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DEPENDENCE ON PARAMETERS FOR THE DIRICHLET PROBLEM WITH SUPERLINEAR NONLINEARITIES

Andrzej Nowakowski — Andrzej Rogowski

Abstract. The nonlinear second order differential equation

$$\frac{d}{dt}h(t,x'(t))+g(t,x(t))=0,\quad t\in[0,T]\text{ a.e.}\qquad x'(0)=x'(T)=0$$

with superlinear function g is investigated. Based on dual variational method the existence of solution is proved. Dependence on parameters and approximation method are also presented.

1. Introduction

We investigate the nonlinear Hamilton equations:

$$\frac{d}{dt}L_{x'}(t,x'(t))+V_x(t,x(t))=0,\quad \text{a.e. in } [0,T] \tag{1.1}$$

where

(H) $T>0$ is arbitrary, $L,V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are convex, Gateaux differentiable in the second variable and measurable in t functions.

We are looking for solutions of (1.1) being a pair (x,p) of absolutely continuous functions $x,p:[0,T]\to\mathbb{R}^n$, with Dirichlet boundary conditions for the second

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function i.e. $p(0) = 0 = p(T)$, such that

$$\begin{aligned}\frac{d}{dt}p(t) + V_x(t, x(t)) &= 0,\\ p(t) &= L_{x'}(t, x'(t)).\end{aligned}$$

Of course, if $L(t, x') = |x'|^2/2$, then for our solution of (1.1) $p = x'$ and thus x belongs to $C^{1,+}([0,T], \mathbb{R}^n)$ of continuously differentiable functions x whose derivatives x' are absolutely continuous. In the sequel we assume that V_x is superlinear. It is clear that (1.1) is the Euler–Lagrange equation to the functional

$$J(x) = \int_0^T (-V(t, x(t)) + L(t, x'(t)))\,dt \tag{1.2}$$

considered on the space A of absolutely continuous functions $x : \mathbb{R} \to \mathbb{R}^n$.

Equations (1.1) with either Dirichlet or periodic boundary conditions were studied in eighties by many authors as well in sublinear case as in superlinear one (see e.g. [6]). We propose to study (1.1) with Dirichlet boundary conditions for the second function of the solution i.e. we can not look for critical points studying functional (1.2) directly. We believe that our paper may contribute some new look at this problem. This is because we propose to study (1.1) by duality methods in a way, to some extend, analogous to the methods developed for (1.1) in sublinear cases [6], [7]. Some cases of (1.1) for superlinear V_x were studied by [5], [6], [2], [9], [1]. It is interesting that the method developed in [5] is based on the dual variational method for the problem, according to the idea developed in [6]. Since functional (1.2) is, in general, unbounded in A_P (especially in superlinear case), therefore it is obvious that we must look for critical points of J of "minmax" type. The main difficulties which appear here are: what kind of sets we should choose over which we wish to calculate "minmax" of J and then to link this value with critical points of J. Of course, we have the mountain pass theorems, the saddle points theorems, the Morse theory, ... (see e.g. [8], [6]) but all these do not exhaust all critical points of J.

Our aim is to find a nonlinear subspace X of A defined by the type of nonlinearity of V (and in fact also L). To be more precise let us set the basic hypothesis we need:

(H1) there exist $0 < \alpha_1, \alpha_2, 2\alpha_2 < 3\alpha_1$ and $d_1, d_2 \in \mathbb{R}$ such that, for $x' \in L^2$,

$$d_1 + \frac{\alpha_1}{2}\|x'\|^2_{L^2} \le \int_0^T L(t, x'(t))dt \le \frac{\alpha_2}{2}\|x'\|^2_{L^2} + d_2, \tag{1.3}$$

$L(t, \cdot\,)$ is strictly convex, $V_x(t, \cdot\,)$ is continuous, $t \in [0,T]$, there exist $0 < \beta_1 < \beta_2$, $q_1 > 1$, $q > 2$, $k_1, k_2 \in \mathbb{R}$ such that for $x \in L^q$

$$k_1 + \frac{\beta_1}{q_1}\|x\|^{q_1}_{L^{q_1}} \le \int_0^T V(t, x(t))dt \le \frac{\beta_2}{q}\|x\|^q_{L^q} + k_2. \tag{1.4}$$

Having the type of nonlinearities of L and V fixed we are able to define nonlinear subspaces $\overline{X}$, $\widetilde{X}$ and X as follows. First we put for a given, arbitrary $k_3 \in \mathbb{R}$

$$\overline{X} = \left\{ v \in A : \int_0^T V(t, v(t))\, dt \leq \frac{1}{2} \int_0^T L(t, v'(t))\, dt + k_3 \right\}.$$

We reduce the space $\overline{X}$ to the set

$$\widetilde{X} = \bigg\{ x(\,\cdot\,) + c_x \in \overline{X} : x \in A_0,\ c_x \in \mathbb{R}^n \text{ is such that } \int_0^T V_x(t, x(t) + c_x)\, dt = 0, \\ \text{and } p(t) = L_{x'}(t, x'(t)),\ t \in [0, T] \text{ belongs to } A_{0,0} \bigg\},$$

where A_0 is the space of absolutely continuous functions $x : [0, T] \to \mathbb{R}^n$ with $x' \in L^2$, $x(0) = 0$, $A_{0,0}$ is the space of absolutely continuous functions $x : [0, T] \to \mathbb{R}^n$ with $x' \in L^2$, $x(0) = 0 = x(T)$. Next we reduce the set $\widetilde{X}$ to the set $X \subset \widetilde{X}$ with the property: for each $v \in X$, there exists (possible another) $\widetilde{v} \in X$ such that $V_x(t, v(t)) = -dL_{x'}(t, \widetilde{v}'(t))/dt$, for a.e. $t \in [0, T]$.

It is clear that, in general, the set X is much smaller than $\widetilde{X}$ and that it depends strongly on the type of nonlinearities V and L. We easily see that X is not in general a closed set in A. As the dual set to X we shall consider the following set

$$X^d = \{ p \in A_{0,0}^{q'} : \text{there exist } v \in X \text{ such that } p(t) = L_{x'}(t, v'(t)), t \in [0, T] \text{ a.e.} \}$$

($A_{0,0}^{q'}$ – the space of absolutely continuous functions with $p' \in L^{q'}$ and $p(0) = 0 = p(T)$).

The constant c_x from the specification of $\widetilde{X}$ posesses very interesting property:

Lemma 1.1. *For any $x \in \widetilde{X}$ the constant c_x from the specification of $\widetilde{X}$ is a minimizer of the functional*

$$c \to \int_0^T V(t, x(t) + c)\, dt. \tag{1.5}$$

Proof. Since the functional (1.5) is convex on $\mathbb{R}^n$, therefore it attains minimum in each point c_x satisfying equality $\int_0^T V_x(t, x(t) + c_x)\, dt = 0$. □

Taking into account the structure of the space X and X^d we shall study the functional

$$J_D(p) = -\int_0^T L^*(t, p(t))\, dt + \int_0^T V^*(t, -p'(t))\, dt$$

on the space X^d.

We shall look for a "min" of J_D over the set X^d i.e.

$$\min_{p\in X^d} J_D(p).$$

To show that element $\overline{p} \in X^d$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D, described in the next section. Just because of the duality theory we are able to avoid in our proof of an existence of critical points the deformation lemmas, the Ekeland variational principle or PS type conditions. One more advantage of our duality results is obtaining for the first time in the superlinear case a measure of a duality gap between primal and dual functional for approximate solutions to (1.1) (for the sublinear case see [7]).

The main result of our paper is the following:

MAIN THEOREM. *Under hypothesis* (H) *and* (H1) *there exists a pair* $(\overline{x},\overline{p})$, $\overline{x} \in X$, $\overline{p} \in X^d$, *being a solution to* (1.1) *and such that*

$$J(\overline{x}) = \min_{x\in X} J(x) = \min_{p\in X^d} J_D(p) = J_D(\overline{p}).$$

We see that our hypotheses on L and V concern only convexity of $L(t,\,\cdot\,)$ or $V(t,\,\cdot\,)$ and that the latter function is of the superquadratic type. We do not assume that $V(t,x) \geq 0$. However we require that the above set X is nonempty, which we must check in each concrete type of equation. Some routine how to do that we show at the end of the paper for the equation

$$x'' + V_x(t,x) = 0.$$

In Section 5 we consider the question of the continuous dependence on parameters of the set of critical points of the functional J and the set of solutions to equation

$$\frac{d}{dt}L_{x'}(t,x'(t)) + V_x(t,x(t),u(t)) = 0, \quad \text{a.e. in } \mathbb{R},$$

where $u \in \mathcal{U} = \{w \in L^\infty(0,T) : u(t) \in U \text{ for a.e. } t \in [0,T]\}$. Here U is a given subset of $\mathbb{R}^m$.

2. Duality results

Because of the type of boundary conditions we deal with, we start from the dual to J functional J_D:

$$J_D(p) = -\int_0^T L^*(t,p(t))\,dt + \int_0^T V^*(t,-p'(t))\,dt.$$

To obtain a duality principle we need a kind of perturbation of J_D. Thus define for each $p \in X^d$ the perturbation of J_D as

$$J_{Dp}(y) = \int_0^T (L^*(t, p(t) + y(t)) - V^*(t, -p'(t)))\, dt \tag{2.1}$$

for $y \in L^2$. Of course, $J_{Dp}(0) = -J_D(p)$. For $x + c_x \in X$ and $p \in X^d$, we define a type of conjugate of J_D by

$$\begin{aligned} J_p^{\#}(x + c_x) = \sup_{y \in L^2} &\bigg\{ \int_0^T \langle y(t), x'(t) \rangle\, dt - \int_0^T L^*(t, p(t) + y(t))\, dt \bigg\} \\ &+ \int_0^T V^*(t, -p'(t))\, dt \end{aligned}$$

By a direct calculation we obtain

$$\begin{aligned} J_p^{\#}(x + c_x) &= \int_0^T \langle x(t) + c_x, p'(t) \rangle\, dt \\ &\quad + \int_0^T L(t, x'(t))\, dt + \int_0^T V^*(t, -p'(t))\, dt \\ &= - \int_0^T \langle x(t) + c_x, -p'(t) \rangle\, dt \\ &\quad + \int_0^T L(t, x'(t)) dt + \int_0^T V^*(t, -p'(t))\, dt. \end{aligned} \tag{2.2}$$

Now we take "min" from $J_p^{\#}(x + c_x)$ with respect to $p \in X^d$ and calculate it. Because X^d is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this effect we use the special structure of the set X. First we observe that for each $x + c_x \in X$ there exists $p_x \in X^d$ such that $p_x'(\,\cdot\,) = -V_x(\,\cdot\,, x(\,\cdot\,) + c_x)$ and therefore

$$\int_0^T \langle -p_x'(t), x(t) + c_x \rangle\, dt - \int_0^T V^*(t, -p_x'(t))\, dt = \int_0^T V(t, x(t) + c_x)\, dt.$$

Next let us note that

$$\begin{aligned} &\int_0^T \langle -p_x'(t), x(t) + c_x \rangle\, dt - \int_0^T V^*(t, -p_x'(t))\, dt \\ &\quad \le \sup_{p \in X^d} \bigg\{ \int_0^T \langle -p'(t), x(t) + c_x \rangle\, dt - \int_0^T V^*(t, -p'(t))\, dt \bigg\} \\ &\quad \le \sup_{p' \in L^2} \bigg\{ \int_0^T \langle -p'(t), x(t) + c_x > dt - \int_0^T V^*(t, -p'(t))\, dt \bigg\} \\ &\quad = \int_0^T V(t, x(t) + c_x)\, dt \end{aligned}$$

and actually all inequalities above are equalities. Therefore we can calculate for $x + c_x \in X$

$$\begin{aligned}(2.3)\qquad \sup_{p\in X^d} -J_p^{\#}(x+c_x) &= \sup_{p\in X^d}\Big\{\int_0^T \langle x(t)+c_x, -p'(t)\rangle\,dt \\ &\qquad -\int_0^T L(t,x'(t))\,dt - \int_0^T V^*(t,-p'(t))\,dt\Big\} \\ &= -\int_0^T L(t,x'(t))dt + \int_0^T V(t,x(t)+c_x)\,dt.\end{aligned}$$

From (2.3) we infer, for $p \in X^d$, that

$$(2.4)\qquad \sup_{p\in X^d} -J_p^{\#}(x+c_x) = -J(x+c_x).$$

We can also define a type of the second conjugate of J_D: for $y \in L^2$, $x \in X$, $p \in X^d$, put

$$\begin{aligned}J_p^{\#\#}(y) = \sup_{x\in X}\Big\{\int_0^T \langle y(t), x'(t)\rangle\,dt + \int_0^T \langle x'(t), p(t)\rangle\,dt \\ -\int_0^T L(t,x'(t))\,dt - \int_0^T V^*(t,-p'(t))\,dt\Big\}.\end{aligned}$$

We assert that $J_p^{\#\#}(0,0) = -J_D(p)$. To prove that, we use the special structure of X^d. First we observe that for each $p \in X^d$ there exists $x_p \in X$ such that

$$p(t) = L_{x'}(t, x_p'(t))$$

and, by classical convex analysis argument,

$$x_p'(t) = L_p^*(t, p(t)),$$

where L^* is a Fenchel conjugate to L. Therefore

$$\int_0^T \langle x_p'(t), p(t)\rangle\,dt - \int_0^T L(t, x_p'(t))\,dt = \int_0^T L^*(t,p(t))\,dt.$$

On the other hand let us note that

$$\begin{aligned}&\int_0^T \langle x_p'(t), p(t)\rangle\,dt - \int_0^T L(t,x_p'(t))\,dt \\ &\quad\le \sup_{x\in X}\Big\{\int_0^T \langle x'(t),p(t)\rangle\,dt - \int_0^T L(t,x'(t))\,dt\Big\} \\ &\quad\le \sup_{x'\in L^2}\Big\{\int_0^T \langle x'(t),p(t)\rangle\,dt - \int_0^T L(t,x'(t))\,dt\Big\} = \int_0^T L^*(t,p(t))\,dt.\end{aligned}$$

Hence we see that, for $p \in X^d$

$$(2.5)\qquad J_p^{\#\#}(0) = \int_0^T (L^*(t,p(t)) - V^*(t,-p'(t)))\,dt = -J_D(p).$$

We easily compute (see (2.4))

$$\sup_{p\in X^d} J_p^{\#\#}(0) = \sup_{p\in X^d}\sup_{x\in X} -J_p^{\#}(x+c_x) = \sup_{x\in X}\sup_{p\in X^d} -J_p^{\#}(x+c_x) = \sup_{x\in X} -J(x). \tag{2.6}$$

Hence, from above and (2.6), we obtain the following duality principle

Theorem 2.1. *For functionals J and J_D we have the duality relation*

$$\inf_{x\in X} J(x) = \inf_{p\in X^d} J_D(p). \tag{2.7}$$

Denote by $\partial J_{Dp}(y)$ the subdifferential of J_{Dp}. In particular,

$$\partial J_{D\overline{p}}(0) = \left\{x' \in L^2 : \int_0^T L^*(t,\overline{p}(t))\,dt + \int_0^T L(t,x'(t))\,dt = \int_0^T \langle \overline{p}(t), x'(t)\rangle\,dt\right\}$$

The next result formulates a variational principle for "minmax" arguments.

Theorem 2.2. *Let $\overline{p}\in X^d$ be such that*

$$\infty > J_D(\overline{p}) = \inf_{p\in X^d} J_D(p) > -\infty$$

and let the set $\partial J_{D\overline{p}}(0)$ be nonempty. Then there exist $\overline{x}'\in\partial J_{D\overline{p}}(0)$ with $\overline{x}(t) = c_{\overline{x}} + \int_0^{t'}\overline{x}'(s)\,ds$ belonging to X, such that $\overline{x}$ satisfies

$$J(\overline{x}) = \inf_{x\in X} J(x).$$

Furthermore

$$J_{D\overline{p}}(0) + J_{\overline{p}}^{\#}(\overline{x}) = 0, \tag{2.8}$$

$$J(\overline{x}) - J_{\overline{p}}^{\#}(\overline{x}) = 0. \tag{2.9}$$

Proof. By Theorem 2.1 to prove the first assertion it suffices to show that $J(\overline{x}) \le J_D(\overline{p})$. Let us observe that $\overline{x}'\in\partial J_{D\overline{p}}(0)$ means, in fact, that $\overline{x}'(t) = L_p^*(t,\overline{p}(t))$, for a.e. $t\in[0,T]$ and therefore we have

$$\begin{aligned}
-J_D(\overline{p}) &= \int_0^T (L^*(t,\overline{p}(t)) - V^*(t,-\overline{p}'(t)))\,dt \\
&= \int_0^T (-V^*(t,-\overline{p}'(t)) - L(t,\overline{x}'(t)))\,dt + \int_0^T \langle \overline{x}(t), -\overline{p}'(t)\rangle\,dt \\
&\le \int_0^T (V(t,\overline{x}(t)) - L(t,\overline{x}'(t)))\,dt = -J(\overline{x}).
\end{aligned}$$

Hence $J(\overline{x}) \le J_D(\overline{p})$ and so $J(\overline{x}) = J_D(\overline{p}) = \inf_{x\in X} J(x)$. The first assertion will be proved if we show that $\overline{x}\in X$.

The second assertion is a simple consequence of two facts: $J_{D\overline{p}}(0) = -J_D(\overline{p})$ so $J_{D\overline{p}}(0) + J_D(\overline{p}) = 0$ and $\overline{x}' \in \partial J_{D\overline{p}}(0)$ i.e. $J_{D\overline{p}}(0) + J^{\#}_{\overline{p}}(\overline{x}) = 0$. Then equality (2.9) implies that

$$\int_0^T (V^*(t, -\overline{p}'(t))) + V(t, \overline{x}(t)))\,dt = \int_0^T \langle \overline{x}(t), -\overline{p}'(t)\rangle\,dt$$

and so $\overline{p}'(t) = -V_x(t, \overline{x}(t))$. By the definition of $\overline{p}$ and Lemma 1.1 we also infer that $\overline{x} \in X$. □

From equations (2.8), (2.9) we are able to derive a dual to (1.1) Euler–Lagrange equations.

COROLLARY 2.1. *Let $\overline{p} \in X^d$ be such that*

$$\infty > J_D(\overline{p}) = \inf_{p \in X^d} J_D(p) > -\infty.$$

Then there exists $\overline{x} \in X$ such that the pair $(\overline{x}, \overline{p})$ satisfies the relations

$$-\overline{p}'(t) = V_x(t, \overline{x}(t)), \tag{2.10}$$

$$\overline{p}(t) = L_{x'}(t, \overline{x}'(t)), \tag{2.11}$$

$$J_D(\overline{p}) = \inf_{p \in X^d} J_D(p) = \inf_{x \in X} J(x) = J(\overline{x}). \tag{2.12}$$

PROOF. By the assumptions on L we see that $y \to \int_0^T L^*(t, y(t))\,dt$ is finite in L^2, convex and lower semicontinuous. Therefore $J_{D\overline{p}}(y)$ is continuous in L^2. Hence $\partial J_{\overline{x}}(0)$ is nonempty and so the existence of x' in Theorem 2.2 is now obvious. Equations (2.8) and (2.9) imply

$$\int_0^T V(t, \overline{x}(t))\,dt + \int_0^T V^*(t, -\overline{p}'(t))\,dt - \int_0^T \langle \overline{x}(t), -\overline{p}'(t)\rangle\,dt = 0,$$

$$\int_0^T L^*(t, \overline{p}(t))\,dt + \int_0^T L(t, \overline{x}'(t))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t)\rangle\,dt = 0,$$

and then (2.10), (2.11). Relations (2.12) are a direct consequence of Theorems 2.1 and 2.2. □

As a direct consequence of the above corollary and definition of X^d we have

COROLLARY 2.2. *By the same assumptions as in Corollary* 2.1 *there exists a pair $(\overline{x}, \overline{p}) \in X \times X^d$ satisfying relations* (2.12), *and the pair $(\overline{x}, \overline{p})$ is a solution to* (1.1). *Conversely, each pair $(\overline{x}, \overline{p})$ satisfying relations* (2.12) *satisfies also equations* (2.10), (2.11).

3. Variational principles and a duality gap for minimizing sequences

In this section we show that a statement similar to Theorem 2.2 is true for a minimizing sequence of J_D.

Theorem 3.1. *Let $\{p_j\}$, $p_j \in X^d$, $j = 1, 2, \ldots$, be a minimizing sequence for J_D and let*

$$\infty > \inf_j J_D(p_j) = a > -\infty.$$

Then there exist $x'_j \in \partial J_{Dp_j}(0)$ with $x_j \in X$ such that $\{x_j\}$ is a minimizing sequence for J i.e.

$$\inf_{x\in X} J(x) = \inf_{x_j\in X} J(x_j) = \inf_{p_j\in X^d} J_D(p_j) = \inf_{p\in X^d} J_D(p).$$

Furthermore

$$J_{Dp_j}(0) + J^{\#}_{p_j}(x_j) = 0, \qquad J(x_j) - J^{\#}_{p_j}(x_j) \le \varepsilon, \qquad 0 \le J_D(p_j) - J(x_j) \le \varepsilon$$

for a given $\varepsilon > 0$ and sufficiently large j.

Proof. We have that $\infty > \inf_{p_j\in X^d} J_D(p_j) = a > -\infty$, and therefore for a given $\varepsilon > 0$ there exists j_0 such that $J_D(p_j) - a < \varepsilon$, for all $j \ge j_0$. Further, the proof is similar to that of Theorem 2.2, so we only sketch it. First, as in the proof of Corollary 2.1, we observe that $\partial J_{Dp_j}(0)$ is nonempty for $j \ge j_0$ and take $x'_j \in \partial J_{Dp_j}(0)$. Accordingly to the definition of X let us take as a primitive of x'_j such x_j that $x_j(0) = c_x$. Therefore, we also have

$$\begin{aligned} -J_D(p_j) &= \int_0^T (-V^*(t, -p'_j(t)) + L^*(t, p_j(t))\, dt \\ &= \int_0^T (-V^*(t, -p'_j(t)) - L(t, x'_j(t)))\, dt + \int_0^T \langle x'_j(t), p_j(t)\rangle\, dt \\ &\le \int_0^T (V(t, x_j(t)) - L(t, x'_j(t)))\, dt = -J(x). \end{aligned}$$

Hence, due to Theorem 2.1,

$$a + \varepsilon \ge J(x_j) \ge a \quad \text{for } j \ge j_0.$$

The second assertion is a simple consequence of two facts: $J_{Dp_j}(0) = -J_D(p_j)$ so $J_{Dp_j}(0) + J_D(p_j) = 0$ and $x'_j \in \partial J_{Dp_j}(0)$ i.e. $J_{Dp_j}(0) + J^{\#}_{p_j}(x_j) = 0$. $\square$

A direct consequence of this theorem is the following corollary.

Corollary 3.1. *Let $\{p_j\}$, $p_j \in X^d$, $j = 1, 2, \ldots$, be a minimizing sequence for J_D and let*

$$\infty > \inf_j J_D(p_j) = a > -\infty.$$

If $x'_j(t) = L^*_p(t, p_j(t))$ then $x_j(t) = c_{x_j} + \int_0^t x'_j(s)\, ds$ belongs to X and $\{x_j\}$ is a minimizing sequence for J i.e.

$$\inf_{x\in X} J(x) = \inf_{x_j\in X} J(x_j) = \inf_{p_j\in X^d} J_D(p_j) = \inf_{p\in X^d} J_D(p).$$

Furthermore

$$J(x_j) - J^{\#}_{p_j}(x_j) \le \varepsilon, \qquad 0 \le J_D(p_j) - J(x_j) \le \varepsilon \tag{3.1}$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

4. The existence of a minimum of J

The last problem which we have to solve is to prove the existence of $\overline{x} \in X$ such that

$$J_D(\overline{p}) = \min_{p \in X^d} J_D(p).$$

To obtain this it is enough to use hypothesis (H1), the results of the former section and known compactness theorems.

THEOREM 4.1. *Under hypothesis* (H1) *there exists* $\overline{p} \in X^d$ *such that* $J_D(\overline{p}) = \min_{p \in X^d} J_D(p)$.

PROOF. Let us observe (see Section 2) that each $p \in X^d$ satisfies the inequality

$$\int_0^T V^*(t, -p'(t))dt \ge \frac{3}{4\alpha_2}\|p\|^2_{L^2} - k_3 - d_2. \tag{4.1}$$

Really, each $x \in \widetilde{X}$ satisfies the inequality

$$\int_0^T V(t, x(t))\, dt \le \frac{1}{2}\int_0^T L(t, x'(t))\, dt + k_3.$$

Then it is enough to add to both sides of this inequality the scalar product $\int_0^T \langle x(t), p'(t)\rangle\, dt$, multiply the obtained inequality by "$-$" and then take superior of both sides over $\widetilde{X}$ and apply (1.3). Hence and by (H1), $J_D(p)$ is bounded below on X^d. Really, by (1.3), (1.4) and (4.1) we obtain:

$$\begin{aligned} J_D(p) &\ge -\int_0^T L^*(t, p(t))\, dt + \frac{3}{4\alpha_2}\|p\|^2_{L^2} - k_3 - d_2 \\ &\ge \frac{3\alpha_1 - 2\alpha_2}{4\alpha_2\alpha_1}\|p\|^2_{L^2} + d_1 - k_3 - d_2 \ge d_1 - k_3 - d_2. \end{aligned} \tag{4.2}$$

From (4.2) we infer the boundedness below of J_D on X^d as well as that the sets $S_b = \{p \in X^d,\ J_D(p) \le b\}$, $b \in \mathbb{R}$ are nonempty for sufficiently large b and bounded with respect to the norm $\|p\|_{L^2}$. Next, analogously as above we get that S_b is bounded with respect to the norm $\|p'\|_{L^{q'}}$ $(1/q + 1/q' = 1)$. The last means that S_b, $b \in \mathbb{R}$ are relatively weakly compact in $A^{q'}_{0,0}$. It is a well known fact that the functional J_D is weakly lower semicontinuous in $A^{q'}_{0,0}$. Therefore there exists a sequence $\{p_n\}$, $p_n \in X^d$, such that $p_n \rightharpoonup \overline{p}$ weakly in $A^{q'}_{0,0}$ with $\overline{p} \in A^{q'}_{0,0}$ and $\liminf_{n\to\infty} J(p_n) \ge J(\overline{p})$. Moreover, we know that $\{p_n\}$ is uniformly convergent to $\overline{p}$. In order to finish the proof we must only show that $\overline{p} \in X^d$.

To prove that we apply the duality results of Section 3. To this effect let us recall from Corollary 3.1 that for

$$x'_n(t) = L^*_p(t, p_n(t)) \tag{4.3}$$

$x_n(t) = c_{x_n} + \int_0^t x'_n(s)\,ds$ belongs to X where c_{x_n} is such that $\int_0^T V_x(t, x_n(t))\,dt = 0$ (see Lemma 1.1). Then $\{x_n\}$ is a minimizing sequence for J. We easily check that $\{c_{x_n}\}$ is a bounded sequence and therefore we may assume (up to a subsequence) that it is convergent. From (4.3) we infer that $\{x'_n\}$ is a bounded sequence in L^2 norm and that it is pointwise convergent to

$$\overline{x}'(t) = L^*_p(t, \overline{p}(t)).$$

Therefore $\{x_n\}$ is uniformly convergent to $\overline{x}$ where $\overline{x}(t) = c_{\overline{x}} + \int_0^t \overline{x}'(s)\,ds$ and $c_{\overline{x}}$ is such that the equality: $\int_0^T V_x(t, \overline{x}(t))\,dt = 0$ holds.

By Corollary 3.1 (see (3.1)) we also have (taking into account (4.3)) that for $\varepsilon_n \to 0$ $(n \to \infty)$

$$0 \le \int_0^T (V^*(t, -p'_n(t)) + V(t, x_n(t)))\,dt - \int_0^T \langle x_n(t), -p'_n(t)\rangle\,dt \quad \le \varepsilon_n$$

and so, taking a limit

$$0 = \int_0^T V^*(t, -\overline{p}'(t))\,dt + \lim_{n\to\infty} \int_0^T V(t, x_n(t))\,dt - \int_0^T \langle \overline{x}(t), -\overline{p}'(t)\rangle\,dt$$

and next, in view of the property of Fenchel inequality,

$$0 = \int_0^T V^*(t, -\overline{p}'(t))\,dt + \int_0^T V(t, \overline{x}(t)))\,dt - \int_0^T \langle \overline{x}'(t), -\overline{p}'(t)\rangle\,dt.$$

We have also $\overline{p}(t) = L_{x'}(t, \overline{x}'(t))$. Thus $\overline{p} \in X^d$ and the proof is completed. $\square$

A direct consequence of Theorem 4.1 and Corollary 2.1 is the following main theorem.

Theorem 4.2. *Under hypothesis* (H) *and* (H1) *there exists a pair* $(\overline{x}, \overline{p})$ *being a solution of* (1.1) *and such that*

$$J(\overline{x}) = \min_{x\in X} J(x) = \min_{p\in X^d} J_D(p = J_D(\overline{p}).$$

5. Dependence on parameters

In this section we consider the question of the continuous dependence on parameters of the set of critical points of the functional J and the set of solutions to equation

$$\frac{d}{dt} L_{x'}(t, x'(t)) + V_x(t, x(t), u(t)) = 0, \qquad \text{a.e. in } \mathbb{R}, \tag{5.1}$$

where $u \in \mathcal{U} = \{w \in L^\infty(0,T) : w(t) \in U \text{ for a.e. } t \in [0,T]\}$. Here U is a given subset of $\mathbb{R}^m$. It is clear that (5.1) is the Euler–Lagrange equation to the functional

$$J(x,u) = \int_0^T (-V(t,x(t),u(t)) + L(t,x'(t)))\,dt \tag{5.2}$$

This problem plays an essential role in applications of differential equations. In the best knowledge of the authors, the problem of the continuous dependence on parameters of solutions of superlinear equation (5.1) has not been investigated up to now. In the seventies, some papers were published which deal with Dirichlet problem for scalar ordinary differential equations. All these works are based on direct methods (cf. [3], [4], [10] and references therein). In the pioneering work [11], sufficient conditions for the continuous dependence on parameters for vector systems of ODE are given. This work is based on variational methods. We also apply the variational approach.

From now, we assume that V has a special structure: $V(t,x,u) = W(t,x) + \langle g(t,u), x\rangle$, $g : [0,T] \times \mathbb{R}^m \to \mathbb{R}$, and W and L satisfy conditions from Section 1, in particular (1.4). The set $\overline{X}$ is now defined as follows:

$$\overline{X} = \left\{v \in A : \int_0^T W(t,v(t)) \le \frac{1}{2}\int_0^T L(t,v'(t))\,dt + k_3\right\},$$

Let us also introduce, for $u \in \mathcal{U}$ the sets $\widetilde{X}_u$

$$\begin{aligned}\widetilde{X}_u = \bigg\{&x(\,\cdot\,) + c_x \in \overline{X} : x \in A_0,\ c_x \in \mathbb{R}^n \text{ is such that}\\ &\int_0^T V_x(t,x(t)+c_x,u(t))\,dt = 0,\\ &\text{and } p(t) = L_{x'}(t,x'(t)),\ t \in [0,T] \text{ belongs to } A_{0,0}\bigg\},\end{aligned}$$

and sets $X_u \subset \widetilde{X}_u$ with the property that for each $v \in X_u$, there exists (possible another) $\widetilde{v} \in X_u$ such that $V_x(t,v(t),u(t)) = -\frac{d}{dt}L_{x'}(t,\widetilde{v}'(t))'$, for a.e. $t \in [0,T]$. Moreover, as the dual set to X_u we shall consider the following set

$$\begin{aligned}X_u^d = \{&p \in A_{0,0} : \text{ there exist } v \in X_u \text{ such that}\\ &p(t) = L_{x'}(t,v'(t)), t \in [0,T] \text{ a.e.}\}.\end{aligned}$$

The above settings allow us to apply, for every $u \in \mathcal{U}$, Theorem 4.1 and Corollary 2.1. We can always assume that $0 \in \mathcal{U}$ and $g(t,0) = 0$.

Theorem 5.1. *Let $\{u_n\}$ be sequence of elements $\mathcal{U}$ such that $\{g(\,\cdot\,,u(\,\cdot\,))\}$ is pointwise convergent in $[0,T]$ to $g(\,\cdot\,,0)$ and $\int_0^T |g(t,u_n(t))|\,dt \le M$ for some M and all $n \in \mathbb{N}$. If $\{x_n,p_n\}$ are solutions (dependent on u_n) to problem* (5.1) *assume also, that $J(x_n, u_n) \le b$ for some b and all $n \in \mathbb{N}$. Then there exist $\overline{x} \in X_0$ and $\overline{p} \in X_0^d$ such that*

$$\begin{cases} -\overline{p}'(t) = W_x(t, \overline{x}(t)), \\ \overline{p}(t) = L_{x'}(t, \overline{x}'(t)), \end{cases}$$

and there exists a subsequence of $\{x_n, p_n\}$ (which we still denote by $\{x_n, p_n\}$) uniformly convergent in $[0,T]$ to $\{\overline{x}, \overline{p}\}$.

Proof. We easily check, since $x_n \in \overline{X}$, that

$$\begin{aligned} b \ge J(x_n, u_n) &\ge \frac{\alpha_1}{4} \int_0^T |x_n'(t)|^2\,dt - \int_0^T |x_n(t)||g(t, u_n(t))|\,dt - K \\ &\ge \frac{\alpha_1}{4} |x_n'|_{L^2}^2 - M|x_n|_{L^\infty} - K \ge \widetilde{\alpha}|x_n'|_{L^2}^2 - \widetilde{K} \end{aligned}$$

for some $\widetilde{\alpha} > 0$, $K, \widetilde{K} \in \mathbb{R}$, and therefore there exists a subsequence of $\{x_n'\}$ (which we still denote by $\{x_n'\}$) weakly convergent in L^2 to some $\overline{x}' \in L^2$. Since $x_n(t) = c_{x_n} + \int_0^t x_n'(s)\,ds$ then $x_n \to \overline{x}$ uniformly in $[0,T]$ and $\overline{x}(t) = c_{\overline{x}} + \int_0^t \overline{x}'(s)\,ds$ belongs to A. From Theorem 4.1 and Corollary 2.1 we also know that

$$\begin{aligned} p_n'(t) &= -W_x(t, x_n(t)) + g(t, u_n(t)) && \text{and} \\ p_n(t) &= L_{x'}(t, x_n(t)) && \text{for } t \in [0,T] \text{ a.e. } n \in \mathbb{N}. \end{aligned} \tag{5.3}$$

Hence $p_n(t) = \int_0^t p_n'(s)\,ds$ belongs to $X_{u_n}^d$. From (5.3) $\{p_n{}'\}$ is bounded in L^2 and pointwise convergent a.e. to

$$\overline{p}'(t) = -W_x(t, \overline{x}(t)). \tag{5.4}$$

Hence $p_n \to \overline{p}$ uniformly in $[0,T]$, where $\overline{p}(t) = \int_0^t \overline{p}'(s)\,ds$. Since p_n satisfies (5.3) therefore

$$0 = \int_0^T L^*(t, p_n(t))\,dt + \int_0^T L(t, x_n'(t)))\,dt - \int_0^T \langle x_n'(t), p_n(t)\rangle\,dt$$

and after taking a limit we get

$$0 = \int_0^T L^*(t, \overline{p}(t))\,dt + \lim_{n\to\infty} \int_0^T L(t, x_n'(t))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t)\rangle\,dt$$

and by Fenchel inequality and Fatou Lemma

$$0 = \int_0^T L^*(t, \overline{p}(t))\,dt + \int_0^T L(t, \overline{x}'(t)))\,dt - \int_0^T \langle \overline{x}'(t), \overline{p}(t)\rangle\,dt.$$

This means that $\overline{p}(t) = L_{x'}(t, \overline{x}(t))$, which together with (5.4) ends the proof. $\square$

An obvious consequence of the above theorem is the following

COROLLARY 5.1. *Under assumptions of Theorem* 5.1 *there exists* $\overline{x} \in X_0$ *and* $\overline{p} \in X_0^d$ *satisfying*

$$\begin{aligned} &\frac{d}{dt}\overline{p}(t) + W_x(t,\overline{x}(t)) = 0, && \text{a.e. in } \mathbb{R}, \\ &\overline{p}(t) = L_{x'}(t,\overline{x}'(t)), && \text{a.e. in } \mathbb{R}. \end{aligned}$$

6. Example

Consider the problem

$$\begin{aligned} &x''(t) + W_x(t,x(t)) = 0, \quad \text{a.e. in } [0,1], \\ &x'(0) = 0 = x'(1), \end{aligned} \tag{6.1}$$

where $W(\,\cdot\,,x)$ is a measurable function in $[0,T]$, $W(t,\,\cdot\,)$, $t \in [0,1]$, is a convex, continuously Frechet differentiable function satisfying the following growth condition:

- there exist $0 < \beta_1 < \beta_2$, $q_1 > 1$, $q > 2$, $k_1 \geq 0$, $k_2 > 0$ such that for $x \in L^q$
$$k_1 + \frac{\beta_1}{q_1}\|x\|^{q_1} \leq W(t,x) \leq \frac{\beta_2}{q}\|x\|^q + k_2.$$

In the notation of the paper we have $L(t,x') = |x'|^2/2$, and $V(t,x) = W(t,x)$. It is easily seen that assumptions (H) and (H1) are satisfied. Therefore what we have to do is to construct a nonempty set X defined in Section 1. To this effect let us take any $k > 0$ and let $\overline{X}$ denote the same as in Section 1 with the new L and V. We assume the following hypothesis:

(H1)' $k_3 > (\beta_2/q)k^q + k_2$,
$k_3 > k((q\beta_2^{1/(q-1)}/(q-1))(k+k_2-k_1)+1)^{q-1} + \int_0^1 W(t,0)\,dt$,
$(q\beta_2^{1/(q-1)}/(q-1))(k+k_2-k_1)+1)^{q-1} \leq \pi k/3$,
$(q_1/q)^{1/q_1}(k/3)^{q/q_1} + ((k_2-k_1)q_1)^{1/q_1} \leq k/3$.

We shall show that the set $X = \{v \in \widetilde{X} : 0 < \|v\|_{L^\infty} \leq k\}$, where

$$\widetilde{X} = \Big\{ x(\,\cdot\,) + c_x \in \overline{X} : x \in A_0,\ c_x \in \mathbb{R}^n \text{ is such that } \int_0^1 W_x(t,x(t)+c_x)dt = 0, \text{ and } p(t) = v'(t),\ t \in [0,1] \text{ belongs to } A_{0,0} \Big\}$$

is a set X which we are looking for. That means: we must prove that for each function $x \in X$ the primitive of the function

$$t \to \int_0^t W_x(\tau,x(\tau))\,d\tau = w'(t), \tag{6.2}$$

belongs to X i.e. $w(t) = c_w + \int_0^t w'(s)\,ds$ with c_w such that $\int_0^1 W_x(\tau, w(\tau))d\tau = 0$. It is obvious that $w' \in A_{0,0}$. Thus what we have to show is that $\|w\|_{L^\infty} \leq k$ because then, by the second of assumptions (H1)' we shall get the inequality $\int_0^1 W(t, w(t))\,dt \leq (1/2)\int_0^1 L(t, w'(t))\,dt + k_3$. If we take $p(t) = w'(t)$ ($w'(t)$ defined by (6.2)) then, by known theorem (taking into account the first of assumptions (H1)'),

$$\|p'\|_{L^\infty} \leq (q\beta_2^{1/(q-1)}/(q-1))(k + k_2 - k_1) + 1)^{q-1}$$

and next applying the estimation for the function by its derivative (for functions with zero at the ends) we have

$$\|w'\|_{L^2} \leq \frac{1}{\pi}\|p'\|_{L^\infty}.$$

Next using the estimations on $W(t, x)$ and the last two assumptions of (H1)' we obtain

$$\|w\|_{L^\infty} \leq k.$$

Therefore w belongs to X. It is clear that the set X is nonempty. Thus all assumptions of Theorem 4.2 are satisfied, so we come to the following theorem with $L = |x'|^2/2$.

Theorem 6.1. *There exists a pair* $(\overline{x}, \overline{p})$ *being a solution of* (6.1) *and such that*

$$J(\overline{x}) = \min_{x \in X} J(x) = \min_{p \in X^d} J_D(p) = J_D(\overline{p}).$$

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ANDRZEJ NOWAKOWSKI AND ANDRZEJ ROGOWSKI
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, POLAND

E-mail address: annowako@math.uni.lodz.pl, arogow@math.uni.lodz.pl