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MORSE RELATIONS FOR GEODESICS ON STATIONARY LORENTZIAN MANIFOLDS WITH BOUNDARY

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction and statement of the results

Morse theory relates the set of critical points of a smooth functional defined on a Hilbert manifold to the topology of the manifold itself. Morse himself gave the first application of his theory to Riemannian geometry (cf. [6, 11, 12]), proving two very nice and famous results. In order to recall them, consider a Riemannian manifold $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle_x)$ with Riemannian structure $\langle\,\cdot\,,\,\cdot\,\rangle_x$.

A curve $\gamma :]a,b[\to \mathcal{M}$ is said to be a *geodesic* if

$$\nabla_s \dot\gamma(s) = 0 \qquad \text{for any } s \in]a,b[,$$

where $\dot\gamma$ is the derivative of γ and $\nabla_s\dot\gamma$ is the covariant derivative of $\dot\gamma$ along γ. It is well known that the geodesic curves joining two given points satisfy a variational principle. Indeed, $\gamma : [0,1] \to \mathcal{M}$ is a geodesic joining x_0 and x_1 (and defined in the interval $[0,1]$) if and only if γ is a critical point of the *action integral*

$$f(x) = \frac{1}{2}\int_0^1 \langle \dot x, \dot x\rangle_x \, ds$$

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defined on the manifold of sufficiently smooth curves joining x_0 and x_1. Thus $\gamma : [0,1] \to \mathcal{M}$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ is a geodesic if and only if

$$\int_0^1 \langle \dot{x}, \nabla_s v \rangle_x \, ds = 0$$

for any smooth vector field v along γ such that $v(0) = v(1) = 0$. (Here $\nabla_s v$ denotes the covariant derivative of v along γ.)

The first result of Morse concerns the Morse index of a geodesic γ as a critical point of the action integral (i.e. the maximal dimension of the subspaces of the tangent space along γ where the Hessian of f is negative definite). Morse proved that it is finite and equal to the number of conjugate points along the geodesic, counted with their multiplicity (cf. Definitions 1.2–1.3 with $\mathcal{M}$ replaced by a Riemannian manifold).

The second result of Morse concerns the so-called Morse relations for the action integral, under certain nondegeneracy assumptions (cf. also [6]).

Theorem 1.1 (Morse relations). *Let $(\mathcal{M}, \langle \cdot\,,\,\cdot \rangle_x)$ be a complete Riemannian manifold, and x_0 and x_1 two nonconjugate points of $\mathcal{M}$ (i.e. they are nonconjugate along every geodesic joining them). Let Ω_0 be the set of continuous curves joining x_0 and x_1, equipped with the uniform topology, and Z the set of geodesics joining x_0 and x_1. Then there exists a formal series $Q(r)$ with natural coefficients (possibly ∞) such that*

$$\sum_{\gamma \in Z} r^{m(\gamma, f)} = P_r(\Omega_0) + (1 + r)Q(r),$$

where $m(\gamma, f)$ is the Morse index of γ as a critical point of f and $P_r(\Omega_0)$ is the Poincaré polynomial of Ω_0 with coefficients in an arbitrary field F.

Recall that denoting by $H_k(\Omega_0, F)$ the kth singular homology group of Ω_0 with coefficients in F, we have

$$P_r(\Omega_0) = \sum_{k \geq 0} \dim H_k(\Omega_0, F) r^k.$$

In this paper we extend Theorem 1.1 to stationary Lorentzian manifolds with boundary which satisfies a convexity property. Some physically relevant cases like *Schwarzschild* and *Reissner–Nordström* space-times are covered by these results. The Morse index of a geodesic (which is always ∞) will be replaced by a geometric index. Indeed, as pointed out in Remark 1.4, the geometric index of a geodesic on a stationary Lorentzian manifold is always finite, and it seems the right tool for extending the Morse relations of Theorem 1.1.

Before stating our results, some recalls of Lorentzian geometry are needed.

A *Lorentzian manifold* is a couple $(\mathcal{M}, \langle \cdot\,,\,\cdot \rangle)$, where $\mathcal{M}$ is a connected finite-dimensional manifold, and $\langle \cdot\,,\,\cdot \rangle$ is a *Lorentzian metric* on $\mathcal{M}$, i.e. a metric tensor having index 1 (cf. [14]). The points of a Lorentzian manifold are often called *events*.

As for Riemannian manifolds, a smooth curve on a Lorentzian manifold $\gamma :]a,b[\to \mathcal{M}$ is said to be a *geodesic* if

$$D_s\dot\gamma = 0, \tag{1.1}$$

where $D_s\dot\gamma$ is the covariant derivative of $\dot\gamma$ with respect to the Lorentzian metric $\langle\,\cdot\,,\,\cdot\,\rangle$.

It is well known that if γ is a geodesic, then there exists a real constant $E(\gamma)$ such that for any $s \in]a,b[$,

$$E(\gamma) = \langle\dot\gamma(s),\dot\gamma(s)\rangle.$$

The geodesic γ is called *timelike*, *lightlike* or *spacelike* according as $E(\gamma)$ is negative, null or positive. The geodesic γ is called *causal* if $E(\gamma) \le 0$. In general relativity a timelike geodesic represents the trajectory of a free falling particle. Null geodesics represent the light rays, while spacelike geodesics, for a suitable local observer, represent Riemannian geodesics consisting of simultaneous events.

We now recall the notions of conjugate point and geometric index for a geodesic, which will be the basic tools for the Morse relations. They are just the extensions to Lorentzian geodesics of well known concepts for Riemannian geodesics.

Definition 1.2. Let $(\mathcal{M},\langle\,\cdot\,,\,\cdot\,\rangle)$ be a Lorentzian manifold and γ a geodesic joining p and q and defined in the interval $[0,1]$. A point $\gamma(s)$, $s \in]0,1]$, is said to be *conjugate to p along* γ if there exists a smooth vector field $v \neq 0$ along γ which is a solution of the problem

$$D_s^2 v + R(\dot\gamma, v)\dot\gamma = 0, \qquad v(0) = v(s) = 0, \tag{1.2}$$

where $R(\,\cdot\,,\,\cdot\,)$ is the curvature tensor for the metric $\langle\,\cdot\,,\,\cdot\,\rangle$ and $D_s^2 v$ is the second covariant derivative of v along γ. The maximal number of linearly independent solutions of (1.2) is called the *multiplicity* of $\gamma(s)$.

Definition 1.3. Let $\gamma : [0,1] \to \mathcal{M}$ be a geodesic. The *geometric index* of γ is the number of conjugate points $\gamma(s)$, $s \in]0,1[$, to p along γ, counted with their multiplicity.

Remark 1.4. The geometric index of a geodesic can be ∞ (see [10]); however, it is always finite for geodesics on stationary Lorentzian manifolds.

Some results of Morse theory for geodesics in globally hyperbolic Lorentzian manifolds (cf. [14] for the definition) have been obtained in [16] (cf. also [2] and the references therein). Note that in [2, 16], only causal geodesics (i.e. geodesics with nonpositive energy) are considered. Therefore, the index of causal geodesics reflects only the topology of the space of causal curves. A smooth curve $z(s)$ is said to be *causal* if $\langle \dot z(s), \dot z(s)\rangle \le 0$ for any s. In the papers quoted above, global hyperbolicity is essential. However, there are physically interesting Lorentzian manifolds which are not globally hyperbolic, because they have a topological boundary (for instance, they are open subsets of a larger manifold).

In this paper we consider stationary Lorentzian manifolds with boundary. $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle_z)$ will be a Lorentzian manifold such that

$$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \tag{1.3}$$

where $(\mathcal{M}_0, \langle\,\cdot\,,\,\cdot\,\rangle_x)$ is a smooth Riemannian manifold, and $\langle\,\cdot\,,\,\cdot\,\rangle_z$ is a stationary metric, i.e. for any $z = (x,t) \in \mathcal{M}_0 \times \mathbb{R}$ and for any $\zeta = (\xi, \tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbb{R}$,

$$\langle \zeta, \zeta\rangle_z = \langle \xi, \xi\rangle_x + 2\langle \delta(x), \xi\rangle_x \tau - \beta(x)\tau^2, \tag{1.4}$$

where $\beta(x)$ is a smooth scalar field on $\mathcal{M}_0$ and $\delta(x)$ is a smooth vector field on $\mathcal{M}_0$.

Classical examples of Lorentzian manifolds satisfying (1.3)–(1.4) are *Schwarzschild*, *Reissner–Nordström* and *Kerr* space-times (cf. [9] for their physical meaning).

The Schwarzschild metric is the solution of the Einstein equations corresponding to the exterior gravitational field produced by a static spherically symmetric massive body. The Schwarzschild metric is given, in polar coordinates, by

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 - \left(1 - \frac{2m}{r}\right) dt^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the standard metric of the unit 2-sphere in the Euclidean 3-space and m represents the mass of the body.

The Schwarzschild space-time is the Lorentzian manifold

$$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \qquad \mathcal{M}_0 = \{(r,\theta,\varphi) : r > 2m\},$$

equipped with the above metric.

The Reissner–Nordström space-time is the solution of the Einstein–Maxwell equations corresponding to the exterior gravitational field produced by a static spherically symmetric charged body. Denoting by m and e respectively the mass and the charge of the body, the metric, in polar coordinates, is given by

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2.$$

Whenever $m^2 > e^2$, the Reissner–Nordström space-time is the Lorentzian manifold

$$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \qquad \mathcal{M}_0 = \{(r,\theta,\varphi) : r > m + \sqrt{m^2 - e^2}\},$$

equipped with the above metric.

The Kerr space-time is the space-time outside an axisymmetric rotating body and its metric is stationary and nonstatic. Whenever there is no rotation of the body, it reduces to the Schwarzschild space-time.

We first consider the case of an open subset $\mathcal{M}$ of a stationary Lorentzian manifold $\widetilde{\mathcal{M}}$. The boundary of $\mathcal{M}$ will satisfy the following convexity property.

Definition 1.5. Let $(\widetilde{\mathcal{M}}, \langle\cdot\,,\cdot\rangle)$ be a Lorentzian manifold and $\mathcal{M}$ an open connected subset of $\widetilde{\mathcal{M}}$ with boundary $\partial\mathcal{M}$. We say that $\mathcal{M}$ has a *convex boundary* $\partial\mathcal{M}$ if any geodesic $z : [a,b] \to \mathcal{M} \cup \partial\mathcal{M}$ with $z(a), z(b) \in \mathcal{M}$ has support $z([a,b]) \subset \mathcal{M}$.

Let $\Phi : \mathcal{M} \to \mathbb{R}$ be a smooth function such that

$$\begin{cases} \Phi(z) = 0 \Leftrightarrow z \in \partial\mathcal{M}, \\ \Phi(z) > 0 \Leftrightarrow z \in \mathcal{M}, \\ \nabla\Phi(z) \neq 0 \ \forall z \in \partial\mathcal{M} \end{cases} \tag{1.5}$$

(whose existence can be proved using the distance from the boundary). Then, if $\partial\mathcal{M}$ is smooth and convex, for any $z \in \partial\mathcal{M}$ and $\zeta \in T_z\partial\mathcal{M}$,

$$H^\Phi(z)[\zeta,\zeta] \leq 0,$$

where $H^\Phi(z) : T_z\mathcal{M} \times T_z\mathcal{M} \to \mathbb{R}$ denotes the Hessian of Φ at the point z. We recall that

$$H^\Phi(z)[\zeta,\zeta] = \left.\frac{d^2\Phi(\gamma(s))}{ds^2}\right|_{s=0},$$

where $\gamma(s)$ is the geodesic such that $\gamma(0) = z$ and $\dot\gamma(0) = \zeta$.

Now, assume that

$$\partial\mathcal{M}_0 \text{ is a smooth submanifold of } \widetilde{\mathcal{M}}_0; \tag{1.7}$$

$$\mathcal{M}_0 \cup \partial\mathcal{M}_0 \text{ is complete with respect to } \langle\cdot\,,\cdot\rangle_x \tag{1.8}$$

(i.e. any geodesic $x :]a,b[\to \mathcal{M}_0$ with respect to the Riemannian structure $\langle\cdot\,,\cdot\rangle_x$ can be extended to a continuous curve $\overline{x} : [a,b] \to \mathcal{M}_0 \cup \partial\mathcal{M}_0$);

$$\partial\mathcal{M} = \partial\mathcal{M}_0 \times \mathbb{R} \text{ is convex;} \tag{1.9}$$

$$\sup_{x\in\mathcal{M}_0} \langle\delta(x), \delta(x)\rangle_x < \infty, \qquad 0 < \inf_{x\in\mathcal{M}_0} \beta(x) \leq \sup_{x\in\mathcal{M}_0} \beta(x) < \infty. \tag{1.10}$$

The following result holds.

Theorem 1.6. *Assume that* (1.7)–(1.10) *hold. Assume also that* $z_0 = (x_0, t_0)$ *and* $z_1 = (x_1, t_1)$ *are nonconjugate* (*i.e. they are nonconjugate along any geodesic joining them*). *Let*

$$Z = \{z = (x,t) : [0,1] \to \mathcal{M} : z \textit{ is a geodesic such that } z(0) = z_0,\ z(1) = z_1\}.$$

Moreover, let Ω be the space of continuous curves joining z_0 and z_1 in $\mathcal{M}$, equipped with the uniform topology. Then

$$\sum_{z\in Z} r^{\mu(z)} = P_r(\Omega,F)+(1+r)Q(r), \tag{1.11}$$

where $Q(r)$ is a formal series with natural coefficients (possibly ∞) and $P_r(\Omega)$ is the Poincaré polynomial of Ω with coefficients in an arbitrary field F.

Corollary 1.7. *Assume that the Riemannian manifold $\mathcal{M}_0$ satisfies* (1.7)–(1.8) *and its boundary is convex* (cf. Definition 1.5 with $\mathcal{M}$ replaced by $\mathcal{M}_0$). *If x_0 and x_1 are nonconjugate points of $\mathcal{M}_0$, then the conclusion of Theorem* 1.1 *holds.*

Remark 1.8. Theorem 1.6 is clearly a generalization of Theorem 1.1 (cf. also Corollary 1.7). It is also a generalization of the Morse relations obtained in [5] in the static case (i.e. $\delta(x)\equiv 0$), with $\mathcal{M}_0$ complete and without boundary.

Now we consider the case in which $\mathcal{M}$ has a topological boundary which is nonsmooth, the metric is not defined on the boundary, $\mathcal{M}_0$ is noncomplete and $\beta(x)$ may approach 0 near the boundary.

This is for instance the case of Schwarzschild and Reissner–Nordström spacetimes (cf. [4] for more details), so Theorem 1.6 cannot be applied to these Lorentzian manifolds. In this case we reinforce a little bit the assumptions on the convexity of the boundary, to gain also control of its nonsmoothness. This kind of assumptions are similar to those used in [4, 7] to study the geodesic connectedness for a class of noncomplete Lorentzian manifolds.

Let $(\mathcal{M},\langle\,\cdot\,,\,\cdot\,\rangle)$ be a stationary Lorentzian manifold and assume that $\mathcal{M}$ has a topological boundary (not necessarily smooth), satisfying: there exists $\varphi\in C^2(\mathcal{M},\mathbb{R}_+\setminus\{0\})$ such that

$$\lim_{z\to z_0\in\partial\mathcal{M}}\varphi(z)=0, \qquad \varphi(z)=\varphi(x,t)=\varphi(x,0)=\varphi(x). \tag{1.12}$$

For any bounded set B in $\mathcal{M}$, there exist positive constants N, L, ν, ϱ such that the function φ of (1.12) satisfies:

$$\begin{aligned} &z\in B,\ \varphi(z)<\varrho \Rightarrow N\ge\langle\nabla\varphi(z),\nabla\varphi(z)\rangle_z\ge\nu,\\ &z\in B,\ \varphi(z)<\varrho,\ \zeta\in T_z\mathcal{M}\Rightarrow\\ &\qquad\qquad H^{\varphi}(z)[\zeta,\zeta]\le L\varphi(z)\ [\langle\xi,\xi\rangle_x+|\langle\delta(x),\xi\rangle_x\tau|+\beta(z)\tau^2]. \end{aligned} \tag{1.13}$$

(Here $\nabla\varphi$ denotes the gradient of φ with respect to the Lorentzian structure of $\mathcal{M}$).

Concerning the Morse relations on stationary Lorentzian manifolds with nonsmooth boundary, we have the following

Theorem 1.9. *Let* $(\mathcal{M},\langle\,\cdot\,,\,\cdot\,\rangle_z)$ *be a stationary Lorentzian manifold satisfying* (1.12)–(1.13). *Moreover, assume that:*

(1.14) *For any* $\eta>0$, *the set* $\{x\in\mathcal{M}_0:\varphi(x)\geq\eta\}$ *is complete with respect to the Riemannian structure of* $\mathcal{M}_0$;

(1.15) β *is bounded on* $\mathcal{M}_0$;

(1.16) *for any bounded subset* B *of* $\mathcal{M}_0$, *there exists* $\varrho_0>0$ *such that for any* $x\in B$ *with* $\varphi(x)<\varrho_0$, $\langle\operatorname{grad}\varphi(x),\operatorname{grad}\beta(x)\rangle_x\geq 0$, *where* grad *denotes the gradient with respect to the Riemannian structure of* $\mathcal{M}_0$;

(1.17) δ/β *is bounded on* $\mathcal{M}_0$;

(1.18) δ/β *and* δ *are uniformly continuous on bounded subsets of* $\mathcal{M}_0$.

Then the assertion of Theorem 1.6 *holds for* $\mathcal{M}$.

Following the computations developed in the appendix of [4], it is not difficult to verify that Schwarzschild and Reissner–Nordström space-times satisfy (1.12)–(1.18), by choosing

$$\varphi(z)=\sqrt{\beta(r)}.$$

In particular, the Morse relations hold for Schwarzschild and Reissner–Nordström space-times (where $\delta\equiv 0$).

The main difficulty in proving Theorems 1.6 and 1.9 is the indefiniteness of the action integral and the lack of compactness due to the presence of the boundary. We overcome the first difficulty by using a variational principle proved in [8], which reduces (in the stationary case) the search for critical points of f to the search for critical points of a suitable functional J depending only on the spatial variable x and bounded from below. As observed in [5] for the static case, the Morse index of a critical point x of J is equal to the geometric index of the corresponding geodesic $(x,t(x))$. This is also true in the stationary case.

Moreover, in order to overcome the lack of compactness, we use a suitable penalizing family of functionals J_ε ($\varepsilon\in[0,1]$) such that $J_0=J$. The convexity (or assumptions (1.12)–(1.13)) of the boundary allows us to prove some a priori estimates on the critical points of the penalizing functionals J_ε. By means of the a priori estimates, we show that the singular homology of the sublevels of J and J_ε coincides (cf. Lemma 4.5 and Propositions 4.8–4.9). This leads to the proof of Theorems 1.6–1.9, by passing to the limit in the Morse relations for J_ε.

2. Some preliminary results

Let $(\mathcal{M},\langle\,\cdot\,,\,\cdot\,\rangle_z)$ be a stationary Lorentzian manifold satisfying (1.3)–(1.4). By the well known Nash embedding theorem (cf. [13]), the Riemannian manifold $(\mathcal{M}_0,\langle\,\cdot\,,\,\cdot\,\rangle_x)$ is isometric to a submanifold of $\mathbb{R}^N$ (with N sufficiently large) equipped with the Euclidean metric. Hence, we can assume that $\mathcal{M}_0$ is a submanifold of $\mathbb{R}^N$ and $\langle\,\cdot\,,\,\cdot\,\rangle_x$ is the Euclidean metric, which will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$.

Set $I=[0,1]$, and for every $m\in\mathbb{N}$, let $H^{1,2}(I,\mathbb{R}^N)$ be the Sobolev space of absolutely continuous curves whose derivative is square summable. It is a Hilbert space with norm

$$\|x\|_1^2=\|x\|^2+\|\dot x\|^2=\int_0^1\langle x,x\rangle\,ds+\int_0^1\langle\dot x,\dot x\rangle\,ds, \tag{2.1}$$

where $\dot x$ denotes the derivative of x and $\|\cdot\|$ the usual norm of $L^2(I,\mathbb{R}^N)$.

Now, let x_0 and x_1 be two points of $\mathcal{M}_0$, and

$$\Omega^1=\Omega^1(x_0,x_1,\mathcal{M}_0)=\{x\in H^{1,2}(I,\mathbb{R}^N):x(I)\subset\mathcal{M}_0,\ x(0)=x_0,\ x(1)=x_1\}.$$

It is well known that Ω^1 is a submanifold of $H^{1,2}(I,\mathbb{R}^N)$ and, for any $x\in\Omega^1$, the tangent space to Ω^1 at x is

$$T_x\Omega^1=\{\xi\in H^{1,2}(I,\mathbb{R}^N):\xi(s)\in T_{x(s)}\mathcal{M}_0 \text{ for any } s\in I,\ \xi(0)=\xi(1)=0\}$$

(cf. e.g. [15]).

On Ω^1 we put the following Riemannian structure:

$$\langle\xi,\xi\rangle_1=\int_0^1\langle\nabla_s\xi,\nabla_s\xi\rangle\,ds,$$

where $\nabla_s\xi$ is the covariant derivative of ξ with respect to the Riemannian structure of $\mathcal{M}_0$.

Now, let t_0 and t_1 be two points of $\mathbb{R}$, and consider

$$H^{1,2}(t_0,t_1)=\{x\in H^{1,2}(I,\mathbb{R}):t(0)=t_0,\ t(1)=t_1\}.$$

Then $H^{1,2}(t_0,t_1)$ is a closed affine submanifold of $H^{1,2}(I,\mathbb{R})$ whose tangent space at every point is

$$H_0^{1,2}(I,\mathbb{R})=\{\tau\in H^{1,2}(I,\mathbb{R}):\tau(0)=\tau(1)=0\}.$$

Finally, let $z_0=(x_0,t_0)$ and $z_1=(x_1,t_1)$ be two points of $\mathcal{M}$, and consider the path space of $H^{1,2}$-curves joining z_0 and z_1 on $\mathcal{M}$,

$$\mathcal{Z}=\mathcal{Z}(z_0,z_1,\mathcal{M})=\Omega^1\times H^{1,2}(t_0,t_1).$$

Obviously, for any $z=(x,t)\in\mathcal{Z}$, the tangent space to z at $\mathcal{Z}$ is

$$T_z\mathcal{Z}=T_x\Omega^1\times H_0^{1,2}(I,\mathbb{R}).$$

In the following we shall also consider the Sobolev space $H^{1/2,2}(I,\mathbb{R}^N)$ consisting of $L^2(I,\mathbb{R}^N)$ curves having square summable derivative of order 1/2. More precisely, let

$$H^{1/2,2}(I,\mathbb{R}^N)=\left\{x\in L^2(I,\mathbb{R}^N):\sum_{k=0}^{\infty}k|x_k|^2<\infty\right\},$$

where $(x_k)_{k\in\mathbb{N}}$ are the Fourier coefficients of x with respect to the usual trigonometric basis of $L^2(I,\mathbb{R}^N)$. It is well known that $H^{1,2}(I,\mathbb{R}^N)$ is compactly embedded in $H^{1/2,2}(I,\mathbb{R}^N)$ (for more details, see [1]).

On the manifold $\mathcal{Z}=\Omega^1\times H^{1,2}(t_0,t_1)$ we consider the action integral

$$f(z)=\frac{1}{2}\int_0^1\langle\dot z,\dot z\rangle_z\,ds. \tag{2.2}$$

It is well known that f is smooth and its critical points are the geodesics joining $z_0=(x_0,t_0)$ and $z_1=(x_1,t_1)$.

REMARK 2.1. Let $\overline{z}=(\overline{x},\overline{t})$ be a geodesic joining z_0 and z_1. For every $s\in\,]0,1]$ consider the functional

$$f_s(z)=\frac{1}{2}\int_0^s\langle\dot z,\dot z\rangle_z\,dr$$

defined on the path space

$$\mathcal{Z}_s=\Omega^1_s(x_0,\overline{x}(s))\times H^{1,2}_s(t_0,\overline{t}(s))$$

of curves joining z_0 and $\overline{z}(s)=(\overline{x}(s),\overline{t}(s))$ and defined in the interval $[0,s]$. Denoting by $\overline{z}_s$ the restriction of $\overline{z}$ to $[0,s]$ and by f''_s the Hessian of f_s, it is not difficult to deduce from Definition 1.2 that

A point $\overline{z}(s)$ is conjugate to z_0 along $\overline{z}$ if and only if $\overline{z}_s$ is a degenerate critical point of f_s, i.e. the linear operator associated with f''_s has a kernel different from $\{0\}$.

Moreover, the dimension of the kernel of f''_s is just the multiplicity of the conjugate point $\overline{z}(s)$.

The search for geodesics joining z_0 and z_1, i.e. critical points of f, is more difficult than in the Riemannian case. Indeed, f is strongly indefinite, and the Morse index of its critical points is ∞.

In this section, we recall a variational principle which allows us to reduce the search for geodesics joining z_0 and z_1 to the search for critical points of a functional defined in Ω^1 and bounded from below.

Consider the action integral (2.2), i.e. the functional

$$f(z)=f(x,t)=\frac{1}{2}\int_0^1\langle\dot z,\dot z\rangle_z\,ds=\frac{1}{2}\int_0^1[\langle\dot x,\dot x\rangle+2\langle\delta(x),\dot x\rangle\dot t-\beta(x)\dot t^2]\,ds. \tag{2.3}$$

Let $f_x(x,t) : T_x\Omega^1 \to \mathbb{R}$ and $f_t(x,t) : H_0^{1,2}(I,\mathbb{R}) \to \mathbb{R}$ be the partial derivatives of f, and consider the set

$$N = \{z = (x,t) \in \mathcal{Z} : f_t(x,t) = 0\}.$$

Consider, for any fixed $x \in \Omega^1$, the problem

$$f_t(x,t) = 0, \quad t(0) = t_0, \quad t(1) = t_1.$$

This problem has a unique solution $t = t(x)$ that can be explicitly evaluated, yielding the following

Lemma 2.2. *N is the graph of the smooth map $\theta : \Omega^1 \to H^{1,2}(t_0,t_1)$ given by*

$$\begin{aligned}\theta(x)(s) = t_0 &+ \int_0^s \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\,dr \\ &+ \left(\int_0^s \frac{1}{\beta(x)}\,dr\right)\biggl((t_1 - t_0) \\ &- \int_0^1 \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\,dr\biggr)\left(\int_0^1 \frac{1}{\beta(x)}\,dr\right)^{-1}.\end{aligned} \tag{2.4}$$

Now consider the restriction of f to the graph of θ, i.e. consider the functional $J : \Omega^1 \to \mathbb{R}$,

$$\begin{aligned}J(x) &= f(x,\theta(x)) \\ &= \frac{1}{2}\int_0^1 \langle\dot{x},\dot{x}\rangle\,ds + \frac{1}{2}\int_0^1 \frac{\langle\delta(x),\dot{x}\rangle^2}{\beta(x)}\,ds \\ &\quad - \frac{1}{2}\,\frac{\left(t_1 - t_0 - \int_0^1 \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\,ds\right)^2}{\int_0^1 \frac{1}{\beta(x)}\,ds}.\end{aligned} \tag{2.5}$$

The following variational principle, proved in Theorem 2.2 of [8], holds.

Theorem 2.3. *Let $z = (x,t) \in \mathcal{Z}$. Then the following statements are equivalent:*

(a) *z is a critical point of f;*
(b) *$t = \theta(x)$ and x is a critical point of J.*

Moreover, if (a) *or* (b) *is true, then*

$$f(z) = J(x). \tag{2.6}$$

Let x be a critical point of J. Analogously to the static case, we have the following "second order variational principle" for the kernels of $J''(x)$ and $f''(x,\theta(x))$ (cf. [5] for the proof in the static case).

THEOREM 2.4. *Let* $x \in \Omega^1$ *be a critical point of* J, $z = (x, \theta(x))$ *and* $\zeta = (\xi, \tau) \in T_z\mathcal{Z}$. *Assume that the second order partial derivative* $f_{tt}(z)$ *is nondegenerate, i.e.*

$$f_{tt}(z)[\tau, \tau'] = 0, \ \forall \tau' \in H_0^{1,2}(I, \mathbb{R}) \Rightarrow \tau = 0.$$

Then the following statements are equivalent:

(a) $\zeta \in \operatorname{Ker} f''(z)$, *i.e.* $f''(z)[\zeta, \zeta'] = 0$ *for all* $\zeta' \in T_z\mathcal{Z}$;

(b) $\tau = \theta'(x)\xi$ *and* $\xi \in \operatorname{Ker} J''(x)$, *i.e.* $J''(x)[\xi, \xi'] = 0$ *for all* $\xi' \in T_x\Omega^1$, *where* θ' *denotes the differential of the map* θ.

REMARK 2.5. From (2.2), for any $z = (x, t) \in \mathcal{Z}$, we have

$$f_{tt}(z)[\tau, \tau] = -\int_0^1 \beta(x)\tau^2 \, ds \leq -\inf_{\mathcal{M}_0} \beta \|\dot{\tau}\|^2; \tag{2.7}$$

therefore, by the Lax–Milgram Theorem, $f_{tt}(z)$ is nondegenerate.

REMARK 2.6. As done in [5] for the static case, we can define θ_s as the analogue of θ with f_s (see Remark 2.1) instead of f, and we can put

$$J_s(y) = f_s(y, \theta_s(y)),$$

obtaining the following

DEFINITION 2.7. Let x be a critical point of J. A point $x(s)$, $s \in]0, 1]$, is said to be *conjugate to* x_0 *along* x if the kernel of $J_s''(x_s)$ is different from $\{0\}$. The dimension of this kernel is called the *multiplicity* of the conjugate point $x(s)$. Moreover, the *geometric index* of x is the number $\mu(x)$ of points conjugate to x_0 along x, counted with their multiplicity.

From Theorem 2.4 it is easy to deduce the following

THEOREM 2.8. *Let* $z = (x, \theta(x))$ *be a geodesic and* $s \in]0, 1]$. *Then:*

(a) $\zeta = (\xi, \tau) \in \operatorname{Ker} f_s''(z_s)$ *if and only if* $\tau = \theta_s'(x_s)\xi$ *and* $\xi \in \operatorname{Ker} J_s''(x_s)$;

(b) $z(s)$ *is conjugate to* z_0 *along* z *if and only if* $x(s)$ *is conjugate to* x_0 *along* x;

(c) $\mu(z) = \mu(x)$, *where the geometric index* $\mu(z)$ *of* z *is defined in Section* 1.

Now the following lemma is needed.

LEMMA 2.9. *Let* x *be a critical point of* J. *Then the linear operator associated* $J''(x)$ *is a compact perturbation of a* (*strictly*) *positive operator on* $T_x\Omega^1$.

Note that from Lemma 2.9, following the proof for Riemannian manifolds (cf. e.g. [6, 11]), we get the equality between the Morse index of x as a critical point of J and the geometric index $\mu(x)$. More precisely, we have the following

Theorem 2.10. *Let x be a critical point of J and $m(x) = m(x,J)$ be the Morse index of x as a critical point of J. Then $m(x) = \mu(x) < \infty$.*

Notice that from Theorems 2.8 and 2.10 we deduce immediately the following

Corollary 2.11. *Let $z = (x, \theta(x))$ be a critical point of f. Then $\mu(z) = m(x,J) < \infty$.*

Proof of Lemma 2.9. Let θ be as in (2.4). By (2.5) we obtain

$$J''(x)[\xi,\xi] = f_{xx}(x,\theta(x))[\xi,\xi] + f_{xt}(x,\theta(x))[\xi,\theta'(x)\xi], \tag{2.8}$$

hence we have to evaluate $f'(z)$ and $f''(z)$. To this end, let Y be a vector field on the Riemannian manifold $\mathcal{M}_0$. The first and second covariant differentials of Y are multilinear maps

$$Y' : \mathcal{X}(\mathcal{M}_0) \to \mathcal{X}(\mathcal{M}_0), \qquad Y'' : \mathcal{X}(\mathcal{M}_0) \times \mathcal{X}(\mathcal{M}_0) \to \mathcal{X}(\mathcal{M}_0)$$

(where $\mathcal{X}(\mathcal{M}_0)$ denotes the set of smooth vector fields on $\mathcal{M}_0$) defined in the following way (see [14] for the details): for any $X, X_1, X_2 \in \mathcal{X}(\mathcal{M}_0)$,

$$Y'[X] = \nabla_X Y, \qquad Y''[X_1,X_2] = \nabla_{X_1}\nabla_{X_2} Y - \nabla_{\nabla_{X_1} X_2} Y,$$

where $\nabla : \mathcal{X}(\mathcal{M}_0) \times \mathcal{X}(\mathcal{M}_0) \to \mathcal{X}(\mathcal{M}_0)$ is the Levi-Civita connection associated with the metric. The main properties of the covariant differential of a vector field are the following.

Let $x :]a,b[\to \mathcal{M}_0$ be a smooth curve and consider the vector field $\xi = Y \circ x$ along the curve x. Then the first and second covariant derivatives of ξ along x are respectively given by

$$\nabla_s \xi = Y'(x(s))[\dot{x}(s)], \tag{2.9}$$

$$\nabla_s^2 \xi = Y''(x(s))[\dot{x}(s),\dot{x}(s)] + Y'(x(s))[\nabla_s \dot{x}(s)]. \tag{2.10}$$

Now, let $z = (x,t) \in \mathcal{Z}$. In order to evaluate $f'(z)$ and $f''(z)$, consider the solution of the Cauchy problem

$$\nabla_\lambda x_\lambda(\lambda,s) = 0, \quad x(0,s) = x(s), \quad x_\lambda(0,s) = \xi(s), \tag{2.11}$$

defined on $]-\lambda_0,\lambda_0[\times I$, with $\lambda_0 > 0$, and consider the curve $z(\lambda,s) = (x(\lambda,s), t(s) + \lambda\tau(s))$ on $\mathcal{Z}$. Then

$$f'(z)[\zeta] = \frac{d}{d\lambda} f(z(\lambda,s))\bigg|_{\lambda=0},$$

while if z is a critical point of f, then

$$f''(z)[\zeta,\zeta] = \frac{d^2}{d\lambda^2} f(z(\lambda,s))\bigg|_{\lambda=0}.$$

Straightforward calculation shows that

$$\begin{aligned} f'(z)[\zeta] &= f'(x,t)[(\xi,\tau)] \\ &= \int_0^1 [\langle \dot x, \nabla_s \xi\rangle + \langle \delta'(x)[\xi], \dot x\rangle \dot t + \langle \delta(x), \nabla_s \xi\rangle \dot t + \langle \delta(x), \dot x\rangle \dot\tau]\, ds \\ &\quad - \int_0^1 \left[\frac{1}{2}\langle \nabla\beta(x), \xi\rangle \dot t^2 - \beta(x)\dot t\dot\tau\right] ds, \end{aligned}$$

where $\nabla\beta(x)$ denotes the gradient of β with respect to the Riemannian metric on $\mathcal{M}_0$.

Now, let $z = (x,t) = (x, \theta(x))$ be a critical point of f. Recalling the well known formula of Riemannian geometry,

$$\nabla_\lambda \nabla_s v = \nabla_s \nabla_\lambda v + R(x_s, x_\lambda) v,$$

where $R(\,\cdot\,,\,\cdot\,)[\,\cdot\,]$ denotes the curvature tensor of the metric on $\mathcal{M}_0$ (see e.g. [14]), using (2.10)–(2.11) and differentiating, we obtain

$$\begin{aligned} f''(z)[\zeta,\zeta] &= f''(z)[(\xi,\tau),(\xi,\tau)] \\ &= \int_0^1 [\langle \nabla_s\xi, \nabla_s\xi\rangle - \langle R(\dot x, \xi)[\dot x + \dot t\delta(x)], \xi\rangle + \langle \delta''(x)[\xi,\xi], \dot x\rangle \dot t]\, ds \\ &\quad + 2\int_0^1 [\langle \delta'(x)[\xi], \nabla_s\xi\rangle \dot t + \langle \delta'(x)[\xi], \dot x\rangle\dot\tau + \langle \delta(x), \nabla_s \xi\rangle\dot\tau]\, ds \\ &\quad - \frac{1}{2}\int_0^1 H^\beta(x)[\xi,\xi]\dot t^2\, ds - 2\int_0^1 \langle \nabla\beta(x), \xi\rangle \dot t\dot\tau\, ds - \int_0^1 \beta(x)\dot\tau^2\, ds, \end{aligned}$$

where $H^\beta(x)$ is the Hessian of β.

Now, setting $\tau = \theta'(x)[\xi]$, we get

$$f_{xt}(x,t)[(\xi,0),(0,\tau)] = 2\int_0^1 [\langle \delta'(x)[\xi], \dot x\rangle\dot\tau + \langle \delta(x), \nabla_s\xi\rangle\dot\tau - \langle \nabla\beta(x), \xi\rangle \dot t\dot\tau]\, ds,$$

and

$$\begin{aligned} f_{xx}(x,t)[(\xi,0),(\xi,0)] = &\int_0^1 [\langle \nabla_s\xi, \nabla_s\xi\rangle - \langle R(\dot x, \xi)[\dot x + \dot t\delta(x)], \xi\rangle \\ &+ \langle \delta''(x)[\xi,\xi], \dot x\rangle \dot t + 2\langle \delta'(x)[\xi], \nabla_s\xi\rangle \dot t]\, ds \\ &- \frac{1}{2}\int_0^1 H^\beta(x)[\xi,\xi]\dot t^2\, ds. \end{aligned}$$

Since the inclusions of $H^{1,2}(I,\mathbb{R}^N)$ in $H^{1/2,2}(I,\mathbb{R}^N)$ and $L^2(I,\mathbb{R}^N)$ respectively are compact, the quadratic form

$$\begin{aligned} &\int_0^1 [2\langle \delta'(x)[\xi], \dot x\rangle\dot\tau - 2\langle \nabla\beta(x), \xi\rangle \dot t\dot\tau - \langle R(\dot x, \xi)[\dot x + \dot t\delta(x)], \xi\rangle]\, ds \\ &\qquad + 2\int_0^1 [\langle \delta'(x)[\xi], \nabla_s\xi\rangle \dot t + \langle \delta''(x)[\xi,\xi], \dot x\rangle \dot t]\, ds - \frac{1}{2}\int_0^1 H^\beta(x)[\xi,\xi]\dot t^2\, ds \end{aligned}$$

defines a compact operator on $T_x\Omega^1$. It remains to study the quadratic form

$$\int_0^1 \langle \delta(x), \nabla_s \xi \rangle \dot{\tau}\, ds.$$

Towards this goal, we have to evaluate $\dot{\tau}$ (recall that $\theta(x) = t$ and $\tau = \theta'(x)[\xi]$). From (2.4), it follows that we can assume β to be a constant function, because the contributions of $\nabla\beta(x)$ to $\int_0^1 \langle \delta(x), \nabla_s \xi \rangle \dot{\tau}\, ds$ are compact. Then, assuming $\beta \equiv 1$, differentiating (2.4) with respect to x and taking the derivative gives

$$\dot{\tau} = \langle \delta(x), \nabla_s \xi \rangle + \langle \delta'(x)[\xi], \dot{x} \rangle - \int_0^1 [\langle \delta(x), \nabla_s \xi \rangle + \langle \delta'(x)[\xi], \dot{x} \rangle]\, ds.$$

Hence we obtain

$$\begin{aligned} \int_0^1 &\langle \delta(x), \nabla_s \xi \rangle \dot{\tau}\, ds \\ &= \int_0^1 \langle \delta(x), \nabla_s \xi \rangle^2\, ds + \int_0^1 \langle \delta(x), \nabla_s \xi \rangle \langle \delta'(x)[\xi], \dot{x} \rangle\, ds \\ &\quad - \bigg(\int_0^1 \langle \delta(x), \nabla_s \xi \rangle\, ds \bigg)^2 - \int_0^1 \langle \delta(x), \nabla_s \xi \rangle\, ds \cdot \int_0^1 \langle \delta'(x)[\xi], \dot{x} \rangle\, ds. \end{aligned} \tag{2.12}$$

Clearly the term

$$\int_0^1 \langle \delta(x), \nabla_s \xi \rangle \langle \delta'(x)[\xi], \dot{x} \rangle\, ds - \int_0^1 \langle \delta(x), \nabla_s \xi \rangle\, ds \cdot \int_0^1 \langle \delta'(x)[\xi], \dot{x} \rangle\, ds$$

defines a compact operator on $T_x\Omega^1$, because of the compact embedding of $H^{1,2}(I, \mathbb{R}^N)$ in $H^{1/2,2}(I, \mathbb{R}^N)$. Moreover, by the Hölder inequality, we have

$$\int_0^1 \langle \delta(x), \nabla_s \xi \rangle^2\, ds - \bigg(\int_0^1 \langle \delta(x), \nabla_s \xi \rangle ds \bigg)^2 \geq 0,$$

hence this difference defines a positive operator.

Thus, we have shown that $J''(x)$ is a compact perturbation of the positive definite quadratic form on $T_x\Omega^1$,

$$\int_0^1 [\langle \nabla_s \xi, \nabla_s \xi \rangle + \langle \delta(x), \nabla_s \xi \rangle^2]\, ds - \bigg(\int_0^1 \langle \delta(x), \nabla_s \xi \rangle ds \bigg)^2.$$

3. The penalization argument and some a priori estimates

In this section we describe the penalization argument that we use together with the a priori estimates needed in the proof of Theorem 1.6.

If assumption (1.10) holds, the functional J is coercive. Indeed, by using the Hölder inequality it is easy to prove the following lemma (see also [8]).

Lemma 3.1. *Assume that* (1.10) *holds. Then*

$$\lim_{\|x\|_* \to \infty} J(x) = \infty, \tag{3.1}$$

where, for every $x \in \Omega^1$,

$$\|x\|_*^2 = \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds. \tag{3.2}$$

In order to apply Morse theory to the functional J, we need the Palais–Smale compactness condition.

We recall that a smooth functional $I : X \to \mathbb{R}$ defined on a Hilbert manifold X is said to satisfy the *Palais–Smale condition* at the level $c \in \mathbb{R}$ ($(\mathrm{P.S.})_c$) if every sequence $\{x_k\}_{k \in \mathbb{N}}$ such that

$$I(x_k) \to c \tag{3.3}$$

and

$$\|I'(x_k)\| \to 0 \tag{3.4}$$

has a convergent subsequence. Here $\| \cdot \|$ denotes the norm induced on $T_x X$ by the Riemannian metric on X and I' the gradient of I.

Since $\mathcal{M}_0$ is not a complete Riemannian manifold (because of the presence of the boundary $\partial\mathcal{M}_0$), $\Omega^1 = \Omega^1(\mathcal{M}_0, x_0, x_1)$ is not complete (indeed, it is an open submanifold of $\widetilde{\Omega}^1 = \Omega^1(\widetilde{\mathcal{M}}_0, x_0, x_1)$). For this reason the functional J does not satisfy the Palais–Smale condition. Indeed, a sequence in Ω^1 which satisfies (3.3) and (3.4) may converge to a curve x which "touches" the boundary of $\mathcal{M}_0$, hence $x \notin \Omega^1$.

In order to overcome this difficulty, we introduce a penalization argument. Since $\partial\mathcal{M}_0$ is a smooth submanifold of $\widetilde{\mathcal{M}}_0$, there exists a smooth function $\phi : \widetilde{\mathcal{M}}_0 \to \mathbb{R}$ such that

$$\mathcal{M}_0 = \{x \in \widetilde{\mathcal{M}} : \phi(x) > 0\}, \tag{3.5}$$

$$\partial\mathcal{M}_0 = \{x \in \widetilde{\mathcal{M}} : \phi(x) = 0\} \tag{3.6}$$

$$\operatorname{grad} \phi(x) \neq 0 \qquad \text{for any } x \in \partial\mathcal{M}_0, \tag{3.7}$$

where $\operatorname{grad} \phi(x)$ denotes the gradient of ϕ at x with respect to the Riemannian structure $\langle \cdot\,, \cdot \rangle$.

Moreover, for any $z \in \mathcal{M}$, we set

$$\Phi(z) = \Phi(x, t) = \phi(x). \tag{3.8}$$

Notice that, denoting by ∇ the gradient of Φ with respect to the Lorentzian structure $\langle \cdot\,, \cdot \rangle_z$, we have

$$\nabla\Phi(z) = \operatorname{grad} \phi(x), 0), \tag{3.8}$$

hence Φ satisfies (1.5). Now, let

$$\psi(\sigma) = e^{\sigma} - (1 + \sigma + \sigma^2/2),$$

and, for every $\varepsilon > 0$,

$$\psi_\varepsilon(\sigma) = \begin{cases} \psi(\sigma - 1/\varepsilon) & \text{if } \sigma \geq 1/\varepsilon, \\ 0 & \text{if } \sigma < 1/\varepsilon. \end{cases}$$

Finally, for every $\varepsilon > 0$, consider the penalized functional $f_\varepsilon : \mathcal{Z} \to \mathbb{R}$,

$$f_\varepsilon(z) = f_\varepsilon(x,t) = f(z) + \int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x)}\right) ds. \tag{3.9}$$

Since the penalization term does not depend on the variable t, the statements of Lemma 2.2 and Theorem 2.3 also hold for the functional f_ε and the functional

$$J_\varepsilon(x) = J(x) + \int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x)}\right) ds, \tag{3.10}$$

where J is defined by (2.5).

As proved in [8], the sublevels of J_ε are complete and J_ε satisfies the Palais–Smale compactness condition at every level c. Indeed, the following theorem holds:

THEOREM 3.2. *Assume that* (1.7)–(1.8) *and* (1.10) *hold. Then:*

(i) *for every $\varepsilon > 0$ and $a \in \mathbb{R}$, the sublevel $J_\varepsilon^a = \{x \in \Omega^1 : J_\varepsilon(x) \leq a\}$ is a complete metric subspace of Ω^1;*

(ii) *for every $\varepsilon > 0$ and $c \in \mathbb{R}$, J satisfies* $(\text{P.S.})_c$.

Now, for every $\varepsilon > 0$, let x_ε be a critical point of J_ε such that

$$J_\varepsilon(x_\varepsilon) \leq M, \tag{3.11}$$

where M is a constant independent on ε. Let $t_\varepsilon = \theta(x_\varepsilon)$ (cf. (2.4) for the definition of θ). Since ϕ does not depend on t, the same proof shows that the assertion of Theorem 2.3 (cf. [8]) also holds for f_ε and J_ε, with the same map θ. Then, since x_ε is a critical point of J_ε, the curve $z_\varepsilon = (x_\varepsilon, t_\varepsilon)$ is a critical point of f_ε and $f_\varepsilon(z_\varepsilon) = J_\varepsilon(x_\varepsilon)$.

Hence, for every $\zeta = (\xi, \tau) \in T_{z_\varepsilon}\mathcal{Z} \equiv T_{x_\varepsilon}\Omega^1 \times H_0^{1,2}(I, \mathbb{R})$,

$$0 = f'_\varepsilon(z_\varepsilon)[\zeta] = f'(z_\varepsilon)[\zeta] - \int_0^1 \psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon)}\right) \cdot \frac{\langle \operatorname{grad} \phi(x_\varepsilon), \xi \rangle}{\phi^3(x_\varepsilon)} ds. \tag{3.12}$$

As proved in [8], z_ε is a smooth curve and satisfies the system of equations

$$-\nabla_s \dot{z}_\varepsilon = \psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon)}\right) \cdot \frac{2}{\phi^3(x_\varepsilon)} \nabla\Phi(z_\varepsilon) \tag{3.13}$$

for any $s \in I$, where $\Phi(x,t) = \phi(x)$ and $\nabla\Phi$ is the gradient of Φ with respect to the Lorentzian metric $\langle \cdot, \cdot \rangle_z$.

Multiplying both sides of (3.13) by $\dot{z}$ gives the existence of a constant H_ε such that, for any $s \in I$,

$$H_\varepsilon = \frac{1}{2}\langle \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s)\rangle_z - \psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right). \tag{3.14}$$

Integrating (3.14) in the interval I, since $\psi_\varepsilon \geq 0$, gives

$$\begin{aligned} H_\varepsilon &= \frac{1}{2}\int_0^1 \langle \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s)\rangle_z \, ds - \int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) ds \\ &\leq f_\varepsilon(z_\varepsilon) - 2\int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) ds \\ &\leq M - \int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) ds \leq M. \end{aligned} \tag{3.15}$$

The following estimates on the family $(z_\varepsilon)_{\varepsilon>0}$ are easy consequences of (3.10), (3.11), (3.1) and (2.4).

Lemma 3.3. *Assume that* (1.10) *holds. For every* $\varepsilon \in \,]0,1]$, *let* x_ε *be a critical point of* J_ε *such that* (3.11) *holds. Moreover, let* $t_\varepsilon = \theta(x_\varepsilon)$ *and* $z = (x_\varepsilon, t_\varepsilon)$. *Then:*

(i) $\sup_{\varepsilon\in]0,1]} \|x_\varepsilon\|_1 < \infty$ (cf. (2.1),
(ii) $\sup_{\varepsilon\in]0,1]} \|t_\varepsilon\|_1 < \infty$ (cf. (2.1) with $m = 1$).

Now consider the multiplier in the equation (3.13),

$$\mu_\varepsilon(s) = \psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon)}\right)\cdot\frac{2}{\phi^3(x_\varepsilon)}. \tag{3.16}$$

The following lemma is needed to prove the a priori estimates on the critical points of J_ε.

Lemma 3.4. *Assume* (1.7)–(1.8), (1.10) *and* (3.11) *hold. Then*

$$\sup_{\varepsilon\in]0,1]} \|\mu_\varepsilon\|_{L^\infty} < \infty. \tag{3.17}$$

Proof. The proof of Lemma 3.4 is contained in [8]. However, we repeat the proof pointing out what is needed in proving Lemma 4.5. For every $\varepsilon > 0$, we put

$$\varrho_\varepsilon(s) = \phi(x_\varepsilon(s)) = \Phi(z_\varepsilon(s)),$$

which is a C^2 function on I. Let s_ε be a minimum point of ϱ_ε. Since the derivative ψ_ε' is nondecreasing, we have

$$\mu_\varepsilon(s) = \frac{2}{\phi^3(x_\varepsilon(s))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) \leq \frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right).$$

Therefore it suffices to prove (3.17) assuming

$$\lim_{\varepsilon\to 0}\phi(x_\varepsilon(s_\varepsilon))=0. \tag{3.18}$$

From Lemma 3.3 we deduce that the families $\{x_\varepsilon\}_{\varepsilon\in]0,1]}$ and $\{t_\varepsilon\}_{\varepsilon\in]0,1]}$ are bounded in $L^\infty(I,\mathbb{R}^N)$ and $L^\infty(I,\mathbb{R})$, respectively. Hence there exists a positive constant c_1 such that

$$H^\Phi(z_\varepsilon(s_\varepsilon))[\dot z_\varepsilon,\dot z_\varepsilon]\le c_1(\langle\dot x_\varepsilon(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle+\dot t_\varepsilon(s_\varepsilon)^2).$$

Now, by (2.4), (1.10) and Lemma 3.3, there exist positive constants c_2 and c_3, independent of ε, such that

$$\dot t_\varepsilon^2\le c_2\langle\dot x_\varepsilon,\dot x_\varepsilon\rangle+c_3,$$

while by (1.4), (2.4), (1.10) and Lemma 3.3, there exists a constant c_4, independent of ε, such that

$$\langle\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)\rangle_z\ge\langle\dot x_\varepsilon(s_\varepsilon),\dot x_\varepsilon(s_\varepsilon)\rangle-c_4.$$

Therefore

$$H^\Phi(z_\varepsilon(s_\varepsilon))[\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)]\le c_5\langle\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)\rangle_z+c_6,$$

where c_5 and c_6 are positive constants independent of ε.

Moreover, if ε is sufficiently small, from (3.18), (3.7) and (3.8), there exists $c_7>0$ such that

$$\langle\nabla\Phi(z_\varepsilon(s_\varepsilon)),\nabla\Phi(z_\varepsilon(s_\varepsilon))\rangle_z\ge c_7.$$

Then, since s_ε is a minimum point for ϱ_ε,

$$\begin{aligned}
0\le\ddot\varrho_\varepsilon(s_\varepsilon)&=H^\Phi(\dot z_\varepsilon(s_\varepsilon))[\dot z_\varepsilon(s_\varepsilon),z_\varepsilon(s_\varepsilon)]+\langle\nabla\Phi(z_\varepsilon(s_\varepsilon)),D_sz_\varepsilon(s_\varepsilon)\rangle_z\\
&=H^\Phi(z_\varepsilon(s_\varepsilon))[\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)]\\
&\quad-\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\langle\nabla\Phi(z_\varepsilon(s_\varepsilon)),\nabla\Phi(z_\varepsilon(s_\varepsilon))\rangle_z\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3.13))}\\
&=c_5\langle\dot z_\varepsilon(s_\varepsilon),\dot z_\varepsilon(s_\varepsilon)\rangle_z+c_6\\
&\quad-c_7\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\\
&\le c_5\left(2H_\varepsilon+\psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\right)+c_6\\
&\quad-c_7\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\qquad\text{(by (3.13))}\\
&\le c_5\left(2M+\psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\right)+c_6\\
&\quad-c_7\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi_\varepsilon'\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\qquad\text{(by (3.11)).}
\end{aligned}$$

Therefore, there exists a positive constant c_8 (independent of ε) such that

$$\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right) \le c_8\left(1+\psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\right).$$

Since the family $\{\psi_\varepsilon\}_{\varepsilon>0}$ of functions has the property $\psi_\varepsilon(s) \le \psi'_\varepsilon(s)$ for any $s \ge 0$, we get

$$\frac{2}{\phi^3(x_\varepsilon(s_\varepsilon))}\psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right) \le c_8\left(1+\psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\right),$$

from which we immediately deduce (3.17).

COROLLARY 3.5.

$$\lim_{\varepsilon\to 0}\int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) ds = 0.$$

PROOF. If $\inf_{\varepsilon>0,\, s\in I}\phi(x_\varepsilon(s)) > 0$, then the proof is obvious. On the other hand, if $\inf_{\varepsilon>0,\, s\in I}\phi(x_\varepsilon(s)) = 0$, then we can assume that (3.18) holds, so with the notations of Lemma 3.4, we have

$$\begin{aligned}\psi_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) \le \psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s))}\right) &\le \psi'_\varepsilon\left(\frac{1}{\phi^2(x_\varepsilon(s_\varepsilon))}\right)\\ &\le c_0\phi^3(x_\varepsilon(s_\varepsilon)) \to 0\end{aligned}$$

as $\varepsilon \to 0$, where c_0 is an upper bound for $\|\mu_\varepsilon\|_{L^\infty}$.

COROLLARY 3.6. *If ε_n tends to zero, then the family of positive real functions*

$$\mu_{\varepsilon_n}(s) = \frac{2}{\phi^3(x_{\varepsilon_n})}\psi'_{\varepsilon_n}\left(\frac{1}{\phi^2(x_{\varepsilon_n}(s))}\right)$$

weakly converges to $\mu(s)$ in $L^2(I,\mathbb{R}^+)$. Moreover, if

$$\inf\{\phi(x_{\varepsilon_n}(s_0)) : n \in \mathbb{N}\} > 0,$$

then $\mu(s) = 0$ in a neighborhood of s_0.

From Lemmas 3.3 and 3.4, it is not difficult to deduce (cf. [8]) the following

PROPOSITION 3.7. *Assume that* (1.7)*,* (1.8) *and* (1.10) *hold. Let x_ε be a critical point of J_ε satisfying* (3.11)*, and set $t_\varepsilon = \theta(x_\varepsilon)$. Then there exists a sequence $\varepsilon_k \to 0$ such that, setting $x_k = x_{\varepsilon_k}$ and $t_k = t_{\varepsilon_k}$, we have:*

(i) *$\{x_k\}_{k\in\mathbb{N}}$ converges to x in $H^{1,2}(I,\mathbb{R}^N)$;*
(ii) *$\{t_k\}_{k\in\mathbb{N}}$ converges to t in $H^{1,2}(I,\mathbb{R})$;*
(iii) *$x(s) \in \Omega^1(\mathcal{M}_0 \cup \partial\mathcal{M}_0, x_0, x_1) \subset \widetilde{\Omega}^1 = \Omega^1(\widetilde{\mathcal{M}}_0 \cup \partial\mathcal{M}_0, x_0, x_1)$* (cf. (1.7))*;*
(iv) *let $z = (x,t)$; then for every $\zeta = (\xi,\tau) \in T_x\widetilde{\Omega}^1 \times H_0^{1,2}(I,\mathbb{R})$,*

$$\int_0^1 \langle \dot z, D_s\zeta\rangle_z\, ds = \int_0^1 \mu(s)\langle\nabla\Phi(z),\zeta\rangle_z\, ds, \tag{3.19}$$

where $\mu \in L^2(I,\mathbb{R}^+)$ and $\mu(s) = 0$ if $\Phi(z(s)) = 0$;

(v) *$x \in H^{2,2}(I,\mathbb{R}^N)$ and $t \in H^{2,2}(I,\mathbb{R})$; in particular x and t are of class C^1.*

Finally, Proposition 3.7 and the convexity of the boundary allow us to get the following a priori estimates on the critical points of J_ε, which play a very important role in the proof of Theorem 1.6. Their proof can be found in [8]. We repeat the proof pointing out what it is needed in the proof of Lemma 4.5.

Lemma 3.8. *Assume that* (1.7)–(1.10) *hold and fix $c \in \mathbb{R}$. Then there exist $\delta_0 = \delta_0(c) > 0$ and $\varepsilon_0 = \varepsilon_0(c) > 0$ such that, for any $\varepsilon \in \,]0,\varepsilon_0]$ and for any critical point x_ε of J_ε satisfying*

$$J_\varepsilon(x_\varepsilon) \le c, \tag{3.20}$$

we have

$$\phi(x_\varepsilon(s)) \ge \delta_0 \qquad \textit{for any } s \in I, \tag{3.21}$$

where ϕ is defined by (3.5)–(3.7).

Proof. Let $t_\varepsilon = \theta(x_\varepsilon)$ (cf. (2.4)) and $z_\varepsilon = (x_\varepsilon,t_\varepsilon)$. If, by contradiction, (3.21) does not hold, by Proposition 3.7 there exist a sequence $\varepsilon_k \to 0$, $x \in H^{2,2}(I,\mathbb{R}^N)$ and $t \in H^{2,2}(I,\mathbb{R})$ ($H^{1,2}$-limits of the sequences $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and $\{t_{\varepsilon_k}\}_{k\in\mathbb{N}}$) such that $z = (x,t)$ satisfies (3.19) and

$$\text{there exists } s_0 \in \,]0,1[\, \text{ such that } \phi(x(s_0)) = 0. \tag{3.22}$$

By equation (3.19) we have

$$D_s\dot z(s) = -\mu(s)\nabla\Phi(z(s)) \qquad \text{for almost every } s \in I. \tag{3.23}$$

Now, if $s \in \,]0,1[$ is such that $\Phi(z(s)) = 0$, it is a minimum point of the real function $\varrho(\,\cdot\,) = \Phi(z(\,\cdot\,))$. Then by (3.23) we have

$$0 \le \ddot\varrho(s) = H^\Phi(z(s))[\dot z(s),\dot z(s)] - \langle\nabla\Phi(z(s)),\mu(s)\nabla\Phi(z(s))\rangle_z,$$

and therefore, for almost every s such that $\Phi(z(s)) = 0$, we have

$$\mu(s)\langle\nabla\Phi(z(s)),\nabla\Phi(z(s))\rangle_z \le H^\Phi(z(s))[\dot z(s),\dot z(s)]. \tag{3.24}$$

Now, by the convexity of the boundary (cf. Definition 1.5 and (1.6)), for any $z \in \partial\mathcal{M}$ and for any $\zeta \in T_z\partial\mathcal{M}$, we have $H^\Phi(z)[\zeta,\zeta] \le 0$. Since z is of class C^1, $\Phi(z(s)) = 0$ implies $z(s) \in T_z\partial\mathcal{M}$; therefore, by (3.24),

$$\mu(s)\langle\nabla\Phi(z(s)),\nabla\Phi(z(s))\rangle_z \le 0$$

for almost every s such that $\Phi(z(s)) = 0$. But, by (3.7), (3.8) and (1.4), $\langle\nabla\Phi(z(s)),\nabla\Phi(z(s))\rangle_z > 0$; therefore, $\mu(s) \le 0$ for almost every s such that

$\Phi(z(s)) = 0$. Then, since μ takes its values in $\mathbb{R}^+$ (cf. (3.19)), $\mu(s) = 0$ for almost every s in I. Consequently, by (3.23),

$$D_s \dot{z}(s) = 0 \qquad \text{for any } s \in I. \tag{3.25}$$

Finally, combining (3.25) with (3.22) gives a contradiction because of the assumption of the convexity of $\partial\mathcal{M}$ (cf. Definition 1.5).

Moreover, we also have the following

Lemma 3.9. *Assume that* (1.7)–(1.10) *hold and fix* $c \in \mathbb{R}$. *Let* $\delta_0 = \delta_0(c)$ *be as in Lemma* 3.8. *Then:*

(i) *for any critical point* x *of* J *satisfying* $J(x) \le c$, *we have*

$$\phi(x(s)) \ge \delta_0 \qquad \textit{for any } s \in I;$$

(ii) *the set*

$$\{x \in \Omega^1 : J(x) \le c \text{ and } J'(x) = 0\}$$

is compact.

Proof. Let x be a critical point of J on Ω^1. Then there exists $\varepsilon \in \,]0, \varepsilon_0[$ (ε_0 as in Lemma 3.8), such that x is a critical point of J_ε and $J_\varepsilon(x) = J(x)$. Then (i) is consequence of (3.21). Moreover, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of critical points of J satisfying $J(x_n) \le c$ for any $n \in \mathbb{N}$, since (i) holds, the same arguments used to prove Proposition 3.7 and Lemma 3.8 show that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging to $x \in \Omega^1$ (critical point of J), and satisfying $\phi(x(s)) \ge \delta_0$ for any $s \in I$.

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6 and get the Morse relations for geodesics joining (x_0, t_0) and (x_1, t_1), assuming that (x_0, t_0) and (x_1, t_1) are nonconjugate.

Remark 4.1. If (x_0, t_0) and (x_1, t_1) are nonconjugate, by Theorem 2.4 it turns out that any critical point of J is nondegenerate.

Unfortunately, we cannot immediately get the Morse relations, because J does not satisfy the Palais–Smale condition. For this reason we use the penalized functional J_ε.

Remark 4.2. Let $c \in \mathbb{R}$ be a regular value of J. Then if ε is sufficiently small, c is a regular value of J_ε. Indeed, if $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a sequence such that $J(x_{\varepsilon_k}) = c$ and $J'_{\varepsilon_k}(x_{\varepsilon_k}) = 0$ ($\varepsilon_k \to 0$), by Proposition 3.7 and Lemma 3.8, we conclude that $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges to a critical point x of J such that $J(x) = c$.

Using the nondegeneracy of the critical points of J, the following lemma can be proved.

Lemma 4.3. *Assume that all the critical points of J are nondegenerate and fix $c \in \mathbb{R}$. Then:*

(i) *there are only a finite number of critical points of J on the sublevel*

$$J^c = \{x \in \Omega^1 : J(x) \leq c\};$$

(ii) *let*

$$\varepsilon_1(c) = \min\{\delta_0^2(c), \varepsilon_0(c)\}, \tag{4.1}$$

where $\delta_0(c)$ and $\varepsilon_0(c)$ are defined in Lemma 3.8; then, for any $\varepsilon \in [0, \varepsilon_1(c)]$, x is a critical point of J in J^c if and only if x is a critical point of J_ε in J_ε^c, and moreover,

$$J_\varepsilon(x) = J(x) \quad \text{and} \quad m(x, J_\varepsilon) = m(x, J). \tag{4.2}$$

Proof. Any critical point of J is isolated (because J is a Morse function). Moreover, by Lemma 3.4, $\{x \in \Omega^1 : J(x) \leq c,\ J'(x) = 0\}$ is compact, so we have (i). On the other hand, Lemma 3.9 and the form of the penalization term in J_ε yield (ii).

From Lemma 4.3 the following Morse relations hold on the sublevels of J_ε.

Theorem 4.4. *Assume that all the critical points of J are nondegenerate. Let c be a regular value of J and set*

$$Z(J, c) = \{x \in J^c : J'(x) = 0\}.$$

Let $\varepsilon_1(c)$ be as in Lemma 4.3. Then, for any $\varepsilon \in \,]0, \varepsilon_1(c)]$,

$$\sum_{x \in Z(J,c)} r^{m(x,J)} = P_r(J_\varepsilon^c) + (1 + r)Q_{\varepsilon,c}(r), \tag{4.3}$$

where $P_r(J_\varepsilon^c)$ is the Poincaré polynomial of J_ε^c and $Q_{\varepsilon,c}(r)$ is a polynomial with positive integer coefficients.

Proof. By Lemma 4.3, for any $\varepsilon \in \,]0, \varepsilon_1(c)]$, c is a regular value for J_ε and J_ε has only nondegenerate critical points below the level c. Since J_ε satisfies the Palais–Smale condition at every level, we have

$$\sum_{x \in Z(J_\varepsilon,c)} r^{m(x,J_\varepsilon)} = P_r(J_\varepsilon^c) + (1 + r)Q_{\varepsilon,c}(r)$$

for a suitable polynomial $Q_{\varepsilon,c}$ with positive integer coefficients (cf. e.g. [3, 6]). Moreover, if $\varepsilon \in \,]0, \varepsilon_1(c)]$, then Lemma 4.3 also gives

$$Z(J_\varepsilon, c) = Z(J, c) \quad \text{and} \quad m(x, J_\varepsilon) = m(x, J) \quad \text{for any } x \in Z(J, c),$$

proving (4.3).

Now, since the critical points of J do not touch the boundary, the form of the penalization term and a priori estimates similar to those of Lemma 3.8 allow us to get the following

Lemma 4.5. *For any $\delta > 0$, let*

$$\mathcal{M}_\delta = \{x \in \mathcal{M}_0 : \phi(x) \geq \delta\}$$

and

$$\Omega^1_\delta = \{x \in \Omega^1 : \phi(x(s)) \geq \delta,\ \forall s \in [0,1]\}.$$

Let c be a regular value of J and $\varepsilon_1 = \varepsilon_1(c)$. Then there exists $\delta_0 = \delta_0(c)$ such that, for any $\delta \in \,]0, \delta_0(c)]$ and $\varepsilon \in \,]0, \varepsilon_1(c)]$,

(i) *$\Omega^1_{\delta/2} \cap J^c$ is a weak deformation retract of J^c;*
(ii) *$\Omega^1_{\delta/2} \cap J^c_\varepsilon$ is a weak deformation retract of J^c_ε.*

(We recall that if (A, B) is a topological pair, then B is called a *weak deformation retract* of A if there exists a continuous map $H : ([0,1] \times A, [0,1] \times B) \to (A, B)$ such that $H(0, \cdot\,)$ is the identity map of A and $H(1, A) \subseteq B$.)

Proof. Let $\delta_0(c)$ be as in Lemma 3.8 and $\varepsilon_1(c)$ as in (4.1). For any real number b, denote by S^b the compact subset of $\overline{\mathcal{M}}_0$ including the support of the curves contained in the sublevel J^b. Choosing a smaller δ_0 if necessary, we can assume that

$$\operatorname{grad} \phi(x) \neq 0 \qquad \text{for any } x \in S^{c+1} \cap \{x \in \overline{\mathcal{M}}_0 : \phi(x) \leq \delta_0\}.$$

For any $\delta \in \,]0, \delta_0[$, let $\chi_\delta \in C^\infty(\mathbb{R}^+, [0,1])$ be a smooth function such that $\chi_\delta(s) = 1$ if $s \in [0, \delta]$, and $\chi_\delta(s) = 0$ if $s \geq 2\delta$, and consider the Cauchy problem

$$\dot{\eta}_\delta(s) = \chi_\delta(\phi(\eta_\delta(s)))\nabla\phi(\eta_\delta), \qquad \eta_\delta(0) = x \in S^{c+1}. \tag{4.4}$$

Clearly there exists $T_\delta > 0$, with $T_\delta \to 0$ as $\delta \to 0$, such that for any $x \in S^{c+1}$,

$$\phi(\eta_\delta(T_\delta, x)) \geq \delta. \tag{4.5}$$

By (4.5) and standard arguments in ordinary differential equations, it follows that

$$\eta_\delta(\sigma T_\delta, \cdot\,) \text{ converges to the identity in } C^1(S^{c+1}) \text{ uniformly in } \sigma. \tag{4.6}$$

Now, put

$$H_\delta(\sigma, x)(\cdot\,) = \eta_\delta(\sigma T_\delta, x(\cdot\,)).$$

Moreover, choose $\mu \in \,]0,1[$ sufficiently small such that $[c, c+\mu]$ does not contain critical values for J and J_ε whenever $\varepsilon < \varepsilon_1(c)$ (cf. (4.1)). By (4.4) and (4.5), the real function

$$\int_0^1 \psi_\varepsilon\left(\frac{1}{\phi^2(H_\delta(\sigma, x))}\right) ds$$

is decreasing with respect to σ, so we can choose δ small enough such that

$$H_\delta(\sigma,x) \in J_\varepsilon^{c+\mu} \text{ for any } \sigma \in [0,1] \text{ and } x \in J_\varepsilon^c, \tag{4.7}$$

$$H_\delta(\sigma,x) \in J^{c+\mu} \text{ for any } \sigma \in [0,1] \text{ and } x \in J^c, \tag{4.8}$$

$$H_\delta(1,x) \in J^{c+\mu} \cap \Omega_\delta^1 \text{ for any } x \in J^c, \tag{4.9}$$

$$H_\delta(1,x) \in J_\varepsilon^{c+\mu} \cap \Omega_\delta^1 \text{ for any } x \in J_\varepsilon^c. \tag{4.10}$$

Now, for any $x \in \Omega^1(\mathcal{M}_0, x_0, x_1)$, put

$$d(x) = \min_{s\in I} \phi(x(s)).$$

It is not difficult to prove that d is a continuous function. Moreover, d is strictly positive on $\Omega^1(x_0, x_1, \mathcal{M}_0)$.

For any $\sigma \in [0,1]$ and $x \in J^c$, let

$$\lambda = \lambda(\sigma,x) = d(H_\delta(\sigma,x)), \qquad m = m(\sigma,x) = \frac{d+\lambda}{2},$$

and consider, for any $\sigma \in [0,1]$ and $x \in J^c$ such that $\lambda(\sigma,x) > d(x)$, the smooth functional

$$J_{\sigma,x}(y) = J(y) + \int_0^1 \psi_{((\lambda-d)/4)^2}\left(\frac{1}{\varphi(y)-m}\right)^2 ds, \tag{4.11}$$

defined on the open set

$$\Omega^1_{\sigma,x} = \{y \in \Omega^1 : \varphi(y(s)) > m,\ \forall s \in I\}.$$

Notice that $H_\delta(\sigma,x) \in \Omega^1_{\sigma,x}$ and $J_{\sigma,x}(H_\delta(\sigma,x)) = J(H_\delta(\sigma,x)) \le c+\mu$.

The same arguments that were used to prove Lemmas 3.4, 3.8 and 3.9 show that (upon choosing δ small enough) $J_{\sigma,x}$ has no critical points in the strip $\{c \le J_{\sigma,x}(y) \le c+\mu\}$ for any $\sigma \in [0,1]$ and $x \in J^c$ such that $\lambda(\sigma,x) > d(x)$ (notice that in this case we have $d(x) < \lambda(\sigma,x) \le \delta$). Moreover, $J_{\sigma,x}$ satisfies (P.S.) in $\Omega^1_{\sigma,x}$ (cf. Theorem 3.2). Consider now the Cauchy problem

$$\dot\Gamma = -\frac{\nabla J_{\sigma,x}(\Gamma)}{\|\nabla J_{\sigma,x}(\Gamma)\|^2}, \qquad \Gamma(0) = y_0. \tag{4.12}$$

Then we have the well defined map Π such that

$$\Pi(y_0) = \begin{cases} \Gamma([J_{\sigma,x}(y_0)-c]^+, y_0) & \text{if } \lambda(\sigma,x) > d(x), \\ y_0 & \text{if } \lambda(\sigma,x) = d(x), \end{cases}$$

which maps $J^{c+\mu}_{\sigma,x}$ to $J^c_{\sigma,x} \subseteq J^c$. Since d is continuous and ∇J is globally Lipschitz continuous on the sublevels of J, it is not difficult to prove, by using the Gronwall lemma in local charts, that the map

$$(\sigma,x) \to \Pi(H_\delta(\sigma,x)) \tag{4.13}$$

is continuous on the open subset $\{(\sigma, x) : \lambda(\sigma, x) > d(x)\}$ (recall that $\lambda(\sigma, x) = d(H_\delta(\sigma, x)))$. Moreover, if $(\sigma_n, x_n) \to (\sigma, x)$ and $\lambda(\sigma, x) = d(x)$, we have

$$\mathrm{dist}(H_\delta(\sigma_n, x_n), x_n) \to 0 \quad \text{and} \quad [J_{\sigma_n, x_n}(x_n) - c]^+ \to 0,$$

from which we can deduce that $\Pi(H_\delta(\sigma_n, x_n)) \to x$.

Thus, (4.13) defines a continuous map. Finally, straightforward calculations show that it is a weak deformation of J^c onto $J^c \cap \Omega^1_{\delta/2}$. Analogously, it is possible to construct a weak deformation of J^c_ε onto $J^c_\varepsilon \cap \Omega^1_{\delta/2}$, completing the proof.

Modifying the above proof in a suitable way, it is possible to get a stronger result:

(i) $\Omega^1_{\delta/2} \cap J^c$ *is a strong deformation retract of* J^c;
(ii) $\Omega^1_{\delta/2} \cap J^c_\varepsilon$ *is a strong deformation retract of* J^c_ε.

We have chosen to deal only with weak deformation retracts, which is sufficient for our purposes. On the other hand, in this case the proof is a little simpler.

In order to prove Theorem 1.6, the following preliminary results are needed. We begin by recalling two simple lemmas in algebraic topology.

Lemma 4.6. *Assume that* (A, B) *is a topological pair such that* B *is a weak deformation retract of* A. *Then* $P_r(A, B) = 0$, *and consequently* (*by the long exact homology sequence for the pair* (A, B)), $P_r(A) = P_r(B)$.

Proof. Let $H : ([0, 1] \times A, [0, 1] \times B) \to (A, B)$ be a continuous map such that $H(0, \cdot\,)$ is the identity map of A and $H(1, A) \subseteq B$, and let $i : B \to A$ be the inclusion map. Since the map $i \circ H(1, \cdot\,) : (A, B) \to (A, B)$ is homotopically equivalent to the identity, it follows that that $H(1, \cdot\,)_* : H_*(A, B) \to H_*(B, B)$ is one-to-one. But $H_*(B, B) = \{0\}$, hence $H_*(A, B) = \{0\}$.

Lemma 4.7. *Let* (X, A) *and* (Y, B) *be topological pairs such that* $B \subset A$ *is a weak deformation retract of* A *and* $Y \subset X$ *is a weak deformation retract of* X. *Then* $P_r(X, A) = P_r(Y, B)$.

Proof. By the exactness of the triple (B, Y, X) and Lemma 4.6 we have $P_r(Y, B) = P_r(X, B)$, while by the exactness of the triple (B, A, X) and Lemma 4.6 we have $P_r(X, B) = P_r(X, A)$.

Note that if $\varepsilon \leq \delta^2 \setminus 4$, then $J_\varepsilon = J$ on $\Omega^1_{\delta/2}$, and therefore, $J^c_\varepsilon \cap \Omega^1_{\delta/2} = J^c \cap \Omega^1_{\delta/2}$. Thus, by Lemmas 4.5–4.7, the following propositions hold.

Proposition 4.8. *For every regular value* c *of* J, *there exists* $\varepsilon(c) > 0$ *such that, for any* $\varepsilon \in\,]0, \varepsilon(c)]$, $P_r(J^c_\varepsilon) = P_r(J^c)$.

Proposition 4.9. *Let $c_2 > c_1$ be regular values for J. Then there exists $\varepsilon(c_1, c_2) > 0$ such that, for any $\varepsilon \in]0, \varepsilon(c_1, c_2)]$, $P_r(J_\varepsilon^{c_2}, J_\varepsilon^{c_1}) = P_r(J^{c_2}, J^{c_1})$.*

Now we are finally ready to prove Theorem 1.6.

Proof of Theorem 1.6. By (4.3) and Proposition 4.8, for any regular value c of J,

$$\sum_{x \in Z(J,c)} r^{m(x,J)} = P_r(J^c) + (1+r)Q_c(r), \tag{4.14}$$

where Q_c is a polynomial with natural coefficients.

Our aim now is to send c to ∞, showing (by a standard argument in algebraic topology and Proposition 4.9) that

$$\sum_{x \in Z(J,\infty)} r^{m(x,J)} = P_r(\Omega^1) + (1+r)Q(r), \tag{4.15}$$

where Q is now a formal series with natural coefficients. Denote by $Z(J)$ the set of critical points of J. Since every critical point of J is nondegenerate, by Lemma 4.3 there exist two sequences $\{b_h\}_{h\in\mathbb{N}}$ and $\{c_h\}_{h\in\mathbb{N}}$ of real numbers such that

- every b_h is a regular value for J,
- $b_0 < \inf J < b_1 < \ldots < b_h < b_{h+1} < \ldots$,
- $\lim_{h\to\infty} b_h = \infty$,
- for any $h \in \mathbb{N}$, $f_{b_h}^{b_{h+1}} \cap Z(J) = \emptyset$ or $= f^{-1}(c_h) \cap Z(J)$ (here $f_a^b = \{x : a \le f(x) \le b\}$),
- for any $h \in \mathbb{N}$, $f^{-1}c_h \cap Z(J)$ is finite.

Now, for any $b \in \mathbb{R}$, the exactness of the triple $(\emptyset, J^b, \Omega^1)$ shows that (cf. [3, Lemma 4.2(iv)]) there exists $Q_b \in \mathcal{S}$ (the set of formal series with coefficients in $\mathbb{N} \cup \{\infty\}$) such that

$$P_r(\Omega^1, J^b) + P_r(J^b) = P_r(\Omega^1) + (1+r)Q_b(r).$$

Therefore, by (4.14), for any $h \in \mathbb{N}$ there exists $Q_h \in \mathcal{S}$ such that

$$\sum_{x \in Z(J,b_h)} r^{m(x,J)} + P_r(\Omega^1, J^{b_h}) = P_r(\Omega^1) + (1+r)Q_h(r). \tag{4.16}$$

Fix $k \in \mathbb{N}$. Our goal is to prove (4.15) by using (4.16) and arguing on the coefficients of each degree of the formal series in (4.15) and (4.16). If the set M_k of points of $Z(J)$ having Morse index k is infinite, then taking the limit in (4.16) as h goes to ∞ gives immediately the proof of (4.15) for the degree k, because the coefficient of degree k of $\sum_{x\in Z(J,b_h)} r^{m(x,J)}$ is nondecreasing with respect to h and tends to ∞.

Now suppose that M_k is finite and consider a regular value b such that

$$b > \max_{M_k} f. \tag{4.17}$$

By (4.16), in order to prove (4.15) (for the degree k and consequently for all degrees) it is sufficient to prove that

(4.18) the coefficient of degree k of $P_r(\Omega^1, J^b)$ is zero

for any b satisfying (4.17).

Let $c > b$ be a regular value for J. By Lemma 4.3 the Morse relations (4.3) can also be written for the strip $\{x \in \Omega^1 : b \leq J_\varepsilon(x) \leq c\}$, by replacing $P_r(J^c_\varepsilon)$ with $P_r(J^c_\varepsilon, J^b_\varepsilon)$. Then by (4.17), Lemma 4.3 and Proposition 4.9 we deduce that

$$H_k(J^c, J^b) = 0. \tag{4.19}$$

Consider now the exact homology sequence

$$\cdots \to H_k(J^c, J^b) \xrightarrow{i^*_k} H_k(\Omega^1, J^b) \xrightarrow{j^*_k} H_k(\Omega^1, J^c) \to \ldots, \tag{4.20}$$

where i^*_k and j^*_k are induced by the inclusion maps. If, by contradiction, (4.18) does not hold, then there exists $\alpha \in H_k(\Omega^1, J^b)$ with $\alpha \neq 0$. Now, denoting by Δ the support of α and choosing $c > \max_\Delta J$ gives $j^*_k(\alpha) = 0$. Then, by the exactness of the homology sequence (4.20), there exists $\beta \in H_k(J^c, J^b)$ such that $i^*_k(\beta) = \alpha$, contrary to (4.19). Thus (4.18) and therefore also (4.15) are proved.

Finally, denoting by Z the set of geodesics $z = (x, t)$ (from I to $\mathcal{M}$) joining z_0 and z_1, by (4.15), Theorem 2.10 and Corollary 2.11 we get

$$\sum_{z \in Z} r^{\mu(z)} = P_r(\Omega^1) + (1 + r)Q(r).$$

Since Ω (cf. the statement of Theorem 1.6) and Ω^1 are homotopically equivalent, we get (1.11), and the proof of Theorem 1.6 is complete.

5. Proof of Theorem 1.9

Let J_ε be as in (3.10). The main difference between the proofs of Theorems 1.6 and 1.9 is in the a priori estimates. They are obtained in the following

Lemma 5.1. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ satisfy* (1.12)–(1.15) *and* (1.17). *Fix $c \in \mathbb{R}$. Then there exist $\delta_0 = \delta_0(c)$ and $\varepsilon_0 = \varepsilon_0(c)$ such that, for any $\varepsilon \in \,]0, \varepsilon_0]$ and for any critical point x_ε of J_ε on $\Omega^1(x_0, x_1, \mathcal{M}_0)$ satisfying*

$$J_\varepsilon(x_\varepsilon) \leq c, \tag{5.1}$$

we have

$$\varphi(x_\varepsilon(s)) \geq \delta_0 > 0 \qquad \textit{for all } s \in I, \tag{5.2}$$

where φ is defined in (1.12)–(1.13).

Proof. Arguing by contradiction, assume that there exists a sequence $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}}$ ($\varepsilon_n \to 0$) such that for any $n \in \mathbb{N}$, x_{ε_n} is a critical point of J_{ε_n}, and

$$\varrho_n((s_n)) = \varphi(x_{\varepsilon_n}(s_n)) \to 0 \qquad \text{for } n \to \infty, \tag{5.3}$$

where s_n is a minimum point for the map

$$\varrho_n(s) = \varphi(x_{\varepsilon_n}(s_n)), \qquad s \in I. \tag{5.4}$$

Now we set

$$z_n(s) = (x_{\varepsilon_n}(s), t_{\varepsilon_n}(s)) \equiv (x_n(s), t_n(s)),$$

where $t_{\varepsilon_n} = \theta(x_{\varepsilon_n})$ and θ is defined by (2.4). Since z_n is a critical point of f_{ε_n}, it satisfies (3.13). Then, for all $s \in [0,1]$, we have

$$\begin{aligned}\varrho_n''(s) &= \frac{d}{ds}[\langle \nabla\varphi(z_n(s)), \dot z_n(s)\rangle_z] \\ &= H^\varphi(z_n(s))[\dot z_n(s), \dot z_n(s)] + \langle \nabla\varphi(z_n(s)), D_s \dot z_n(s)\rangle_z \\ &= H^\varphi(z_n(s))[\dot z_n(s), \dot z_n(s)] - \mu_{\varepsilon_n}(s)\langle \nabla\varphi(z_n(s)), \nabla\varphi(z_n(s))\rangle_z,\end{aligned}$$

where $\mu_{\varepsilon_n}(s)$ is defined by (3.16). Then, by assumption (1.13),

$$\varrho_n''(s) \le H^\varphi(z_n(s))[\dot z_n(s), \dot z_n(s)].$$

Now, by (2.5), (1.15), (1.17), and (5.1), there exists a constant c_1 (independent of n) such that

$$\int_0^1 \langle \dot x_n, \dot x_n\rangle\, ds \le c_1. \tag{5.5}$$

Therefore, by (1.12) and (1.13), there exists a positive constant c_0 (independent of n) such that

$$\varrho_n''(s) \le c_0 \varrho_n(s)[\langle \dot x_n, \dot x_n\rangle + |\langle \delta(x_n), \dot x_n\rangle \dot t_n| + \beta(x_n)\dot t_n^2]. \tag{5.6}$$

Passing to a subsequence if necessary, let $s_0 = \lim_{n\to\infty} s_n$. Since $\varphi(z_n(0)) = \varphi(x_0) > 0$, and $\varphi(z_n(1)) = \varphi(x_1) > 0$, by (5.5) we deduce that $s_0 \ne 0$ and $s_0 \ne 1$. Therefore, for any n suffciently large we have $\varrho_n'(s_n) = 0$. By (5.3), (5.5) and (5.6), in order to get a contradiction (proving Lemma 5.1) it suffices to apply the Gronwall lemma to $\varrho_n(s)$, after having proved the existence of two positive constants c_2 and c_3 (independent of n) such that

$$\int_0^1 |\langle \delta(x_n), \dot x_n\rangle \dot t_n|\, ds \le c_2 \tag{5.7}$$

and

$$\int_0^1 \beta(x_n)\dot t_n^2\, ds \le c_3. \tag{5.8}$$

Indeed, in this case we have $\varrho_n(s_n) \to 0$, $\varrho_n'(s_n) = 0$, and $0 \le \varrho_n''(s) \le c_0 \varrho_n(s) u_n(s)$, where $u_n(s)$ is a function such that $\int_0^1 u_n(s)\,ds$ is uniformly bounded with respect to n. Therefore, by the Gronwall lemma, we have $\varrho_n(s) \to 0$, uniformly in $[0,1]$, contrary to $\varrho_n(0) = \varphi(x_0) > 0$.

In order to prove (5.8) set

$$K(x) = \frac{t_1 - t_0 - \int_0^1 \frac{\langle \delta(x), \dot{x}\rangle}{\beta(x)}\,ds}{\int_0^1 \frac{1}{\beta(x)}\,ds} \tag{5.9}$$

and notice that by (2.4),

$$\dot{t}_n = \frac{\langle \delta(x_n), \dot{x}_n\rangle + K(x_n)}{\beta(x_n)}.$$

Therefore, we have

$$\begin{aligned} \int_0^1 \beta(x_n)\dot{t}_n^2\,ds & \\ = \int_0^1 & \frac{\langle \delta(x_n), \dot{x}_n\rangle^2}{\beta(x_n)}\,ds \\ & - \left(\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n\rangle}{\beta(x_n)}\,ds\right)^2 \left(\int_0^1 \frac{1}{\beta(x_n)}\,ds\right)^{-1} \\ & - 2(t_1 - t_0)\left(\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n\rangle}{\beta(x_n)}\,ds\right)\left(\int_0^1 \frac{1}{\beta(x_n)}\,ds\right)^{-1} \\ & + (t_1 - t_0)^2\left(\int_0^1 \frac{1}{\beta(x_n)}\,ds\right)^{-1} + 2(t_1 - t_0)\left(\int_0^1 \frac{1}{\beta(x_n)}\,ds\right)^{-1}. \end{aligned} \tag{5.10}$$

Now, combining (1.15), (1.17), (2.5), (3.10) and (5.1) gives (using also the Hölder inequality) the existence of a positive constant c_4 (independent of n) such that

$$\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n\rangle^2}{\beta(x_n)}\,ds - \left(\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n\rangle}{\beta(x_n)}\,ds\right)^2 \left(\int_0^1 \frac{1}{\beta(x_n)}\,ds\right)^{-1} \le c_4. \tag{5.11}$$

Then, from (5.10), (5.11), (1.15) and (1.17) we deduce (5.8).

In order to prove (5.7), notice that (since $f_\varepsilon(\,\cdot\,, \theta(\,\cdot\,)) = J_\varepsilon(\,\cdot\,)$ and (5.1) holds), by (3.15) there exists a bounded sequence $\{H_n\}_{n\in\mathbb{N}}$ of real constants such that

$$\langle \dot{x}_n, \dot{x}_n\rangle + 2\langle \delta(x_n), \dot{x}_n\rangle \dot{t}_n - \beta(x_n)\dot{t}_n^2 = 2H_n + 2\psi_{\varepsilon_n}\left(\frac{1}{\varphi^2(x_n)}\right),$$

and so

$$2\langle \delta(x_n), \dot{x}_n\rangle \dot{t}_n = 2H_n + 2\psi_{\varepsilon_n}\left(\frac{1}{\varphi^2(x_n)}\right) - \langle \dot{x}_n, \dot{x}_n\rangle + \beta(x_n)\dot{t}_n^2.$$

Since, for any $\varepsilon > 0$, ψ_ε is a positive function, integrating on the interval I and combining (5.1), (5.5) and (5.8), we obtain (5.7).

Proof of Theorem 1.9. Thanks to Lemma 5.1, the proof of Theorem 1.9 is the same as the proof of Theorem 1.6, except for the proof of (4.7) and (4.8) in Lemma 4.5. However, these are simple consequences of (1.16) and (1.18).

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