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A MINIMAX THEOREM FOR MARGINALLY UPPER/LOWER SEMICONTINUOUS FUNCTIONS

Rodney C. Bassanezi — Gabriele H. Greco

Dedicated to Ky Fan

Let X, Y be nonempty convex subsets of real separated topological vector spaces. Sion [3] proved that every u.s.c./l.s.c. quasi-concave-convex function $f : X \times Y \to \overline{\mathbb{R}}$ has a saddle value, whenever either X or Y is compact.

Recall that f is said to be *u.s.c./l.s.c.* (resp. *quasi-concave-convex*) if for every $x_0 \in X$, $y_0 \in Y$ and $r \in \mathbb{R}$ the sets $\{x \in X : f(x, y_0) \geq r\}$ and $\{y \in Y : f(x_0, y) \leq r\}$ are closed (resp. convex). Moreover, f is said to have a *saddle value* if $\inf_Y \sup_X f = \sup_X \inf_Y f$.

The purpose of this note is to show an improvement of the finite-dimensional version of the Sion Theorem, by replacing the upper/lower semicontinuity of f with the marginal upper/lower semicontinuity. A function f is said to be *marginally u.s.c./l.s.c.* if for every $r \in \mathbb{R}$, open subset U of X and open subset V of Y, the sets

$$\{x \in X : \inf_{y \in V} f(x, y) \geq r\} \quad \text{and} \quad \{y \in Y : \sup_{x \in U} f(x, y) \leq r\} \quad \text{are closed.}$$

It is clear that every u.s.c./l.s.c. function is marginally u.s.c./l.s.c. The following example gives a function which is marginally u.s.c./l.s.c. but not u.s.c./l.s.c.

Example 1. Let $X = Y = [0, 1]$. Let $\Omega = \{(x, y) \in X \times Y : y \geq 2x$ and $0 \leq x < 1/2\} \cup \{(x, y) \in X \times Y : y < 2(x - 1/2)$ and $1/2 < x \leq 1\}$.

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Its indicator function ψ_Ω on $X \times Y$ (equal to 0 on Ω, and to $+\infty$ otherwise) is quasi-concave-convex; it is marginally u.s.c./l.s.c., but not u.s.c./l.s.c. $\square$

Example 2 [2]. Let $f : X \times Y \to \overline{\mathbb{R}}$ be a quasi-concave-convex function. Define the functions f^{+-}, $f^{-+} : X \times Y \to \overline{\mathbb{R}}$ by

$$f^{+-}(x,y) = \liminf_{y' \to y} \limsup_{x' \to x} f(x',y') \quad \text{and} \quad f^{-+}(x,y) = \limsup_{x' \to x} \liminf_{y' \to y} f(x',y').$$

Then f^{+-} and f^{-+} are quasi-concave-convex and marginally u.s.c./l.s.c. $\square$

Minimax Theorem. *Let X, Y be nonempty convex subsets of real separated locally convex topological vector spaces such that either X or Y is finite-dimensional and compact. Then every quasi-concave-convex, marginally u.s.c./l.s.c. function on $X \times Y$ has a saddle value.*

The proof of this Minimax Theorem requires additional terminology, some well known elementary properties of multifunctions and two preparatory lemmas. Let A, B be two sets and Γ be a multifunction from A to B, denoted by $A \twoheadrightarrow B$. If Γ' is another multifunction from A to B, the inclusion $\Gamma \subset \Gamma'$ means "$\Gamma x \subset \Gamma' x$, for every $x \in A$". If A and B are convex sets, then Γ is said to be *concave-convex* whenever the values of Γ are convex and, for every $y \in Y$, the set $\{x \in X : y \notin \Gamma x\}$ is convex (see [2]).

Moreover, if A and B are topological spaces, the multifunction $\mathbf{Li}\,\Gamma : A \twoheadrightarrow B$, called the *Kuratowski lower limit* of Γ, is defined for every $x \in A$ by $\mathbf{Li}\,\Gamma x = \{y \in B$: for every neighbourhood V of y, there is a neighbourhood U of x such that for every $x' \in U$ the sets $\Gamma x'$ and V intersect$\}$. As usual, Γ is called a *lower semicontinuous multifunction* if $\Gamma \subset \mathbf{Li}\,\Gamma$; in other words, Γ is lower semicontinuous if and only if, for every open subset V of B, $\{x \in X : \Gamma x \cap V \neq \emptyset\}$ is an open subset of A.

Let $\overline{\Gamma} : A \twoheadrightarrow B$ denote the multifunction defined by $\overline{\Gamma} x = \overline{\Gamma x}$, where $\overline{\Gamma x}$ is the closure of Γx in B. Then $\mathbf{Li}\,\Gamma = \mathbf{Li}\,\overline{\Gamma} \subset \overline{\Gamma}$; hence, Γ is lower semicontinuous if and only if $\overline{\Gamma} = \mathbf{Li}\,\overline{\Gamma}$. Observe that if a topological subspace B' of B contains all the values of Γ, then the multifunction $\Gamma' : A \twoheadrightarrow B'$ defined by $\Gamma' x = \Gamma x$ is lower semicontinuous if and only if Γ is lower semicontinuous.

Let Γ be lower semicontinuous. Recall that, for any open subset V of B, the multifunction $\Gamma' : A \twoheadrightarrow B$ defined by $\Gamma' x = \Gamma x \cap V$ is lower semicontinuous; but, generally, Γ' is not lower semicontinuous when V is closed.

Lemma 1. *Let A be a topological space and let $\Gamma : A \twoheadrightarrow L$ be a lower semicontinuous multifunction with convex values in a real separated locally convex topological vector space L. Let H be a closed affine hyperplane in L and $A_H := \{x \in A : H$ strictly separates two distinct points of $\Gamma x\}$. Then A_H is an* *open subset of* A *and the multifunction* $\Gamma' : A_H \twoheadrightarrow L$ *defined by* $\Gamma' x := \Gamma x \cap H$ *is lower semicontinuous and has nonempty values.*

Proof. Denote by H_+ and H_- the open half-spaces corresponding to H. By the definition of A_H, a point x belongs to A_H if and only if

$$\text{(i)} \qquad \Gamma x \cap H_+ \neq \emptyset \quad \text{and} \quad \Gamma x \cap H_- \neq \emptyset.$$

By the lower semicontinuity of Γ, the sets $\Gamma^- H_+ := \{x \in A : \Gamma x \cap H_+ \neq \emptyset\}$ and $\Gamma^- H_- := \{x \in A : \Gamma x \cap H_- \neq \emptyset\}$ are open subsets of A; hence $A_H = \Gamma^- H_+ \cap \Gamma^- H_-$ is open. Moreover, by the lower semicontinuity of Γ, the multifunctions Γ_-, $\Gamma_+ : A_H \twoheadrightarrow L$ defined by $\Gamma_- x = \Gamma x \cap H_-$ and $\Gamma_+ x = \Gamma x \cap H_+$ are lower semicontinuous, because H_- and H_+ are open; in other words,

$$\text{(ii)} \qquad \overline{\Gamma_- x} = \mathbf{Li}\,\overline{\Gamma_-} x \quad \text{and} \quad \overline{\Gamma_+ x} = \mathbf{Li}\,\overline{\Gamma_+} x.$$

For every pair of convex subsets C, D of a separated topological vector space for which $C \cap \operatorname{int} D \neq \emptyset$, one has the known equality $\overline{C \cap \operatorname{int} D} = \overline{C} \cap \overline{D}$ (see, for example, [1]). Hence, for every $x \in A_H$, from (i) it follows that

$$\text{(iii)} \qquad \overline{\Gamma_- x} = \overline{\Gamma x \cap \overline{H_-}} \quad \text{and} \quad \overline{\Gamma_+ x} = \overline{\Gamma x \cap \overline{H_+}}.$$

Therefore, by combining (ii) and (iii), the multifunctions $\Gamma_1, \Gamma_2 : A_H \twoheadrightarrow L$ defined by

$$\text{(iv)} \qquad \Gamma_1 x = \Gamma x \cap \overline{H_-} \quad \text{and} \quad \Gamma_2 x = \Gamma x \cap \overline{H_+}$$

are lower semicontinuous. Now, to show that Γ' is lower semicontinuous, pick an $x_0 \in A_H$, a $y_0 \in \Gamma' x_0$ and an open convex neighbourhood V of y_0. We must find a neighborhood U of x_0 such that, for every $x \in U$, $\Gamma' x \cap V \neq \emptyset$. Since

$$\text{(v)} \qquad \Gamma' x = \Gamma_1 x \cap \Gamma_2 x,$$

by the lower semicontinuity of the multifunctions Γ_1 and Γ_2, there is a neighborhood $U \subset A_H$ of x_0 such that, for every $x \in U$,

$$\text{(vi)} \qquad V \cap \Gamma x \cap \overline{H_-} \neq \emptyset \quad \text{and} \quad V \cap \Gamma x \cap \overline{H_+} \neq \emptyset.$$

Now, using the fact that $V \cap \Gamma x$ is convex and that $H = \overline{H_-} \cap \overline{H_+}$, from (vi) it follows that, for every $x \in U$, $V \cap \Gamma x \cap H \neq \emptyset$. This shows the lower semicontinuity of Γ'. Obviously, the values of Γ' are nonempty. □

The following lemma is an immediate consequence of [2, Theorem 2.5].

Lemma 2. *Let X be a convex subset of a real topological vector space and let Y be a compact convex subset of a real locally convex topological vector space. If $\Omega : X \twoheadrightarrow Y$ is a lower semicontinuous concave-convex multifunction with nonempty values then $\bigcap_{x \in X} \overline{\Omega x} \neq \emptyset$.*

Proof. By virtue of [2, Theorem 2.5] we need only verify that, for every $x \in X$, $\mathbf{Li}\,\overline{\Omega}x \neq \emptyset$. This is a consequence of the nonemptiness of the values of Ω and of the lower semicontinuity of Ω which amounts to "$\overline{\Omega x} = \mathbf{Li}\,\overline{\Omega}x$, for every $x \in X$". □

Intersection Theorem. *Let X be a convex subset of a real locally convex topological vector space and let Y be a compact convex subset of $\mathbb{R}^n$. Let $\Delta : X \twoheadrightarrow Y$ be a multifunction such that, for every open subset U of X, $\bigcap_{x \in U} \Delta x$ is a closed subset of Y. If there is a lower semicontinuous concave-convex multifunction $\Omega : X \twoheadrightarrow Y$ with nonempty values such that $\Omega \subset \Delta$, then $\bigcap_{x \in X} \Delta x \neq \emptyset$.*

Proof. We will prove this theorem by induction on the dimension m of Y. For $m = 0$, the assertion of the theorem is trivial. So suppose that the Theorem holds true if $\dim Y \leq m$, and assume that $\dim Y = m+1$. By Lemma 2, we have $\bigcap_{x \in X} \overline{\Omega x} \neq \emptyset$. Then choose $y_0 \in \bigcap_{x \in X} \overline{\Omega x}$ and $x_0 \in X$. In order to prove that the required set intersection is nonempty, we need only show that $y_0 \in \Delta x_0$.

If $y_0 \in \Omega x_0$, it is clear that $y_0 \in \Delta x_0$, because $\Omega \subset \Delta$. Hence suppose that $y_0 \notin \Omega x_0$. Then choose an open ball B_0 in $\mathbb{R}^n$ with center at a point of the nonempty value Ωx_0 such that $y_0 \notin \overline{B_0}$. Since $\Omega x_0 \cap B_0 \neq \emptyset$ and B_0 is open, from the lower semicontinuity of Ω it follows that there is an open neighborhood U_0 of x_0 in X such that the multifunction $\Omega' : U_0 \twoheadrightarrow Y$ defined by $\Omega' x := \Omega x \cap B_0$ is lower semicontinuous and has nonempty values. Since X is a convex subset of a locally convex topological vector space, we can suppose that U_0 is convex. Then, since Ω' is concave-convex, by Lemma 2, one obtains $\bigcap_{x \in U_0} \overline{\Omega x \cap B_0} \neq \emptyset$. Now, pick $y_1 \in \bigcap_{x \in U_0} \overline{\Omega x \cap B_0}$. Since $y_0 \notin \overline{B_0}$ and $y_0 \in \bigcap_{x \in X} \overline{\Omega x}$, we have

$$y_0 \neq y_1 \quad \text{and} \quad [y_0, y_1] \subset \bigcap_{x \in U_0} \overline{\Omega x}, \tag{1}$$

where $[y_0, y_1]$ is the closed segment joining y_0 and y_1. Let B be an open ball in $\mathbb{R}^n$ centered at y_0 and let H_B be an affine hyperplane in $\mathbb{R}^n$ such that $y_0 \notin H_B$ and $(y_0, y_1) \cap B \cap H_B \neq \emptyset$, where (y_0, y_1) denotes the open segment joining y_0 and y_1. We derive from (1) that, for every $x \in U_0$, there exist two distinct points of $\Omega x \cap B$ strictly separated by the hyperplane H_B. Therefore, by Lemma 1, the concave-convex multifunction $\Omega_B : U_0 \twoheadrightarrow Y \cap H_B$ defined by $\Omega_B x = \Omega x \cap B \cap H_B$ is lower semicontinuous and has nonempty values. On the other hand, the multifunction $\Delta_B : U_0 \twoheadrightarrow Y \cap H_B$ defined by $\Delta_B x = \Delta x \cap \overline{B} \cap H_B$ satisfies $\Omega_B \subset \Delta_B$ and, for every open subset U of U_0, the set $\bigcap_{x \in U} \Delta_B x$ is closed. Therefore, since the dimension of $Y \cap H_B$ is $\leq m$, the inductive hypothesis entails $\bigcap_{x \in U_0} \Delta_B x \neq \emptyset$. Hence, by the definition of Δ_B, we have $\overline{B} \cap \bigcap_{x \in U_0} \Delta x \neq \emptyset$. Since $\bigcap_{x \in U_0} \Delta x$ is a closed set and B is an arbitrary open ball centered at y_0, it follows that $y_0 \in \bigcap_{x \in U_0} \Delta x$. Finally, since $x_0 \in U_0$, we have $y_0 \in \Delta x_0$. This completes the proof of the theorem. □

Proof of the Minimax Theorem. Without loss of generality, assume that Y is finite-dimensional and compact. Let $f : X \times Y \to \overline{\mathbb{R}}$ be a marginally u.s.c/l.s.c. quasi-concave-convex function. Since $\inf_Y \sup_X f \geq \sup_X \inf_Y f$, it is enough to prove that, for every real number $r > \sup_X \inf_Y f$, the inequality $r \geq \inf_Y \sup_X f$ holds. Therefore, let $r > \sup_X \inf_Y f$ and consider the multifunctions $\Omega, \Delta : X \twoheadrightarrow Y$ defined by

$$\Omega x = \{y \in Y : f(x,y) < r\} \quad \text{and} \quad \Delta x = \{y \in Y : f(x,y) \leq r\}. \tag{2}$$

Observe that, by (2), $\Omega \subset \Delta$. For every open subset U of X, $\bigcap_{x \in U} \Delta x = \{y \in Y : \sup_{x \in U} f(x,y) \leq r\}$ is closed, because f is marginally u.s.c./l.s.c. The multifunction Ω is lower semicontinuous, since, for every open set V in Y, the set $\{x \in X : \Omega x \cap V \neq \emptyset\} = \{x \in X : \inf_{y \in V} f(x,y) < r\}$ is open, because f is maginally u.s.c./l.s.c. Since $r > \sup_X \inf_Y f$, the values of Ω are nonempty. Moreover, f being quasi concave-convex, the multifunction Ω is concave-convex. Hence, from the Intersection Theorem it follows that $\bigcap_{x \in X} \Delta x \neq \emptyset$; that is, there is $y_0 \in Y$ such that, for every $x \in X$, $f(x, y_0) \leq r$; thus $r \geq \inf_Y \sup_X f$. This completes the proof. □

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Rodney C. Bassanezi
Instituto de Matemática
Universidade Estadual de Campinas
13081 Campinas-SP, BRASIL

Gabriele H. Greco
Dipartimento di Matematica
Università di Trento
38050 Povo, ITALY