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CLASSIFICATION OF POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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To Jean Leray, with profound admiration for a life filled with great discoveries

1. Introduction

We are interested in positive solutions of the quasilinear elliptic equation

$$\Delta u + f(|x|, u, |\nabla u|) = 0, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.0}$$

where $n > 2$ and f is a smooth function of its arguments.

When f is a pure power u^p ($p > 1$), then (1.0) reduces to

$$\Delta u + u^p = 0, \qquad x \in \mathbb{R}^n \setminus \{0\}. \tag{I}$$

It was shown by Fowler that (I) does not admit any positive radial solutions when $p \leq l_1$, and no bounded positive radial solutions when $l_1 < p < l$, where

$$l_1 = \frac{n}{n-2}, \qquad l = \frac{n+2}{n-2}.$$

When $p \geq l$, (I) admits infinitely many bounded positive solutions, which may be extended to regular solutions of (I) on the entire space $\mathbb{R}^n$. On the other hand, (I) admits infinitely many unbounded positive solutions when $l_1 < p < l$ and exactly one unbounded positive solution when $p > l$. We refer to these unbounded solutions as *singular*.

For $p > l_1$, let

$$\alpha = \frac{2}{p-1}, \qquad \lambda^{p-1} = \alpha(n-2-\alpha). \tag{1.1}$$

Then it is easy to see that (I) has a singular solution of the form

$$U(r) = \lambda r^{-\alpha}. \tag{1.2}$$

Thus the unique singular solution when $p > l$ has exactly the form (1.2). For $l_1 < p < l$, the function $U(r)$ also has an exceptional role: it is the *only* positive solution of (I) with *slow decay* at infinity

$$\overline{\lim_{r\to\infty}}\, r^{\alpha} u(x) > 0, \qquad r = |x|. \tag{1.3}$$

When $p > l$ all positive solutions of (I) other than $U(r)$ are still *singular at infinity*, i.e. have slow decay:

$$\lim_{r\to\infty} r^{\alpha} u = \lambda,$$

but are *regular* (bounded) at the origin. When $l_1 < p < l$ the conclusion is reversed: all positive solutions of (I) other than $U(r)$ are still *singular* at the origin:

$$\lim_{r\to 0} r^{\alpha} u = \lambda,$$

but are *regular* at infinity:

$$\lim_{r\to\infty} r^{n-2} u = c,$$

where c is a positive constant (the entire set of positive values c is attained in this way). For these results, see the proposition in Section 3.

In this paper, we shall, among other things, reprove Fowler's results concerning positive radial solutions of (I) when $p > l_1$ and also provide some generalizations for quasilinear equations.

Our uniqueness results depend only on the behavior of solutions at infinity when $p < l$, and at the origin when $p > l$. In particular, the exterior problem

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0, \quad |x| > r_0 > 0, \\ \overline{\lim}_{r\to\infty} r^{\alpha} u(r) > 0 \end{cases}$$

has no radial solution when $l_1 < p < l$ other than $u = U(r)$, and a corresponding result holds when $p > l$.

When $p < l$, it was proved by Gidas and Spruck in [5] that positive solutions of (I) with a non-removable singularity at the origin and slow decay at infinity necessarily have the form (1.2). However, their proof depends on the requirement that both the singularity at the origin and the slow decay at infinity are non-removable and thus cannot be directly extended to our case.

More recently, Bideaut-Veron studied the Emden-Fowler equation (I) with the Laplacian replaced by the "p-Laplacian". She classified all solutions of this equation by transforming it to an autonomous first order system (phase plane analysis). As was the case for Fowler's earlier analysis, however, her arguments depend crucially on the fact that for the Emden-Fowler equation the function f

is a pure power. Our approach, on the other hand, does not require autonomy and moreover generalizes to quasilinear functions f.

Thus for (1.0) we are able to obtain similar results for suitable functions f, including, in particular, the Chipot-Weissler case

$$f(|x|, u, |\nabla u|) = u^p - |\nabla u|^q, \qquad p > 0,\ q > 0,$$

and the canonical example

$$f(|x|, u, |\nabla u|) = u^p + u^q, \qquad l_1 < p < q.$$

The Chipot-Weissler equation, that is,

$$\Delta u + u^p - |\nabla u|^q = 0, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

has been extensively studied in [11]. The presence of the term $-|\nabla u|^q$ in this equation is the source of major difference. Nevertheless, the uniqueness results for the Lane-Emden equation hold in this case (see Theorem 4.3 for details of this conclusion).

The modified Lane-Emden equation equation

$$\Delta u + u^p + u^q = 0, \qquad x \in \mathbb{R}^n \setminus \{0\},\ l_1 < p < q, \tag{1.4}$$

has drawn much attention recently. When p and q are in the range $l_1 < p < l < q$, the mixed growth structure (supercritical for u large and subcritical for u small) has a profound impact on the existence and non-existence theory, and changes the outcome for the Lane-Emden equation. The analysis is surprisingly difficult, and there is no definitive result as yet.

When either $p \geq l$ or $q \leq l$, the answer is complete and we are able to classify all positive solutions of (1.4) (see (o)-(iv) of Proposition 5.1).

As noted above, the situation when $l_1 < p < l < q$ is complicated. The only known fact is that (1.4) admits a unique positive slow decay (regular at the origin) solution for some (p, q) (see in particular Section 5, where explicit solutions are given for $q = 2p - 1$ and $p > n/(n-2) = l_1$; see also [6]). On the other hand, it is unknown if (1.4) has *any* positive solution at all for other (p, q) values. Finally, it is not even known whether there are *any* fast decay solutions.

One may view the asymptotic behavior at a singularity of (1.4) as the effect of a perturbation to the Lane-Emden equation (I). If, say, the origin is a singularity, then the term u^q is dominant and the term u^p is thus a small perturbation. Therefore, the outcome ought to be essentially the same as that of (I) with p replaced by q. At infinity, the situation is just reversed. The term u^p is dominant and the term u^q is a small perturbation. This, in particular, explains the difficulty when $l_1 < p < l < q$.

The situation for (p, q) in the parameter domain $1 < p < q$ is illustrated in the following Figure. In the region (i) there is a single positive slow decay

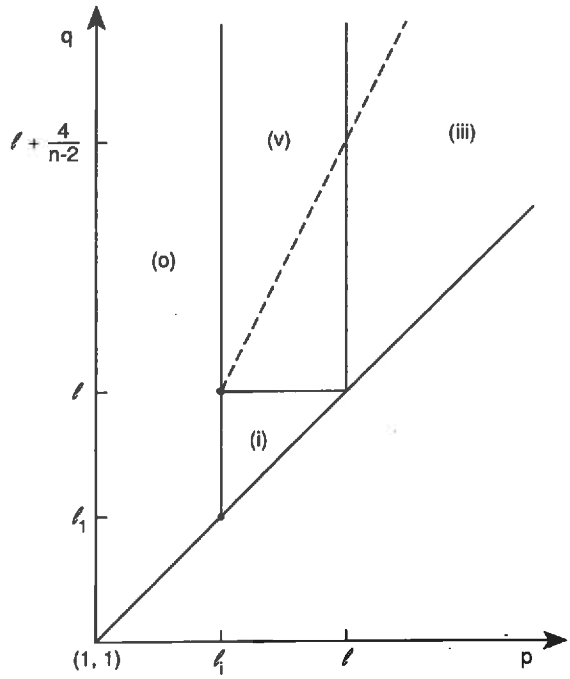

FIGURE. The parameter domain $1 < p < q$. The regions (i), (iii), (v) correspond to the cases of Theorem 5.2. The region (o) where $1 < p < l_1$, on the other hand, allows no positive solutions whatsoever. The dashed line is $q = 2p - 1$, $p > l_1$.

solution and a family of singular positive fast decay solutions, while in (iii) there is a unique positive singular solution and a family of regular positive slow decay solutions. On the other hand, for (p, q) on the line $q = 2p - 1$ in (v) there is a unique positive (regular) slow decay solution.

2. Preliminaries

In this section we present some preliminary results for radial ground states $u(r)$ of (I), where $r = |x|$ is the radius. Obviously we can consider $u(r)$ to be a solution of the ordinary differential equation

$$u''(r) + \frac{n-1}{r} u'(r) + u^p(r) = 0, \qquad 0 < r < \infty, \tag{2.0}$$

with

$$u(r) > 0 \qquad \text{for all } r > 0. \tag{2.1}$$

Put

$$\alpha = \frac{2}{p-1}, \qquad \lambda^{p-1} = \alpha(n-2-\alpha).$$

Using the decay estimates obtained in [7] and [8] and the Pokhozhaev identity, we can prove the following result.

LEMMA 2.0. *Solutions of* (2.0) *are necessarily positive for all* $r > 0$ *if they are positive near the origin if* $p > l$. *On the other hand, when* $p < l$ *they are positive for all* $r > 0$ *if they are positive near infinity.*

PROOF. We only sketch the proof for $p > l$. Let u be a solution of (2.0) which is positive near the origin. It is obvious that u can be extended as long as u is positive. If u never reaches zero at a finite r_0, then we are done. Otherwise, let $R < \infty$ be the first zero of u. Then u is a positive solution of the equation

$$\begin{cases} \Delta u + u^p = 0, & x \in B_R \setminus 0, \\ u = 0, & x \in \partial B_R. \end{cases}$$

On the other hand, by the estimates of [7], p. 381, one has

$$u \leq cr^{-\alpha}, \qquad |u'| \leq cr^{-\alpha-1}$$

for $0 < r < R$. Thus the standard Pokhozhaev identity applies since $p > l$. It implies that $u \equiv 0$, a contradiction. The proof is complete.

When p is in the range

$$p > \frac{n}{n-2}, \qquad p \neq l = \frac{n+2}{n-2}, \tag{2.2}$$

one can prove the following exact limit properties.

THEOREM 2.1. *Let u be a solution of* (2.0), *and suppose that* (2.2) *is satisfied. Then we have*

$$\lim_{r\to\infty} r^\alpha u = \lambda, \quad \textit{or} \quad \lim_{r\to\infty} r^{n-2} u = c > 0 \tag{2.3}$$

for some constant $c > 0$. *Moreover,*

$$\lim_{r\to 0} r^\alpha u = \lambda, \quad \textit{or} \quad \lim_{r\to 0} u = c_1 > 0 \tag{2.4}$$

for some constant $c_1 > 0$.

To prove this result, we need some technical lemmas. Let u be a solution of (2.0) and set $v(r) = r^\alpha u(r)$, where $\alpha = 2/(p-1)$.

LEMMA 2.1. *The function v satisfies*

$$v'' + \frac{n-1-2\alpha}{r} v' + \frac{v^p}{r^2} - \frac{\lambda^{p-1} v}{r^2} = 0, \qquad 0 < r < \infty, \tag{2.5}$$

and

$$|v^{(\tau)}(r)| \leq \frac{c}{r^\tau}, \qquad 0 < r < \infty, \tag{2.6}$$

for all non-negative integers τ and some $c > 0$, where $^{(\tau)}$ denotes the τ-th derivative. Moreover, we have

$$v'^2 r \in L^1(0,\infty), \qquad v''^2 r^3 \in L^1(0,\infty). \tag{2.7}$$

PROOF. Equation (2.5) arrives by direct calculation. The proof of (2.6) is equivalent to showing

$$|u^{(\tau)}(r)| \le \frac{c}{r^{\tau+\alpha}}, \qquad 0 < r < \infty.$$

For a proof, we refer the reader to [7], p. 381, and [8], p. 235. To prove $(2.7)_1$, multiply (2.5) by $v'r^2$ and integrate from $t > 0$ to $T > t$ to obtain

$$\frac{v'^2 r^2}{2}\bigg|_t^T + (n-2-2\alpha)\int_t^T v'^2 r + \frac{v^{p+1}}{p+1}\bigg|_t^T - \frac{\lambda^{p-1} v^2}{2}\bigg|_t^T = 0. \tag{2.8}$$

It follows from (2.6) that there exists a positive constant c such that

$$\int_t^T v'^2 r \le c, \qquad \text{for all } T > t > 0$$

since $n - 2 - 2\alpha \ne 0$ by (2.2), and $(2.7)_1$ follows. One may prove $(2.7)_2$ similarly by differentiating (2.5). The proof is complete.

LEMMA 2.3. *We have*

$$\lim_{r\to 0} rv' = \lim_{r\to\infty} rv' = 0, \qquad \lim_{r\to 0} r^2 v'' = \lim_{r\to\infty} r^2 v'' = 0. \tag{2.9}$$

PROOF. We only prove $(2.9)_1$. Suppose for contradiction that it is not true. Then there exist a sequence $\{r_k\}$ and a constant $c > 0$ such that

$$|v'(r_k) r_k| \ge c.$$

From (2.6), one obviously has

$$|(v'^2 r^2)'| \le \frac{M}{r},$$

for some $M > 0$. Combining the above yields

$$|v'^2 r^2 - v'(r_k)^2 r_k^2| \le M|r - r_k| \max\left(\frac{1}{r}, \frac{1}{r_k}\right)$$

and so

$$v'^2 r^2 \ge \frac{c^2}{2}, \qquad r \in [(1+\varepsilon)^{-1} r_k, (1+\varepsilon) r_k], \qquad \varepsilon = \frac{c^2}{2M}.$$

Choosing an infinite subsequence, if necessary, of $\{r_k\}$ so that the intervals above are disjoint, it follows that

$$\int_0^\infty v'^2 r \geq \sum_{k=1}^\infty \int_{(1+\varepsilon)^{-1}r_k}^{(1+\varepsilon)r_k} v'^2 r$$
$$\geq \sum_{k=1}^\infty \int_{(1+\varepsilon)^{-1}r_k}^{(1+\varepsilon)r_k} \frac{c^2}{2r} = c^2 \sum_{k=1}^\infty \ln(1+\varepsilon) = \infty.$$

This contradicts $(2.7)_1$ and completes the proof.

LEMMA 2.3. *There exist two non-negative numbers* λ_1 *and* λ_2 *such that*

$$\lim_{r\to\infty} v = \lambda_1, \qquad \lim_{r\to 0} v = \lambda_2. \tag{2.10}$$

PROOF. Consider the function

$$a(r) = \frac{v^{p+1}}{p+1} - \frac{\lambda^{p-1}v^2}{2}.$$

By Lemmas 2.1 and 2.2, we have, for fixed $t > 0$,

$$\frac{v'^2T^2}{2} \to 0, \qquad \int_t^T v'^2 r \to c, \qquad \text{as } T \to \infty$$

for some constant c. This implies, by (2.8), that $a(r)$ must tend to a finite limit as $r \to \infty$. In turn v approaches a finite limit since the limit set of v is connected. Similarly, we conclude that v approaches a finite limit as $r \to 0$. The lemma is proved.

If $\lambda_i = 0$ $(i = 1, 2)$ occurs in (2.10), equation (2.5) suggests that v tends to zero at an algebraic rate. Indeed, since $p > 1$ and v tends to zero, v is expected to satisfy asymptotically (near the origin or infinity respectively) the equation

$$v'' + \frac{n-1-2\alpha}{r}v' - \frac{\lambda^{p-1}v}{r^2} = 0. \tag{2.5$'$}$$

Therefore v should have one of the two asymptotical behaviors:

$$v \approx r^\alpha \qquad \text{or} \qquad v \approx r^{-(n-2-\alpha)}.$$

We have the following lemma.

LEMMA 2.4. *If* $\lambda_1 = 0$ *in* (2.10), *then*

$$\lim_{r\to\infty} r^{n-2}u(r) = c_1, \tag{2.11}$$

for some positive constant c_1. *If* $\lambda_2 = 0$ *in* (2.10), *then*

$$\lim_{r\to 0} u(r) = c_2, \tag{2.12}$$

for some positive constant c_2.

PROOF. We only prove (2.12), the demonstration of (2.11) being the same. By assumption, for any $\varepsilon > 0$ there exists a positive number r_ε such that v satisfies,

$$v'' + \frac{n-1-2\alpha}{r}v' - \frac{(\lambda^{p-1}-\varepsilon)v}{r^2} \geq 0, \qquad 0 < r < r_\varepsilon. \tag{2.13}$$

The characteristic equation of (2.13) has the two characteristic values

$$a_1 = \alpha - \frac{n-2-\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha + O(\varepsilon),$$

$$a_2 = \alpha - \frac{n-2+\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha + 2 - n + O(\varepsilon).$$

Rewrite (2.13)

$$\left(D - \frac{a_2-1}{r}\right)\left(D - \frac{a_1}{r}\right)v \geq 0,$$

and so

$$\left[r^{n-1-\alpha+O(\varepsilon)}\left(D - \frac{a_1}{r}\right)v\right]' \geq 0. \tag{2.14}$$

Observe that, for ε small enough,

$$\lim_{r\to 0} r^{n-1-\alpha+O(\varepsilon)}\left(D - \frac{a_1}{r}\right)v = 0$$

by (2.6), since $n-1-\alpha > 1$. It follows from (2.14) that

$$\left(D - \frac{a_1}{r}\right)v \geq 0, \qquad 0 < r < r_\varepsilon.$$

Integrating once from r to r_ε yields

$$v \leq c_\varepsilon r^{a_1} = c_\varepsilon r^{\alpha+O(\varepsilon)}, \qquad 0 < r < r_\varepsilon.$$

Thus for ε sufficiently small,

$$v'' + \frac{n-1-2\alpha}{r}v' - \frac{\lambda^{p-1}v}{r^2} = g(r) \tag{2.5''}$$

with

$$g(r) = \frac{v^p}{r^2} = o(1).$$

Now using the representation of solutions of (2.5″), we immediately infer that v is bounded by r^α, and in turn u is bounded. By standard theory, we get (2.12).

PROOF OF THEOREM 2.1. If $\lambda_1 = 0$ in (2.10), then $(2.3)_2$ holds by Lemma 2.4. It remains to show that $(2.3)_1$ is true if $\lambda_1 > 0$. To do this, we first observe by (2.9) that

$$\lim_{r\to\infty} rv' = \lim_{r\to\infty} r^2 v'' = 0.$$

Therefore, from (2.5), λ_1 satisfies

$$\lambda_1^p - \lambda^{p-1}\lambda_1 = 0,$$

that is, $\lambda_1 = \lambda$ since $\lambda_1 > 0$. The same argument proves (2.4) and the proof is complete.

3. A uniqueness result

In this section, we shall prove a uniqueness result for radial positive singular solutions of (I) when $p > l_1$. More precisely, we require that solutions be singular *at infinity for* $p \le l$, and *at the origin for* $p \ge l$.

Recall that a positive solution u has slow decay at infinity if

$$\overline{\lim_{r\to\infty}}\, r^\alpha u(r) > 0,$$

and singular at the origin if

$$\overline{\lim_{r\to 0}}\, u(r) = \infty.$$

If u has slow decay at infinity, we also say that u is singular at infinity. The above definitions will be used through the rest of the paper.

By the results of the previous section, we have for such a singular solution with $p \neq l$,

$$\lim_{r\to\infty} r^\alpha u(r) = \lambda, \qquad \lim_{r\to\infty} r^{\alpha+1} u'(r) = -\alpha\lambda, \qquad \text{if } l_1 < p < l, \tag{3.1}$$

and

$$\lim_{r\to 0} r^\alpha u(r) = \lambda, \qquad \lim_{r\to 0} r^{\alpha+1} u'(r) = -\alpha\lambda, \qquad \text{if } p > l. \tag{3.2}$$

THEOREM 3.1. *Equation* (I) *admits exactly one radial positive solution,*

$$U(r) = \lambda r^{-\alpha}$$

satisfying (3.1) *for* $l_1 < p \le l$, *and* (3.2) *for* $p \ge l$.

Consider first the case when $p \ge l$. Let u be a solution of (I) different from $U(r)$. We introduce the function

$$w(r) = r^\alpha u(r) - r^\alpha U(r) = r^\alpha u(r) - \lambda \to 0, \quad \text{as } r \to 0^+ \tag{3.3}$$

and show that w is identically zero. We require the following lemma.

LEMMA 3.1. *The function w satisfies*

$$w'' + \frac{n-1-2\alpha}{r} w' + \frac{2(n-2-\alpha)}{r^2} w + \frac{f(r)}{r^2} = 0, \tag{3.4}$$

where

$$f(r) = (\lambda + w(r))^p - \lambda^p - p\lambda^{p-1} w(r) = \lambda^p \sum_{k=2}^{\infty} \frac{(p-k+1)!}{k!} \left(\frac{w(r)}{\lambda}\right)^k. \tag{3.5}$$

The proof of the lemma is straightforward, using (2.0), (2.4), (3.2), and the relation $\alpha + 2 = \alpha p$.

PROOF OF THEOREM 3.1. By the above lemma, it suffices to show that any solution of (3.4), (3.5) satisfying (3.3) is identically zero. For $T > 0$ and $h \in (0, T)$, multiply (3.4) by $r^2 w'$ and integrate from h to T to obtain

$$\left[\frac{r^2 w'^2}{2} + (n-2-\alpha)w^2\right]_h^T + (n-2-2\alpha)\int_h^T rw'^2 + \int_h^T f(r)w' = 0. \tag{3.6}$$

By (3.5), one has

$$\begin{aligned}\int_h^T f(r)w' &= \lambda^{p-1}\sum_{k=3}^{\infty}\frac{(p-k+2)!}{k!}\left(\frac{w(r)}{\lambda}\right)^k\Bigg|_h^T \\ &= \lambda^{p-1}\sum_{k=3}^{\infty}\frac{(p-k+2)!}{k!}\left[\left(\frac{w(T)}{\lambda}\right)^k - \left(\frac{w(h)}{\lambda}\right)^k\right].\end{aligned}$$

It is also clear that

$$\liminf_{r\to 0^+} |rw'| = 0.$$

For small T, by letting $h \to 0$ (along some sequence) in (3.6) and using (3.3), it follows that

$$T^2 w'^2 + 2(n-2-\alpha)w^2(T) \le M|w(T)|^3,$$

since $n-2-2\alpha \ge 0$. Hence $w(T) \equiv 0$ for all sufficiently small T, which implies that w is identically zero for all $r > 0$ by the uniqueness of the initial value problem for ordinary differential equations. This finishes the proof for $p \ge l$.

When $l_1 < p \le l$, we use the transformation

$$w(t) = r^\alpha u(r) - r^\alpha U(r), \qquad r = 1/t.$$

Then w satisfies

$$\ddot{w} + \frac{3+2\alpha-n}{t}\dot{w} + \frac{2(n-2-\alpha)}{t^2}w + \frac{f(t)}{t^2} = 0, \qquad \dot{} = \frac{d}{dt},$$

where

$$f(t) = (\lambda + w(t))^p - \lambda^p - p\lambda^{p-1}w(t) = \lambda^p\sum_{k=2}^{\infty}\frac{(p-k+1)!}{k!}\left(\frac{w(t)}{\lambda}\right)^k.$$

The above equation is almost the same as (3.4), except for the coefficient of the second term. On the other hand, one easily sees that the key ingredient in the proof for the case $p \ge l$ is that the coefficient of the term w'/r is not less than 1 and of the term w/r^2 is positive. Fortunately, we have

$$3 + 2\alpha - n \ge 1, \quad \text{and} \quad n - 2 - \alpha > 0$$

when $l_1 < p \le l$. Hence the proof above carries over immediately.

Finally, we have the following corollary.

PROPOSITION 3.1 (Classification of positive radial solutions of (I)).

(o) *If* $p \le l_1$, *then* (I) *admits no positive solution.*

(i) *If* $p < l$, *then* (I) *admits exactly one solution,* $U(r)$, *with slow decay at infinity and a family of positive solutions satisfying*

$$\lim_{r\to 0} r^\alpha u = \lambda, \qquad \lim_{r\to\infty} r^{n-2} u = c \tag{3.7}$$

for all $c > 0$. *Moreover,* (I) *does not have any other positive solutions.*

(ii) *If* $p > l$, *then* (I) *admits exactly one solution,* $U(r)$, *singular at the origin and a family of positive solutions satisfying*

$$\lim_{r\to 0} u = c, \qquad \lim_{r\to\infty} r^\alpha u = \lambda \tag{3.8}$$

for all $c > 0$. *Moreover,* (I) *does not have any other positive solutions.*

(iii) *If* $p = l$, *then* (I) *admits a family of singular solutions, each of the form*

$$u(r) = U(r)h(r), \tag{3.9}$$

where h *oscillates endlessly, both near the origin and at infinity, between fixed values* λ_1 *and* λ_2 *satisfying*

$$0 < \lambda_1 \le 1 \le \lambda_2, \qquad b(\lambda_1) = b(\lambda_2),$$

with

$$b(h) = \frac{h^{l+1}}{l+1} - \frac{h^2}{2}.$$

In particular, λ_1 *can take any value in* $(0,1]$, *thus determining the corresponding value* λ_2; *when* $\lambda_1 = \lambda_2 = 1$, *we get the exact solution*

$$u(r) = U(r).$$

All other positive solutions are regular at both the origin and infinity and have the form

$$u(r) = \left(\frac{\delta\sqrt{n(n-2)}}{\delta^2 + r^2}\right)^{(n-2)/2}, \qquad \delta = const. > 0. \tag{3.10}$$

PROOF. We first prove (i). By Lemmas 2.3 and 2.4, either $(3.7)_1$ or $(3.8)_1$ holds at the origin. One immediately excludes $(3.8)_1$ by using the Pokhozhaev identity and the fact that $p < l$. On the other hand, by Theorem 3.1, (I) has exactly one solution singular at infinity, which is obviously $U(r)$. Using Lemmas 2.3 and 2.4 again, we obtain $(3.7)_2$. The existence of solutions satisfying $(3.7)_2$ is obtained by a shooting argument from infinity (asymptotic integration, see [12]).

The proof of (ii) is essentially the same as that of (i).

To prove (iii), we need only consider the case when (3.10) fails. Then u is singular at the origin, and we can set $u(r) = U(r)h(r)$. Thus $h(r) = v(r)/\lambda$, and satisfies the equation

$$h'' + \frac{1}{r}h' + \frac{\lambda^{p-1}}{r^2}(h^p - h) = 0, \qquad 0 < r < \infty,$$

see (2.5) and recall (2.8). In particular, by Lemma 2.1, h is bounded. Moreover, as in (2.8), we have

$$\frac{1}{2}\alpha^2 r^2 h'^2 + \frac{h^{p+1}}{p+1} - \frac{h^2}{2} = \text{const.} = c.$$

Hence $b(h) \le c$, and since $h > 0$ it is easy to see that necessarily $c \le 0$. At any critical point r_0 of h we have

$$b(h) = \frac{h^{p+1}}{p+1} - \frac{h^2}{2} = c,$$

so $h = \lambda_1$ or $h = \lambda_2$ (suppose $c < 0$), and $h'' > 0$ when $h = \lambda_1$ and $h'' < 0$ when $h = \lambda_2$. Thus h is oscillatory, between λ_1 and λ_2. By changing variables $r \to 1/t$, we see that h is also oscillatory as $r \to 0$.

[The case $c = 0$ is special, because $\lambda_1 = 0$. In this case, since $b(h) \approx -h^2/2$ for small h, we have $h' \approx \pm\alpha h/r$ as $h \to 0$, so in turn by integration h decays algebraically (r^α) to 0 as $r \to 0$. Thus $u(r) = h(r)U(r)$ is bounded at the origin, and so regular there. But this is just the case given in (3.10), completing the proof.]

REMARK. The results (i) and (ii) are *dual*, in the sense that the classification of radial solutions of (I) in either the case $l_1 < p < l$, or the case $p > l$, yields a complete classification for the other case. Specifically, the change of variables

$$w(t) = r^{n-2}u(r), \qquad r = (\mu/t)^\mu, \tag{3.11}$$

$$\mu = \frac{2}{(n-2)p - n}, \qquad p > l_1,$$

transforms (2.1) into

$$\ddot{w} + \frac{m-1}{t}\dot{w} + w^p = 0, \qquad 0 < t < \infty, \tag{3.12}$$

where

$$m - 2 = (n-2)\mu. \tag{3.13}$$

The relation (3.13) can be rewritten

$$\frac{n+2}{n-2} - p = p - \frac{m+2}{m-2},$$

showing that when p is subcritical for (2.1) it is supercritical for (3.12). Thus a classification for $p \in (l_1, l)$ produces through the transform (3.11) and the identification $m \to n$, $t \to r$ a classification for $p \in (l, \infty)$.

For example, the limit conditions (3.7) respectively transform into

$$\lim_{t\to\infty} t^\alpha w = \overline{\lambda}, \qquad \lim_{t\to 0} w = c,$$

where $\overline{\lambda} = \alpha(m-2-\alpha)$, which is just (3.8) with m in place of n, and t instead of r.

4. Extensions to quasilinear equations

In this section, we generalize the above results to equations of quasilinear type. We particularly consider the model equation of Chipot and Weissler

$$\Delta u + u^p - |\nabla u|^q = 0, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{II}$$

and the radial version

$$u'' + \frac{n-1}{r}u' + u^p - |u'|^q = 0, \qquad 0 < r < \infty. \tag{4.1}$$

We first consider solutions singular at infinity. It was proved in [10] that positive solutions of (4.1) decay no slower than $r^{-2/(p-1)}$ at infinity when $q \geq 2p/(p+1)$ and $p > 1$. On the other hand, we have (see [10]), for the same (p,q) value,

LEMMA 4.1. *Let u be a positive solution of* (4.1). *Then u' is ultimately negative and*

$$u = O(r^{-\alpha}), \qquad u' = O(r^{-\alpha-1}), \qquad \textit{as } r \to \infty.$$

When p and q are in the range

$$q > \frac{2p}{p+1}, \qquad p > l_1 = \frac{n}{n-2}, \qquad p \neq l = \frac{n+2}{n-2} \tag{4.2}$$

the following stronger limit property holds, where as before

$$\alpha = \frac{2}{p-1}, \qquad \lambda^{p-1} = \alpha(n-2-\alpha).$$

THEOREM 4.1. *Let u be a solution of* (4.1). *Suppose that* (4.2) *is satisfied. Then either*

$$\lim_{r\to\infty} ur^\alpha = \lambda \tag{4.3}$$

or

$$\lim_{r\to\infty} r^{n-2}u(r) = c > 0 \tag{4.4}$$

for some constant $c > 0$.

To prove this result, we need some technical lemmas. Let u be a solution of (4.1) and set $v(r) = r^\alpha u(r)$.

LEMMA 4.2. *The function v satisfies*

$$v'' + \frac{n-1-2\alpha}{r} v' + \frac{v^p}{r^2} - \frac{\lambda^{p-1} v}{r^2} - \frac{|\alpha v - r v'|^q}{r^{(\alpha+1)q-\alpha}} = 0, \qquad r > 0. \tag{4.5}$$

Moreover, we have

$$v'^2 r \in L^1(0,\infty), \qquad [|\alpha v - r v'|^q - (\alpha v)^q] v' \in L^1(0,\infty). \tag{4.6}$$

PROOF. The proof of (4.5) is by direct calculation. To prove $(4.6)_1$ we multiply (4.5) by $v' r^2$ and integrate from 0 to R to obtain

$$\begin{aligned}(n-2-2\alpha) \int_0^R v'^2 r \\ - \int_0^R \frac{|\alpha v - r v'|^q v'}{r^{(\alpha+1)q-\alpha-2}} + \frac{v'^2 R^2}{2} + \frac{v^{p+1}(R)}{p+1} - \frac{\lambda^{p-1} v^2(R)}{2} = 0.\end{aligned} \tag{4.7}$$

By Lemma 4.1, all non-integral terms are bounded on $(0, \infty)$. Also it is easy to check that

$$|\alpha v - r v'|^q |v'| r^{\alpha+2-(\alpha+1)q} = O(r^{-(\alpha+1)(q-1)}) \qquad \text{as } r \to \infty.$$

It follows that

$$\int_0^\infty |\alpha v - r v'|^q |v'| r^{\alpha+2-(\alpha+1)q} < \infty.$$

Hence, from (4.7),

$$\int_0^\infty v'^2 r < \infty,$$

since $n - 2 - 2\alpha \neq 0$. This completes the proof of $(4.6)_1$.

To prove $(4.6)_2$, we see from the mean value theorem that

$$[(\alpha v - r v')^q - (\alpha v)^q] v' = -q\zeta(r) r v'^2,$$

where ζ is between $(\alpha v - r v')^{q-1}$ and $(\alpha v)^{q-1}$, and hence bounded. Therefore

$$|[(\alpha v - r v')^q - (\alpha v)^q] v'| \le M r v'^2$$

for some constant $M > 0$. Thus $(4.6)_2$ follows from $(4.6)_1$ and the proof of the lemma is complete.

LEMMA 4.3. *There exists a non-negative number $\widetilde{\lambda}$ such that*

$$\lim_{r\to\infty} v = \widetilde{\lambda}. \tag{4.8}$$

PROOF. First, we have (cf. Lemma 2.3)

$$\lim_{r\to\infty} r v' = 0. \tag{4.9}$$

Now consider the function

$$a(r) = \frac{v^{p+1}}{p+1} - \frac{\lambda^{p-1}v^2}{2}.$$

By (4.6) and (4.9), we have

$$\frac{v'^2R^2}{2} \to 0, \qquad \int_0^R v'^2 r \to c_1, \qquad \int_0^R (\alpha v - rv')^q v' r^{\alpha+2-(\alpha+1)q} \to c_2$$

as $R \to \infty$ for some constants c_1 and c_2. From (4.7), it follows that $a(r)$ tends to a finite limit as $r \to \infty$. Now we infer that v must approach a finite limit since the limit set of v is connected.

LEMMA 4.4. *If $\widetilde{\lambda} = 0$, then $v' < 0$ ultimately.*

PROOF. Suppose not. Then either $v' \geq 0$ ultimately or v' changes sign infinitely many times. The first obviously cannot happen for then v would not go to zero at all. Suppose that the second occurs. Then there is a sequence $\{r_k\}$, which tends to infinity, such that v assumes local maximal values at each r_k. Therefore

$$\lim_{k\to\infty} v(r_k) = 0, \qquad v'(r_k) = 0, \qquad v''(r_k) \leq 0.$$

It follows from (4.5) that $v(r_k)^{p-1} \geq \lambda^{p-1}$, which is a contradiction.

LEMMA 4.5. *Suppose that $\widetilde{\lambda} = 0$. Then necessarily*

$$r^{n-2}u(r) \leq c, \qquad r^{n-1}|u'(r)| \leq c \tag{4.10}$$

for some constant $c > 0$.

PROOF. For $k = 2(n-1-\delta)$, one has (cf. [10])

$$[r^k H(r)]' = r^{k-1}\left\{-\delta u'^2(r) + \frac{k}{p+1}u^{p+1}(r) - r|u'(r)|^{q+1}\right\},$$

where

$$\delta = \frac{n-2}{2} - \frac{n-1}{p+1} > 0, \qquad H(r) = \frac{1}{2}u'^2 + \frac{1}{p+1}u^{p+1}.$$

By Lemma 4.4,

$$\alpha u = r^{1-\alpha}v' - ru' \leq r|u'|$$

for r large. It follows that

$$u^{p+1} = u^2 r^{-2}(r^{2/(p-1)}u)^{p-1} \leq \alpha^{-2}|u'|^2 v^{p-1}.$$

Therefore

$$[r^k H(r)]' \leq r^{k-1}u'^2(r)\left\{-\delta + \alpha^{-2}v^{p-1}\right\} < 0$$

for r sufficiently large, since v goes to zero as $r \to \infty$. It follows that $r^k H(r)$ is ultimately decreasing, and in turn

$$|u'| \leq cr^{-k/2}, \qquad u \leq cr^{1-k/2}. \tag{4.11}$$

Now for $\bar{k} = 2(n-1)$, using (4.11) we obtain

$$[r^{\bar{k}} H(r)]' = r^{\bar{k}-1}\left\{\frac{\bar{k}}{p+1}u^{p+1}(r) - r|u'(r)|^{q+1}\right\} \le \frac{\bar{k}}{p+1}r^{\bar{k}-1}u^{p+1}(r)$$
$$\le cr^{2n-3-(n-2)(p+1)+(n-1-\bar{k}/2)(p+1)} = cr^{-1-\delta(p+1)},$$

where $\delta > 0$. Therefore $r^{2n-2}H(r)$ is bounded, which yields $(4.10)_2$. Finally, $(4.10)_1$ immediately follows from $(4.10)_2$.

LEMMA 4.6. *Suppose that $\widetilde{\lambda} = 0$ in (4.8). Then*

$$\lim_{r\to\infty} r^{n-2}u(r) = c, \tag{4.12}$$

for some positive constant c.

PROOF. From Lemma 4.5, $|u'|^{q-1} \le cr^{-1-\delta}$ where $\delta = q(n-1) - n > 0$. Hence

$$\int_0^r |u'|^{q-1} < \infty. \tag{4.13}$$

From (4.1), one has

$$\left(u'r^{n-1}e^{\int_0^r |u'|^{q-1}}\right)' = -u^p r^{n-1}e^{\int_0^r |u'|^{q-1}} < 0.$$

Hence the function $u'r^{n-1}e^{\int_0^r |u'|^{q-1}}$ is negative, decreasing and bounded below. It follows that

$$\lim_{r\to\infty} u'r^{n-1}e^{\int_0^r |u'|^{q-1}} = -c$$

for some $c > 0$, and in turn

$$\lim_{r\to\infty} u'r^{n-1} = -c_1 \tag{4.14}$$

for some $c_1 > 0$ by (4.13). Integrating (4.14) yields (4.12).

PROOF OF THEOREM 4.1. If $\widetilde{\lambda} = 0$ in (4.8), then (4.4) holds by Lemma 4.6. It remains to show that (4.3) is true if $\widetilde{\lambda} > 0$. To do this, we first observe that

$$\lim_{r\to\infty} r^{\alpha+1}u' = -\alpha\widetilde{\lambda}$$

since $rv' = r^{\alpha+1}u' + \alpha r^{\alpha}u \to 0$ as $r \to \infty$. From (4.1), we see that $r^{\alpha+2}u''$ has a finite limit as $r \to \infty$. Using L'Hospital's rule, we then get

$$\lim_{r\to\infty} r^{\alpha+2}u'' = -(\alpha+1)\lim_{r\to\infty} r^{\alpha+1}u' = \alpha(\alpha+1)\widetilde{\lambda}.$$

Putting the above two limits into (4.5), we immediately obtain (4.3).

If solutions have a singularity at the origin, we have the following corresponding result.

THEOREM 4.2. *Let u be a solution of* (4.1), *and suppose that*

$$q < \frac{2p}{p+1}, \qquad p > \frac{n}{n-2}, \qquad p \neq l = \frac{n+2}{n-2},$$

Then either

$$\lim_{r \to 0} u r^{\alpha} = \lambda \tag{4.15}$$

or

$$\lim_{r \to 0} u(r) = c > 0 \tag{4.16}$$

for some constant $c > 0$.

Finally, as in Section 3, we derive the following uniqueness result.

THEOREM 4.3. *If $l_1 < p \leq l$ and $q > 2p/(p+1)$, then equation* (II) *admits at most one radial positive solution with slow decay at infinity. If $p \geq l$ and $q < 2p/(p+1)$, then equation* (II) *admits at most one radial positive solution with a singularity at the origin.*

REMARK. In contrast to the case for the Emden equation (I), there exist solutions of (II) when $l_1 < p \leq l$ and $q > 2p/(p+1)$ which have fast decay at infinity but do *not* have the singular behavior $(3.7)_1$ at the origin. An explicit example is the function $u(r) = [(n-2)r^{-1}]^{(n-2)}$ when $q = (n-1)p/(n-2)$.

We only prove the first case of the theorem, i.e., $l_1 < p \leq l$ and $q > 2p/(p+1)$. Let u be a solution of (4.1) with slow decay at infinity and set

$$\bar{v}(r) = v(r) - \lambda = r^{\alpha} u(r) - \lambda \to 0 \qquad \text{as } r \to \infty. \tag{4.17}$$

LEMMA 4.7. *We have*

$$\bar{v}'' + \frac{(n-1-2\alpha)\bar{v}'}{r} + \frac{2(n-2-\alpha)\bar{v}}{r^2} + \frac{f(r)}{r^2} - \frac{|\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q}{r^{2-\delta}} = 0, \qquad r > 0, \tag{4.18}$$

where $\delta = (\alpha+1)q - \alpha - 2$ and

$$f(r) = (\lambda + \bar{v}(r))^p - \lambda^p - p\lambda^{p-1}\bar{v}(r) = \lambda^p \sum_{k=2}^{\infty} \frac{(p-k+1)!}{k!} \left(\frac{\bar{v}(r)}{\lambda}\right)^k.$$

LEMMA 4.8. *For any $\varepsilon \in (0, \delta)$, we have*

$$\bar{v} = O(r^{-\varepsilon}), \qquad \bar{v}' = O(r^{-\varepsilon-1}), \qquad \textit{as } r \to \infty. \tag{4.19}$$

PROOF. For $T > 0$ and $t \in (0, T)$, multiply (4.18) by $r^2\bar{v}'$ and integrate from t to T to obtain

$$\left[\frac{r^2\bar{v}'^2}{2} + (n-2-\alpha)\bar{v}^2\right]_t^T + (n-2-2\alpha)\int_t^T r\bar{v}'^2 + \int_t^T f(r)\bar{v}' - \int_t^T |\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q \bar{v}' r^{-\delta} = 0. \tag{4.20}$$

By (4.9) and (4.17), it is easy to see that

$$T^2\bar{v}'^2(T) \to 0, \qquad \bar{v}^2(T) \to 0, \qquad \text{as } T \to \infty,$$

and

$$\begin{aligned}\int_t^T f(r)\bar{v}' &= \lambda^{p-1}\sum_{k=3}^{\infty}\frac{(p-k+2)!}{k!}\left(\frac{\bar{v}(r)}{\lambda}\right)^k\Bigg|_t^T \\ &= \lambda^{p-1}\sum_{k=3}^{\infty}\frac{(p-k+2)!}{k!}\left[\left(\frac{\bar{v}(T)}{\lambda}\right)^k - \left(\frac{\bar{v}(t)}{\lambda}\right)^k\right] \\ &= -\lambda^{p-1}\sum_{k=3}^{\infty}\frac{(p-k+2)!}{k!}\left(\frac{\bar{v}(t)}{\lambda}\right)^k, \qquad \text{as } T \to \infty.\end{aligned}$$

It follows by letting $T \to \infty$ in (4.20) that

$$t^2\bar{v}'^2 + 2(n-2-\alpha)\bar{v}^2 \le M|\bar{v}|^3 + \left|\int_t^\infty |\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q \bar{v}' r^{-\delta}\right|,$$

since $n-2-2\alpha \le 0$. It follows that for large t,

$$t^2\bar{v}'^2 + 2(n-2-\alpha)\bar{v}^2 \le M\left|\int_t^\infty |\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q \bar{v}' r^{-\delta}\right|. \tag{4.21}$$

On the other hand, by (4.9) and (4.17) again, we have

$$\left|\int_t^T |\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q \bar{v}' r^{-\delta}\right| \le M\int_t^T r^{-\delta-1} \le Mt^{-\delta}.$$

Hence by (4.21),

$$r|v'| + |v| \le Mr^{-\delta/2}.$$

Therefore

$$\left|\int_t^T |\alpha\bar{v} + \alpha\lambda - r\bar{v}'|^q \bar{v}' r^{-\delta}\right| \le M\int_t^T r^{-3\delta/2-1} \le Mt^{-3\delta/2},$$

and so by (4.21),

$$r|v'| + |v| \le Mr^{-3\delta/4}.$$

Thus for any $m > 0$, using a simple iteration of m-step in (4.21) yields

$$r|v'| + |v| \leq Mr^{-(2^m-1)\delta/2^m},$$

and (4.19) follows by taking m large.

We are ready to prove the theorem. Let u_1 and u_2 be two different solutions of (II). We introduce the function

$$w(r) = r^\alpha u_1(r) - r^\alpha u_2(r) = \bar{v}_1 - \bar{v}_2$$

and show that w is identically zero.

PROOF OF THEOREM 4.3. Clearly w satisfies

$$w'' + \frac{(n-1-2\alpha)w'}{r} + \frac{2(n-2-\alpha)w}{r^2} + \frac{f_1(r) - f_2(r)}{r^2} = \frac{[\alpha\bar{v}_1 + \alpha\lambda - r\bar{v}_1']^q - [\alpha\bar{v}_2 + \alpha\lambda - r\bar{v}_2']^q}{r^{2-\delta}} \tag{4.22}$$

where

$$|f_1(r) - f_2(r)| \leq \lambda^p \sum_{k=2}^{\infty} \frac{(p-k+1)!}{k!} \left| \left(\frac{\bar{v}_1(r)}{\lambda}\right)^k - \left(\frac{\bar{v}_2(r)}{\lambda}\right)^k \right|$$
$$\leq M|w||\bar{v}_1 + \bar{v}_2| \leq Mr^{-\varepsilon}|w|$$

for any $\varepsilon \in (0, \delta)$, and

$$|[\alpha\bar{v}_1 + \alpha\lambda - r\bar{v}_1']^q - [\alpha\bar{v}_2 + \alpha\lambda - r\bar{v}_2']^q| \leq M[|w| + r|w'|].$$

For $T > 0$ and $t \in (0, T)$, multiply (4.22) by $r^2 w'$ and integrate from t to T to obtain

$$\left[\frac{r^2 w'^2}{2} + (n-2-\alpha)w^2\right]_t^T + (n-2-2\alpha)\int_t^T rw'^2 + \int_t^T [f_1(r) - f_2(r)]w' - \int_t^T r^{-\delta}[[\alpha\bar{v}_1 + \alpha\lambda - r\bar{v}_1']^q - [\alpha\bar{v}_2 + \alpha\lambda - r\bar{v}_2']^q]w' = 0. \tag{4.23}$$

We have the following estimates:

$$\left|\int_t^T [f_1(r) - f_2(r)]w'\right| \leq M\int_t^T |ww'|r^{-\varepsilon} \leq M\int_t^T r^{-\varepsilon-1}[w^2 + w'^2 r^2],$$

and

$$\left|\int_t^T r^{-\delta}[[\alpha\bar{v}_1 + \alpha\lambda - r\bar{v}_1']^q - [\alpha\bar{v}_2 + \alpha\lambda - r\bar{v}_2']^q]w'\right| \leq M\int_t^T r^{-\delta-1}[w^2 + w'^2 r^2].$$

For large t, it follows, by letting $T \to \infty$ in (4.23) and using Lemma 4.8, that

$$t^2 w'^2 + 2(n-2-\alpha)w^2 \leq M\int_t^\infty r^{-\varepsilon-1}[w^2 + w'^2 r^2],$$

since $n-2-2\alpha \le 0$. Hence $w(r) \equiv 0$ for all sufficiently large r by the Gronwall inequality. Thus w is identically zero for all $r > 0$ and the proof is complete.

5. Further examples

As mentioned in the introduction, the arguments we used can be applied to more general functions f. We shall give two examples to demonstrate the generality of the methods and leave the proofs to the reader.

First consider the case when

$$f(r,u,|\nabla u|) = u^p + u^q, \qquad l_1 < p < q,$$

and so (1.0) takes the form

$$\text{(III)} \qquad \Delta u + u^p + u^q = 0, \qquad x \in \mathbb{R}^n \setminus \{0\},\ l_1 < p < q.$$

Put

$$\alpha_1 = \frac{2}{p-1}, \quad \alpha_2 = \frac{2}{q-1}; \qquad \lambda_i^{p-1} = \alpha_i(n-2-\alpha_i), \quad i = 1,2$$

(note that $\alpha_1 > \alpha_2$). Then we have the following results.

PROPOSITION 5.1 (Classification of positive radial solutions of (III)).

(o) *If $p \le l_1$, then* (III) *admits no positive solution.*

(i) *If $q < l$, then* (III) *admits exactly one solution with slow decay at infinity. This solution has the following exact limits:*

$$\lim_{r\to\infty} r^{\alpha_1} u = \lambda_1, \qquad \lim_{r\to 0} r^{\alpha_2} u = \lambda_2.$$

Moreover, (III) *admits a family of positive solutions with fast decay at infinity satisfying*

$$\lim_{r\to\infty} r^{n-2} u = c, \qquad \lim_{r\to 0} r^{\alpha_2} u = \lambda_2$$

for all $c > 0$. Finally, (III) *does not have any other positive solutions.*

(ii) *If $q = l$, then* (III) *admits exactly one solution with slow decay at infinity. This solution has the exact limit*

$$\lim_{r\to\infty} r^{\alpha_1} u = \lambda_1.$$

Moreover, (III) *admits a family of positive solutions with fast decay at infinity satisfying*

$$\lim_{r\to\infty} r^{n-2} u = c$$

for all $c > 0$. Moreover, (III) *does not have any other positive solutions.*

All solutions are singular at the origin, and exactly one solution has the exact limit

$$\lim_{r\to 0} r^{(n-2)/2} u = \left(\frac{n-2}{2}\right)^{(n-2)/2}.$$

Other solutions must have the form

$$u(r) = \left(\frac{n-2}{2r}\right)^{(n-2)/2} h(r),$$

where h oscillates endlessly near the origin between two sequences $\{\mu_{1,i}\}$ and $\{\mu_{2,i}\}$ satisfying $0 < \mu_{1,i} < \mu_{2,i}$ and

$$\lim_{i\to\infty} \mu_{1,i} = \mu_1, \qquad \lim_{i\to\infty} \mu_{2,i} = \mu_2,$$

where μ_1 and μ_2 are fixed values satisfying

$$0 < \mu_1 < 1 < \mu_2, \qquad b(\mu_1) = b(\mu_2),$$

with

$$b(h) = \frac{h^{l+1}}{l+1} - \frac{h^2}{2}.$$

In particular, μ_1 can take any value in $(0, 1)$, thus determining the corresponding value μ_2.

(iii) *If $p > l$, then* (III) *admits exactly one solution singular at the origin. This solution has the exact limits*

$$\lim_{r\to 0} r^{\alpha_2} u = \lambda_2, \qquad \lim_{r\to\infty} r^{\alpha_1} u = \lambda_1.$$

Moreover, (III) *admits a family of positive solutions regular at the origin satisfying*

$$\lim_{r\to 0} u = c, \qquad \lim_{r\to\infty} r^{\alpha_1} u = \lambda_1$$

for all $c > 0$. Finally, (III) *does not have any other positive solutions.*

(iv) *If $p = l$, then* (III) *admits exactly one solution singular at the origin. This solution has the exact limit*

$$\lim_{r\to 0} r^{\alpha_2} u = \lambda_2.$$

Moreover, (III) *admits a family of positive solutions regular at the origin satisfying*

$$\lim_{r\to 0} u = c$$

for all $c > 0$. Finally, (III) *does not have any other positive solutions.*

In addition, all solutions have slow decay at infinity, and exactly one solution has the exact limit

$$\lim_{r\to\infty} r^{(n-2)/2} u = \left(\frac{n-2}{2}\right)^{(n-2)/2}$$

Other solutions necessarily have the form

$$u(r) = \left(\frac{n-2}{2r}\right)^{(n-2)/2} h(r),$$

where h oscillates endlessly near infinity between two sequences $\{\mu_{1,i}\}$ and $\{\mu_{2,i}\}$ satisfying $0 < \mu_{1,i} < \mu_{2,i}$ and

$$\lim_{i\to\infty} \mu_{1,i} = \mu_1, \qquad \lim_{i\to\infty} \mu_{2,i} = \mu_2,$$

where μ_1 and μ_2 are fixed values satisfying

$$0 < \mu_1 < 1 < \mu_2, \qquad b(\mu_1) = b(\mu_2),$$

with

$$b(h) = \frac{h^{l+1}}{l+1} - \frac{h^2}{2}.$$

In particular, μ_1 can take any value in $(0,1)$, thus determining the corresponding value μ_2.

(v) *If $p < l < q$, then* (III) *admits at most one solution with slow decay at infinity. If so, then this solution has the exact limit*

$$\lim_{r\to\infty} r^{\alpha_1} u = \lambda_1. \tag{5.1}$$

(III) *also admits at most one solution singular at the origin. If it does, then this solution has the exact limit*

$$\lim_{r\to 0} r^{\alpha_2} u = \lambda_2. \tag{5.2}$$

REMARKS. 1. In case (v) the situation is complicated. The only known fact is that (III) admits a unique positive slow decay (regular at the origin) solution for some (p,q). When $q = 2p-1$ and $p > n/(n-2) = l_1$, in particular, there is such a solution, having the explicit form

$$u(r) = \left(\frac{2p}{p-1} \cdot \frac{\beta}{\beta^2 + pr^2}\right)^{1/(p-1)}$$

with

$$\beta = \frac{(n-2)p - n}{p-1}.$$

On the other hand, it is unknown if (III) has *any* positive solution at all for other (p,q) values. Finally, it is not even known whether there are *any* fast decay solutions.

The situation for (p,q) in the parameter domain $1 < p < q$ is illustrated in Figure. Recall that in the region (i) there is a single positive slow decay solution and a family of singular positive fast decay solutions, while in (iii) there is a unique positive singular solution and a family of regular positive slow decay solutions. On the other hand, for (p,q) on the line $q = 2p-1$ in (v) there is a unique positive (regular) slow decay solution. This being the case, it is tempting to conjecture that for each (p,q) in the region (v) there exist a unique positive (regular) slow decay solution and a unique positive (singular) fast decay solution.

2. For oscillatory solutions, we refer the reader to [10] in which more general cases were discussed.

3. It is always possible to obtain local existence of positive solutions at infinity or at the origin by using asymptotic integration (see also [12]). Global existence on the entire interval $(0, \infty)$ is obvious in cases (i)-(iv) (local existence plus an application of the Pokhozhaev identity). However, this seems a much harder problem in case (v) (cf. Remark 1).

4. It is interesting to notice the different outcomes for equations (I), (II) and (III). The asymptotic behavior at a singularity of solutions of (III) can be viewed as the effect of a perturbation to the Lane-Emden equation (I). If, say, the origin is a singularity, then the term u^q is dominant and the term u^p is thus a small perturbation. Therefore, the outcome ought to be essentially the same as that for (I) with p replaced by q. At infinity, the situation is just reversed. The term u^p is dominant and the term u^q is a small perturbation. Similarly the presence of the gradient term $-|\nabla u|^q$ in the Chipot-Weissler equation is the source of major differences. One obvious fact is that (I) does not have singular solutions with fast growth at the origin but (II) has.

The second case we shall discuss is when

$$f(r, u, |\nabla u|) = r^\sigma u^p, \qquad p > 1,\ \sigma > -2,$$

so that (1.0) takes the form

$$\text{(IV)} \qquad \Delta u + r^\sigma u^p = 0, \qquad x \in \mathbb{R}^n \setminus \{0\},\ p > 1.$$

We introduce the following modified definitions for the numbers l_1, l, α and λ (see (1.1) and (1.2)),

$$\text{(5.3)}\ \ l_1 = \frac{n+\sigma}{n-2}, \qquad l = \frac{n+2+2\sigma}{n-2}, \qquad \alpha = \frac{2+\sigma}{p-1}, \qquad \lambda^{p-1} = \alpha(n-2-\alpha),$$

and we set

$$U(r) = \lambda r^{-\alpha}.$$

Then we have the following proposition.

PROPOSITION 5.2 (Classification of positive radial solutions of (IV)). *The solutions of* (IV) *have exactly the classification given earlier in Proposition* 3.1 *for solutions of* (I), *with the values* l_1, l, α *and* λ *now given by* (5.1). *In place of* (3.10), *moreover, we have*

$$u(r) = \left(\frac{\delta\sqrt{(n+\sigma)(n-2)}}{\delta^2 + r^{2+\sigma}} \right)^{(n-2)/(2+\sigma)}, \qquad \delta = const. > 0.$$

This can be proved by the same procedure as before, but in fact because of the special form of (IV) an easier proof is available. Making the change of variable

$$y = \frac{2}{2+\sigma} r^{(2+\sigma)/2},$$

equation (IV) becomes in the radial case

$$\ddot{u} + \frac{n'-1}{y}\dot{u} + u^p = 0, \qquad \dot{} = \frac{d}{dy},$$

where $n' = 2(n+\sigma)/(2+\sigma)$. We can then apply Proposition 3.1 directly, once we notice that

$$\frac{n'}{n'-2} = \frac{n+\sigma}{n-2}, \qquad \frac{n'+2}{n'-2} = \frac{n+2+2\sigma}{n-2},$$

and that

$$y^{2/(p-1)} = \left(\frac{2}{2+\sigma}\right)^{2/(p-1)} r^{(2+\sigma)/(p-1)}.$$

REMARK. The methods above also allow us to treat, under appropriate conditions, the function

$$f(r,u,|\nabla u|) = \sum_{i=1}^{k} r^{\sigma_i} u^{p_i} - \sum_{j=1}^{l} r^{\gamma_j} |\nabla u|^{q_j},$$

where k and l are positive integers, $\sigma_i, \gamma_j \in \mathbb{R}$, and p_i, q_j are positive numbers. One also could treat general forms of f under suitable assumptions, for instance,

$$f = u^p + g(u),$$

where $g(u)$ is a perturbation satisfying suitable conditions (not necessarily a pure power).

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