{E_0}$ is induced by $i_0: F_0 \hookrightarrow E_0$. Denote by $i_{0\#_{K_0}}$ the induced map on the quotient, the set of orbits of the respective Reidemeister action by $\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, and the respective cardinality, $\#\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, by $\mathrm{R}_{K_0}(f_0')$. We will denote the set of orbits of the Reidemeister action of φ' which are in the image of the orbits of the Reidemeister action of φ_0' under $\pi_{F_0}/K_0 \to \pi_F/K$ by $\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and the respective cardinality, $\#\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, by $\mathrm{R}_{K,K_0}(f', f_0')$. Moreover, we can suppose, without loss of generality, that $i_{\#_K}$ is the inclusion map (notice that $\pi_F/K \cong \operatorname{Im} i_{\#_K} < \pi_E$).

We have the following

PROPOSITION 3.2. *Let (f, f_0) be a fiber map of the pair, $K = \ker i_\#$ and $K_0 = \ker i_{0\#}$. Then we have the inclusion $\mathcal{R}(\varphi, \varphi_0) \subseteq \widehat{p}^{\,-1}\mathcal{R}(\overline{\varphi}, \overline{\varphi_0})$. Also, we have the inequalities*

$$\mathrm{R}(f; E - E_0) \geq \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B) - \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}} \overline{\varphi}) : p_\# \ (\mathrm{Fix}(\tau_{\beta\alpha} \varphi))]},$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$ *and* $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$, *and*

$$\begin{aligned}\mathrm{R}(f; E, E_0) \geq & \sum_{[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})} \sum_{[\beta_0]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\beta_0\alpha_0}\varphi_0))]} \\ & + \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B) - \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]},\end{aligned}$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$, $[\beta_0] \in \mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0', \pi_{F_0})$, $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ *and* $[\alpha_0] \in \widehat{p_0}^{\,-1}([\overline{\alpha_0}])$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\varphi_0, \pi_{E_0}) & \xrightarrow{\widehat{i_{E,E_0}}} & \mathcal{R}(\varphi, \pi_E) \\ {\scriptstyle \widehat{p_0}}\downarrow & & \downarrow{\scriptstyle \widehat{p}} \\ \mathcal{R}(\overline{\varphi_0}, \pi_{B_0}) & \xrightarrow[\widehat{i_{B,B_0}}]{} & \mathcal{R}(\overline{\varphi}, \pi_B) \end{array}$$

where $\widehat{p}$ and $\widehat{p_0}$ are surjective.

Let $[\alpha] \in \mathcal{R}(\varphi, \varphi_0) \subset \mathcal{R}(\varphi, \pi_E)$. Then there is $[\alpha_0] \in \mathcal{R}(\varphi_0, \pi_{E_0})$ such that $\widehat{i_{E,E_0}}[\alpha_0] = [\alpha]$. By the commutativity of the above diagram,

$$\widehat{p}[\alpha] = \widehat{p}\widehat{i_{E,E_0}}[\alpha_0] = \widehat{i_{B,B_0}}\widehat{p_0}[\alpha_0] \in \mathrm{Im}\, \widehat{i_{B,B_0}}.$$

Since $\mathrm{R}(f; E - E_0) = \mathrm{R}(f) - \mathrm{R}(f, f_0)$, using Theorem 2.1, the inclusion proven above, plus the algebraic definition of the Reidemeister number and properties of the sequence (2.1), we have

$$\begin{aligned}\mathrm{R}(f; E - E_0) &= \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} - \#\mathcal{R}(\varphi, \varphi_0) \\ &\geq \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} - \#\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi}, \overline{\varphi_0})) \\ &= \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} \\ &\quad - \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} \\ &= \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B) - \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \, (\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]}.\end{aligned}$$

Again, using the definition of $\mathrm{R}(f; E, E_0)$, Theorem 2.1 and the inequality just proven for the relative Reidemeister number on the complement, we obtain the desired inequality for the relative Reidemeister number. □

It is clear that if $\mathcal{R}(\varphi,\varphi_0) \supseteq \widehat{p}^{\,-1}\mathcal{R}(\overline{\varphi},\overline{\varphi_0})$ then equality for $\mathrm{R}(f;E-E_0)$ will hold. But the following example shows that in general a class in the complement may project to a common class.

EXAMPLE 3.3. Let (E,p,B) be a product fibration with $B = D^2 \times S^1$ and $F = S^1 \times S^1$. Let $\overline{f}\colon B \to B$ be $\overline{f} = \overline{f_1} \times \overline{f_2}$ where $\overline{f_1} = \mathrm{id}_{D^2}$ and $\overline{f_2}$ be the flipping across the x-axis. Let $f'\colon F \to F$ be $f' = f_1' \times f_2'$ where f_1' is the flipping across the x-axis and $f_2' = \mathrm{id}_{S^1}$.

Let (E_0,p_0,B_0) also be a product fibration with $B_0 = D^2 \times I_1$, where I_1 is the segment in the circle that goes counterclockwise from $-i$ to i, and $F_0 = I_2 \times S^1$ where I_2 is the segment in the circle that goes counterclockwise from i to $-i$. Let $\overline{f_0} = \overline{f}|_{B_0}$ and let $f_0' = f'|_{B_0}$.

The base space B has two distinct classes ($[(d,-1)]$ and $[(d,1)]$) and only $[(d,1)]$ is weakly common. In the total space E we have four distinct classes ($[1,1]$, $[1,-1]$, $[-1,1]$ and $[-1,-1]$) and only one of them is weakly common, namely, $[1,-1]$. Both $[1,1]$ and $[1,-1]$ project to the weakly common class of B so $\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi},\overline{\varphi_0})) \supset \mathcal{R}(\varphi,\varphi_0)$, but $\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi},\overline{\varphi_0})) \not\subset \mathcal{R}(\varphi,\varphi_0)$.

However, even without the reversed inclusion of Proposition 3.2, under suitable hypotheses, we may still have equality in the above formulas. To begin, let us assume that $|\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi})| = |\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0})| = 1$, for all $\overline{\alpha} \in \mathcal{R}(\overline{\varphi},\pi_B)$, respectively $\overline{\alpha_0} \in \mathcal{R}(\overline{\varphi_0},\pi_{B_0})$.

Let $[\beta] \in \mathcal{R}(\varphi,\varphi_0)$ be a weakly common class. This means $[\beta] \in \mathcal{R}(\varphi,\pi_E)$ and there exists a $[\beta_0] \in \mathcal{R}(\varphi_0,\pi_{E_0})$ such that $\widehat{i_{E,E_0}}[\beta_0] = [\beta]$. We have already seen that there are correspondences

$$\begin{aligned}\mathcal{R}(\varphi,\pi_E) &\leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p}^{\,-1}([\overline{\alpha}]) \\ &\leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p_\alpha}^{\,-1}[\overline{e}] = \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}\mathcal{R}(\tau_\alpha\varphi',\pi_F).\end{aligned}$$

And similar ones for $\mathcal{R}(\varphi_0,\pi_{E_0})$.

Also due to the hypotheses $|\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi})| = 1$, for all $\overline{\alpha} \in \mathcal{R}(\overline{\varphi},\pi_B)$ and, respectively $|\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0})| = 1$, for all $\overline{\alpha_0} \in \mathcal{R}(\overline{\varphi_0},\pi_{B_0})$, the maps induced by the inclusions are, in fact, injective and the diagram below is commutative.

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{R}_{K_0}(\varphi_0',\pi_{F_0}) \arrow[r, "\widehat{i_{K_0}}"] \arrow[d, "\widehat{i_{F,F_0}}"'] & \mathcal{R}(\varphi_0,\pi_{E_0}) \arrow[r, "\widehat{p_0}"] \arrow[d, "\widehat{i_{E,E_0}}"] & \mathcal{R}(\overline{\varphi_0},\pi_{B_0}) \arrow[r] \arrow[d, "\widehat{i_{B,B_0}}"] & 1 \\
1 \arrow[r] & \mathcal{R}_K(\varphi',\pi_F) \arrow[r, "\widehat{i_K}"'] & \mathcal{R}(\varphi,\pi_E) \arrow[r, "\widehat{p}"'] & \mathcal{R}(\overline{\varphi},\pi_B) \arrow[r] & 1
\end{tikzcd}$$

There is a corresponding class $[\beta'] \in \mathcal{R}_{K,K_0}(\tau_\alpha\varphi',\tau_{\alpha_0}\varphi_0')$ to each such $[\beta] \in \mathcal{R}(\varphi,\varphi_0)$. To see this, first note that $[\beta] = \widehat{i_{E,E_0}}([\beta_0])$ for some $[\beta_0] \in \mathcal{R}(\varphi_0,\pi_{E_0})$.

By Corollary 2.5 $[\beta_0]$ corresponds to some $[\beta_0{}'] \in \mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0', \pi_{F_0})$. Then, following the above commutative diagram, $[\beta'] = \widehat{i_{F,F_0}}([\beta_0{}'])$. So we have

$$\#\mathcal{R}(\varphi, \varphi_0) = \sum_{\overline{\alpha}\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')$$

where $\widehat{p}\,([\alpha]) = [\overline{\alpha}]$ and $\widehat{i_{F,F_0}}[\alpha_0] = [\alpha]$.

Also, if instead of having the hypotheses $|\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi})| = |\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0})| = 1$, for all $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)$, respectively $[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})$, we have that f is globally Fix group uniform then

$$\#\mathcal{R}(\varphi, \varphi_0) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \frac{\#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')}{[\mathrm{Fix}(\overline{\varphi}) : p_\# \ \mathrm{Fix}(\varphi)]}. \tag{3.1}$$

Although $\widehat{p}^{\,-1}\mathcal{R}(\overline{\varphi}, \overline{\varphi_0}) \supsetneq \mathcal{R}(\varphi, \varphi_0)$, we still have $\mathcal{R}(\varphi, \varphi_0)$ mapped surjectively onto $\mathcal{R}(\overline{\varphi}, \overline{\varphi_0})$. This is the reason why some of the terms in the summand in (3.1) may be equal to zero.

Similar to $\mathcal{N}_K$ and N_K, we can define $\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$ as the set of essential $\alpha' \in \mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and $\mathrm{N}_{K,K_0}(f', f_0') = \#\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$.

With the equality (3.1) and the calculation of $\mathrm{R}(f; E - E_0)$ from Proposition 3.2, we obtain the following

Theorem 3.4. *Let (f, f_0) be a fiber map of the pair. Let $K = \ker i_\#$ and $K_0 = \ker i_{0\#}$. Suppose f is globally Fix group uniform and let $s = [\mathrm{Fix}(\overline{\varphi}) : p_\# \ (\mathrm{Fix}(\varphi))]$, then*

$$\begin{aligned}\mathrm{R}(f; E - E_0)& \\ = \frac{1}{s}\bigg\{&\sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')\bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{R}(f; E, E_0) = &\sum_{[\overline{\alpha_0}]\in\mathcal{R}(\overline{\varphi_0},\pi_{B_0})} \frac{\#\mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0', \pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0}\varphi_0))]} \\ &+ \frac{1}{s}\bigg\{\sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')\bigg\}.\end{aligned}$$

Also, if (E, E_0) is a Jiang-type pair with nonzero Lefschetz numbers, $\mathrm{L}(f) \cdot \mathrm{L}(f_0) \neq 0$, then we have the respective Nielsen numbers

$$\begin{aligned}\mathrm{N}(f; E - E_0)& \\ = \frac{1}{s}\bigg\{&\sum_{[\overline{\alpha}]\in\mathcal{N}(\overline{\varphi},\pi_B)} \#\mathcal{N}_K(\tau_\alpha\varphi', \pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{N}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{N}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')\bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{N}(f;E,E_0)&\\ &= \sum_{[\overline{\alpha_0}]\in\mathcal{N}(\overline{\varphi_0},\pi_{B_0})} \frac{\#\mathcal{N}_{K_0}(\tau_{\alpha_0}\varphi_0',\pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}):p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0}\varphi_0))]}\\ &+\frac{1}{s}\bigg\{\sum_{[\overline{\alpha}]\in\mathcal{N}(\overline{\varphi},\pi_B)}\#\mathcal{N}_K(\tau_\alpha\varphi',\pi_F)-\sum_{[\overline{\alpha}]\in\mathcal{N}(\overline{\varphi},\overline{\varphi_0})}\#\mathcal{N}_{K,K_0}(\tau_\alpha\varphi',\tau_{\alpha_0}\varphi_0')\bigg\}.\end{aligned}$$

Example 3.5. Let E be the three dimensional solvmanifold given by the relation $(x,y,z)\sim(x+d,(-1)^a y+b,(-1)^a z+c)$ on the simply connected solvable Lie group $\mathbb{R}^3$ where $a,b,c,d\in\mathbb{Z}$. There is a Mostow fibration $F\hookrightarrow E\xrightarrow{p}B$ where F is a 2-torus and B is the unit circle so that $\pi_1(E)$ admits a semi-direct decomposition

$$0\to\mathbb{Z}^2\to\pi_1(E)\to\mathbb{Z}\to 0.$$

The fibration p is given by the projection $[x,y,z]\mapsto[x]$ on the first coordinate. Let $F_0=\{[0,y,z]\}$, $E_0=\{[0,y,z]\}$ and $B_0=\{[0]\}$. The map f given by $[x,y,z]\mapsto[-x,y+z,y]$ is a fiber preserving map of the fiber pair (E,E_0). The induced map $\overline{f}\colon B\to B$ is the map given by $[x]\to[-x]$ such that $\mathrm{Fix}(\overline{f})=\{[0],[1/2]\}$. The restriction $f'_{[0]}$ over $[0]\in B_0$ has Lefschetz number -1 and the restriction $f'_{[1/2]}$ over $[1/2]\in\mathrm{Fix}(\overline{f})$ has Lefschetz number 1. Thus, $\mathrm{L}(f)=0$.

Note that $K,K_0,\mathrm{Fix}(\overline{\varphi})=\mathrm{Fix}(\overline{f}_\#)$ are all trivial so that $s=1$ and f is globally Fix group uniform (see e.g. [7]). The formulas in Theorem 3.4 yield $\mathrm{R}(f;E-E_0)=1$ and $\mathrm{R}(f;E,E_0)=2$. We should point out that we also have $\mathrm{R}(f;E-E_0)=\mathrm{N}(f;E-E_0)$ and $\mathrm{R}(f;E,E_0)=\mathrm{N}(f;E,E_0)$ even though $\mathrm{L}(f)=0$, i.e. the pair (E,E_0) is not a Jiang-type pair. These equalities hold because for solvmanifolds, $\mathrm{R}(f)<\infty\Rightarrow\mathrm{R}(f)=\mathrm{N}(f)$ (see [6]).

In the absolute case (i.e. $A=\emptyset$), results such as Theorem 4.10 of [10] or Theorem 5.9 of [15] concerning the computation of $\mathrm{N}(f)$ can be obtained without eventually commutativity of f, as done, for instance, in [7]. Similarly, it is easy to show that Schusteff's results (Theorem III.2.3 and III.2.19 of [13]) can be obtained and therefore generalized by replacing eventually commutativity with mod K fiber uniform and globally Fix group uniform conditions. To further illustrate this, we give an example in the case of a pair of nilmanifolds.

Example 3.6. Consider the subgroup of $GL(3,\mathbb{R})$,

$$G=\left\{\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}\;\middle|\;x,y,z\in\mathbb{R}\right\}$$

and the subgroup of $GL(3,\mathbb{Z})$,

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}.$$

The coset space $E = G/\Gamma$ is a nilmanifold, which fibers over the 2-torus $B = T^2$, with the circle $F = S^1$ as a typical fiber. We have a subfibration with $E_0 = T^2 \subset E$,

$$E_0 = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \Gamma \;\middle|\; y, z \in \mathbb{Z} \right\},$$

considering the usual fibration of T^2 by S^1.

Consider the fiber map $f\colon (E, E_0) \to (E, E_0)$ defined by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -a & -2b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix},$$

which has Nielsen numbers $\mathrm{N}(f) = 6$, $\mathrm{N}(\overline{f}) = 2$ and $\mathrm{N}(f_b) = 3$, for any $b \in B$.

On the subfibration, we have Nielsen numbers $\mathrm{N}(f_0) = 3$, $\mathrm{N}(f_{0b}) = 3$ and $\mathrm{N}(\overline{f_0}) = 1$. The respective Nielsen numbers for the essentially common classes (see [12]) are $\mathrm{N}(f, f_0) = 3$, $\mathrm{N}(f_b, f_{0b}) = 3$ and $\mathrm{N}(\overline{f}, \overline{f_0}) = 1$. Therefore we obtain the relative Nielsen numbers $\mathrm{N}(f; E, E_0) = 6$, $\mathrm{N}(f_b; F, F_0) = 3$ and $\mathrm{N}(\overline{f}; B, B_0) = 2$ so that the naïve product formula holds. Note that $\mathrm{N}(f_b) = \mathrm{N}_{K,K_0}(f_b; F, F_0)$ since $K = K_0 = \{e\}$. We should point out that the hypotheses of Theorem III.2.19 of [13] are satisfied except that the map f is not eventually commutative.

To see that, observe

$$f_\#^n \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-1)^n a & (-1)^n 2^n b \\ 0 & 1 & 2^n c \\ 0 & 0 & 1 \end{pmatrix}.$$

So, for any two elements

$$P = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

of $\pi_1(E)(\equiv G)$, for any $n \geq 1$ we have that $f_\#^n(P)$ commutes with $f_\#^n(Q)$ if and only if $az = cx$.

Remark 3.7. Theorem III.2.23 of [13] also holds for Jiang-type spaces.

4. Estimation of asymptotic Nielsen type numbers

In this final section, we make use of the computational techniques for calculating or approximating the relative Reidemeister number on the complement for a fiber-preserving map of pairs to estimate the asymptotic Nielsen type number introduced by Jiang in [9].

In dynamics, one is interested in the asymptotic behavior of the growth of the number of periodic points. N. Ivanov in [8] introduced the asymptotic Nielsen number to be the growth rate of $\{\mathrm{N}(f^n)\}$, i.e. $\mathrm{N}^\infty(f) = \mathrm{Growth}_{n\to\infty}\mathrm{N}(f^n) := \max\{1, \limsup \mathrm{N}(f^n)^{1/n}\}$ for any selfmap $f\colon X \to X$ on a compact connected polyhedron X. He showed that $h(f) \geq \log \mathrm{N}^\infty(f)$ where $h(f)$ denotes the topological entropy of f. While the Nielsen number $\mathrm{N}(f^n)$ gives a lower bound for the number of n-periodic points of f, it is often a poor estimate. On the other hand, the Nielsen type number $(1/n)NP_n(f)$ (see [9]), which measures the number of periodic orbits of least period n, yields better information so that the growth rate of $\{(1/n)NP_n(f)\}$, denoted by $\mathrm{NI}^\infty(f)$, was introduced by Jiang to study the asymptotic behavior of the growth of the periodic orbits.

In [14], it was shown that

$$\mathrm{NI}^\infty(f) \leq \operatorname*{Growth}_{n\to\infty} \mathrm{R}(g_n(f); Y_{n_{(1)}})$$

where $\mathrm{R}(g_n(f); Y_{n_{(1)}})$ is the relative Reidemeister number on the complement $Y_{n_{(1)}}$. Here, $Y_n = X \times \ldots \times X$ is the n-fold product with a cyclic action of $\mathbb{Z}_n$, $g_n(f)$ is the associated $\mathbb{Z}_n$-map given by

$$g_n(f)(x_1, \ldots, x_n) = (f(x_n), f(x_1), \ldots, f(x_{n-1})),$$

$Y_{n_{(1)}} = Y_n - Y_n^{>(1)}$ where $Y_n^{>(1)}$ is the closed subspace consisting of points of non-trivial isotropy and $\mathrm{R}(g_n(f); Y_{n_{(1)}})$ is the relative Reidemeister number on the complement. Furthermore, if X is a Jiang-type space then we have equality.

Suppose $\xi\colon X \to \overline{X}$ is a Hurewicz fibration and $f\colon X \to X$ is a fiber-preserving map with induced map $\overline{f}\colon \overline{X} \to \overline{X}$. Let p be a prime. Then $(g_{p^r}(f), g_{p^{r-1}}(f))$ is a fiber map of the pair $(Y_{p^r}, Y_{p^{r-1}})$, in the sense of Section 3, with induced maps $(g_{p^r}(\overline{f}), g_{p^{r-1}}(\overline{f}))$. We have the following commutative diagram

$$\begin{array}{ccc}
Y_{p^r}, Y_{p^{r-1}} & \xrightarrow{g_{p^r}(f)} & Y_{p^r}, Y_{p^{r-1}} \\
{\scriptstyle q}\big\downarrow{\scriptstyle q_0} & & {\scriptstyle q}\big\downarrow{\scriptstyle q_0} \\
\overline{Y}_{p^r}, \overline{Y}_{p^{r-1}} & \xrightarrow[g_{p^r}(\overline{f})]{} & \overline{Y}_{p^r}, \overline{Y}_{p^{r-1}}
\end{array}$$

where $q = \xi \times \ldots \times \xi$ (p^r-fold product) and q_0 is the p^{r-1}-fold product of ξ.

Next, we estimate the asymptotatic Nielsen number $\mathrm{NI}^\infty(f)$.

Theorem 4.1. *Suppose* $\mathrm{N}(f^n) = \mathrm{R}(f^n)$ *for all* $n \geq 1$. *Then, for any prime* p,

$$\begin{aligned}\mathrm{NI}^\infty(f) &\geq \operatorname*{Growth}_{r\to\infty} \mathrm{R}(g_{p^r}(f); Y_{{p^r}_{(1)}}) \\ &\geq \operatorname*{Growth}_{r\to\infty} \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi_r},\pi_{B_r})-\mathcal{R}(\overline{\varphi_r},\overline{\varphi_{0r}})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi_r}) : q_{\#}(\mathrm{Fix}(\tau_{\beta\alpha}\varphi_r))]}\end{aligned}$$

where φ_r *and* $\overline{\varphi_r}$ *are the homomorphisms induced by* $g_{p^r}(f)$ *and* $g_{p^r}(\overline{f})$ *respectively at the fundamental group level. Here,* $B_r = \overline{Y_{p^r}}$ *and* $F_r = q^{-1}(b)$, $b \in \mathrm{Fix}(g_{p^r}(\overline{f})) \cap (\overline{Y_{p^r}} - \overline{Y_{p^{r-1}}})$.

Proof. The inequality $\mathrm{NI}^\infty(f) \geq \mathrm{Growth}_{r\to\infty}\mathrm{R}(g_{p^r}(f); Y_{{p^r}_{(1)}})$ follows from Theorem 4.2 of [14]. The last inequality follows from Proposition 3.2. □

Fiberwise techniques have proven to be useful in Nielsen theory for periodic points, especially for maps on solvmanifolds and nilmanifolds. Estimation of $\mathrm{NI}^\infty(f)$ has been obtained in [14] for a selfmap f on a nilmanifold. We end this paper with an application of our results to estimating $\mathrm{NI}^\infty(f)$ for selfmaps on solvmanifolds.

Recall that every selfmap $f\colon M \to M$ of a solvmanifold M is homotopic to a fiber-preserving map with respect to a Mostow fibration $F \hookrightarrow M \to B$ where the typical fiber F is a nilmanifold and the base B is a torus. Furthermore, if $\mathrm{R}(f) < \infty$ then $\mathrm{N}(f) = \mathrm{R}(f)$ (see [6]).

Corollary 4.2. *If* X *is a solvmanifold and* $\mathrm{R}(f^n) < \infty$ *for all* n, *then for any prime* p,

$$\mathrm{NI}^\infty(f) \geq \operatorname*{Growth}_{r\to\infty} \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi_r},\pi_{B_r})-\mathcal{R}(\overline{\varphi_r},\overline{\varphi_{0r}})} \#\mathcal{R}(\tau_\alpha\varphi_r', \pi_{F_r})$$

where f *is a fiber-preserving map of a Mostow fibration of* X.

Proof. Since $\mathrm{R}(f^n) < \infty$, then $\mathrm{R}(\overline{f}^n) < \infty$. Since the base of the fibration of X is a torus, it follows that $|\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi_n})| = |\mathrm{Fix}(\overline{\varphi_n})| = 1$. Moreover, K is trivial. Now the inequality follows from Theorem 4.1. □

Remark 4.3. In Corollary 4.2 the fiber is a nilmanifold, so $\#\mathcal{R}(\tau_\alpha\varphi_r', \pi_{F_r})$ coincides with the corresponding Lefschetz number on the fiber and is therefore computable.

References

[1] R. F. Brown, *The Nielsen number of a fiber map*, Ann. Math. **85** (1967), 483–493.

[2] F. Cardona, *Reidemeister theory for maps of pairs*, Far East J. Math. Sci. Special Volume, Part I (Geometry and Topology) (1999), 109–136.

[3] F. Cardona and P. Wong, *On the computation of the relative Nielsen number*, Proceedings of "Theory of Fixed Points and its Applications" (São Paulo, 1999) (D. Gonçalves and P. Wong, eds.); Topology Appl. **116** (2001), 29–41.

[4] E. Fadell, *Natural fiber splittings and Nielsen numbers*, Houston J. Math. **2** (1976), 71–84.

[5] D. Gonçalves, *Coincidence Reidemeister classes on nilmanifolds and nilpotent fibrations*, Topology Appl. **83** (1998), 169–186.

[6] D. Gonçalves and P. Wong, *Homogeneous Spaces in Coincidence Theory II*, preprint.

[7] P. Heath, E. Keppelmann and P. Wong, *Addition formulae for Nielsen numbers and for Nielsen type numbers of fiber-preserving maps*, Topology Appl. **67** (1995), 133–157.

[8] N. V. Ivanov, *Entropy and the Nielsen numbers*, Soviet Math. Dokl. **26** (1982), 63–66.

[9] B. Jiang, *Estimation of the number of periodic orbits*, Pacific J. Math. **172** (1996), 151–185.

[10] ———, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math., vol. 14, Amer. Math. Soc., Providence, Rhode Island, 1983.

[11] T. Kiang, *The Theory of Fixed Point Classes*, Springer–Verlag, 1989.

[12] H. Schirmer, *A relative Nielsen number*, Pacific J. Math. **122** (1986), 459–473.

[13] A. Schusteff, *Product formulas for relative Nielsen numbers of fiber map pairs*, doctoral dissertation, UCLA (1990).

[14] P. Wong, *Estimation of Nielsen type numbers for periodic points* II, Proceedings of the XI Encontro Brasileiro de Topologia (Rio Claro, 1998) (S. Firmo, D. Gonçalves and O. Saeki, eds.), World Scientific Publ., 2000, pp. 23–32.

[15] C. You, *Fixed point classes of a fibre map*, Pacific J. Math. **100** (1982), 217–241.

[16] X. Zhao, *A relative Nielsen number for the complement*, Topological Fixed Point Theory and Applications (B. Jiang, ed.), Tianjin, 1988; Lecture Notes in Math. vol 1411 (1989), 189–199.

Manuscript received November 5, 2001

Dernanda S. P. Cardona
Dept. de Matemática
IME – USP
Caixa Postal 66.281 – CEP
05389-970, São Paulo – SP, BRASIL

E-mail address: cardona@ime.usp.br

Peter N.-S. Wong
Department of Mathematics
Bates College
Lewiston, ME 04240, USA

E-mail address: pwong@bates.edu