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# GLOBAL BIFURCATION OF PERIODIC SOLUTIONS

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*Dedicated to the memory of Karol Borsuk*

## 0. Introduction

In this paper we study periodic solutions of a family of autonomous differential equations

$$u'(t) = \phi(u(t), a)$$

where $a \in \mathbb{R}$, $U \subset \mathbb{R}^m$ is an open subset and, the function $\phi : U \times \mathbb{R} \to \mathbb{R}^m$, is assumed to be of class $C^1$ and to satisfy some natural conditions.

This work was inspired by a paper by Mallet-Paret and Yorke [15]. Our main results are contained in Theorems 1.3, 1.5 and 1.8. The major difference between Mallet-Paret and Yorke's result and our Theorem 1.5 is that we discuss the general case, and not only some generic one. This is possible owing to purely topological methods of proof (see also Fiedler [7]for a complete review of previous works).

The principal tools used in the present paper are the $S^1$-equivariant degree (defined in [5]) and the complementary function method (introduced by Ize [10], [11]).

Let us briefly illustrate the geometric essence of the method, using the classical Brouwer degree. Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open bounded set and $f : \overline{\Omega} \to \mathbb{R}^n$ a continuous map. Further, assume $U_+$, $U_-$ are disjoint open subsets of $\partial\Omega$ such that $f^{-1}(0) \cap \partial\Omega \subset U_+ \cup U_-$. We call $\theta : \overline{\Omega} \to \mathbb{R}$ a complementing function if $\theta(x) < 0$ for $x \in U_-$ and $\theta(x) > 0$ for $x \in U_+$. Setting

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$F(x) = (f(x), \theta(x))$, we obtain a continuous map $F : \overline{\Omega} \to \mathbb{R}^{n+1}$ with $F \neq 0$ on $\partial\Omega$. Therefore the classical Brouwer degree $\deg(F, \Omega)$ is defined. On the other hand, the following two quantities may be associated with $f$: the number of zeros of $f$ "entering $\Omega$ through $U_-$" and the number of zeros of $f$ "leaving $\Omega$ through $U_+$." It turns out that the two quantities are equal to $\deg(F, \Omega)$, which fact plays a key role in the proof of the classical theorems of bifurcation theory.

In our paper we replace the classical Leray-Schauder degree by the $S^1$-equivariant degree and a version of the complementing function method. In this way, for bifurcations of periodic solutions we obtain the analogues of two classical theorems; the (local) Krasnoselskii theorem—Theorem 1.3, and the (global) Rabinowitz theorem—Theorem 1.5. In the first case, it yields a new proof of the Hopf theorem in the version of Chow, Mallet-Paret and Yorke [2]. In the second case, as already noted, we derive the main theorem of [2], but in general case. It states that the sum of local invariants (of bifurcation of periodic solutions) is equal to zero, where the summation is over all centers lying on the bounded component branch of periodic solutions.

Further, we apply Theorem 1.3 to the equation

$$u'(t) = \psi(u(t), a), \quad \psi : U \to \mathbb{R}^m,$$

admitting a first integral $G : \mathbb{R}^m \to \mathbb{R}$, by perturbing the equation with the gradient of $G$ (Th. 1.8). The last theorem has a particularly simple formulation for a Hamiltonian system

$$u'(t) = J \ \operatorname{grad} H(u(t))$$

whose energy function $H$ is a Morse function (Th. 1.9).

The organization of the paper is as follows. First, by replacing the classical Brouwer degree by the finite-dimensional $S^1$-degree we obtain finite-dimensional versions of the main theorems (Thms. 2.2 and 2.3). These versions have natural generalizations to the Hilbert space setting (Theorems 3.4 and 3.5). In Section 5 we prove the main results of the paper by applying Theorems 3.4 and 3.5 to $S^1$-equivariant maps determined by a given family of differential equations. The paper also contains a detailed proof of equality of two commonly used invariants: the crossing number and the homotopy obstruction $\gamma_f$ (Sections 4 and 6).

## 1. The formulation of results

Assume that $U$ is an open subset of $\mathbb{R}^m \times \mathbb{R}$ and $\varphi : U \to \mathbb{R}^m$ is a $C^1$-map. We investigate periodic solutions of the equation

$$(*) \qquad u'(t) = \varphi(u(t), a).$$

In this paper we use the following terminology:

- $(x_0, a_0)$ is a *stationary point of* $\varphi$ *if* $\varphi(x_0, a_0) = 0$,
- $(x_0, a_0)$ is a *nonsingular stationary point of* $\varphi$ *if* $\varphi(x_0, a_0) = 0$ and $D_x\varphi(x_0, a_0) : \mathbb{R}^m \to \mathbb{R}^m$ (the derivative with respect to $x$) is a linear isomorphism,
- $(x_0, a_0)$ is a *center for* $\varphi$ if it is a nonsingular stationary point of $\varphi$ and $\sigma(D_x\varphi(x_0, a_0)) \cap i\mathbb{R} \neq \emptyset$, where $\sigma(L)$ denotes the spectrum of a linear map $L$.
- $(x_0, a_0)$ is an *isolated center for* $\varphi$ if there exists a neighbourhood of $(x_0, a_0)$ in $U$ in which $(x_0, a_0)$ is the only center.

We say the $(x, a) \in U$ is a *nonsingular* (or *nontrivial*) *periodic point of* $(*)$ if $\varphi(x, a) \neq 0$ and there exists a solution $u(t)$ of $(*)$ such that $u(0) = u(p) = x$ for some $p > 0$. In this case we call $p$ a *period of* $(x, a)$. A period is called the *minimal period of* $(x, a)$ if $u(t) \neq u(0)$ for all $t \in (0, p)$. Clearly the minimal period always exists. Moreover, if $p$ is the minimal period then any other period is of the form $kp$ for some $k \in \mathbb{N}$. We say that a stationary point $(x_0, a_0)$ is a *bifurcation point for* $(*)$ if it belongs to the closure of the set of nontrivial periodic points of $(*)$.

Together with $U$, which is the *phase space* for $(*)$, we consider $U \times (0, \infty)$ which we call the *Fuller space* of $(*)$ (cf. [8]). We say that $(x_0, a_0, p_0) \in U \times (0, \infty)$ is a *Fuller center for* $(*)$ if $(x_0, a_0)$ is a center for $(*)$ and there exists $\beta > 0$ and $k \in \mathbb{N}$ such that $i\beta \in \sigma(D_x\varphi(x_0, a_0))$ and $p_0 = 2k\pi\beta^{-1}$. We say that $(x, a, p) \in U \times (0, \infty)$ is a *nonstationary* (or *nontrivial*) periodic point of $(*)$ if $(x, a)$ is a nonstationary periodic point for $(*)$ and $p$ is a period of $(x, a)$. Finally, we call a stationary point $(x_0, a_0, p_0) \in U \times (0, \infty)$ a *bifurcation point for* $(*)$ if it belongs to the closure of all nonstationary periodic points of $(*)$ in the Fuller space $U \times (0, \infty)$.

As an easy consequence of the Implicit Function Theorem, one gets the following statement which gives a necessary condition for $(x_0, a_0, p_0)$ to be a bifurcation point.

PROPOSITION 1.1. *Assume that* $(x_0, a_0)$ *is a stationary point for* $(*)$. *If* $(x_0, a_0, p_0)$ *is not a Fuller center then* $(x_0, a_0, p_0)$ *is not a bifurcation point.*

For the proof of 1.1 see, e.g. [1], [2], [16]. Note that Proposition 1.1 gives a necessary condition for bifurcation in the Fuller space, but not in the phase space. To give an example, consider the following perturbed Hamiltonian system:

$$u'(t) = -H_y(u(t), v(t)) + a\, H_x(u(t), v(t)),$$
$$v'(t) = +H_x(u(t), v(t)) + a\, H_y(u(t), v(t)),$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is the energy function $H(x,y) = (x^2-1)(y^2-1)$ and $H_x, H_y$ are the partial derivatives of $H$. By a general argument ( see the proof of Theorem 1.8 ) if $((x,y),a)$ is a nontrivial periodic point of this system then $a = 0$. For $a = 0$ it is not difficult to check that

$$((0,0),0),\ ((1,1),0),\ ((1,-1),0),\ ((-1,1),0),\ ((-1,-1),0)$$

are all stationary points of this system with only $((0,0),0)$ being a center. Moreover, all those points are bifurcation points in the phase space $\mathbb{R}^2 \times \mathbb{R}$. On the other hand, only $((0,0),k\pi)$, $k \in \mathbb{N}$, is the bifurcation point in the Fuller space $(\mathbb{R}^2 \times \mathbb{R}) \times (0,\infty)$.

Assume now that $(x_0, a_0)$ is an isolated center for $(*)$. Then there exist $\delta > 0$ and a continuous map $\eta : [a_0 - \delta, a_0 + \delta] \to \mathbb{R}^m$ such that

(i) $\eta(a_0) = x_0$,

(ii) $(\eta(a), a) \in U$ and $\varphi(\eta(a), a) = 0$ for all $a \in [a_0 - \delta, a_0 + \delta]$,

(iii) $\sigma(D_x\varphi(\eta(a), a)) \cap i\mathbb{R} = \emptyset$ for all $a \neq 0$.

Define $A : [-1,1] \to \mathrm{GL}(m, \mathbb{R})$ by $A(\alpha) = D_x\varphi(\eta(a_0 + \alpha\delta), a_0 + \alpha\delta)$. We call $A$ a *characteristic map for the center* $(x_0, a_0)$. Suppose further that $\beta > 0$ and $i\beta \in \sigma(A(0))$ (where $A(0) = D_x\varphi(x_0, a_0)$). We let $c(x_0, a_0, i\beta)$ denote the algebraic number of eigenvalues of $A(\alpha)$ crossing $i\mathbb{R}$ through $i\beta$ at $\alpha = 0$. We give a rigorous definition of the crossing number in Section 4.

DEFINITION 1.2. Assume that $(x_0, a_0)$ is an isolated center for $(*)$. For $p \in (0,\infty)$ and $k \in \mathbb{N}$ we let

$$r_k(x_0, a_0, p) = \begin{cases} c(x_0, a_0, 2k\pi i p^{-1}), & \text{if } 2k\pi i p^{-1} \in \sigma(D_x\varphi(x_0, a_0)), \\ 0, & \text{otherwise.} \end{cases}$$

We have (cf. [2], [11], [16]).

THEOREM 1.3. *Assume that $(x_0, a_0)$ is an isolated center for $(*)$ and $p_0 > 0$. If there exists $k \in \mathbb{N}$ such that $r_k(x_0, a_0, p_0) \neq 0$ then $(x_0, a_0, p_0)$ is a bifurcation point in the Fuller space $U \times (0,\infty)$. Moreover, there exists a sequence $\{(x_n, a_n, p_n)\}$ converging to $(x_0, a_0, p_0)$ such that $p_n k^{-1}$ is an integer multiple of the minimal period of $(x_n, a_n)$.*

We will present a proof of the above version of the Hopf Bifurcation Theorem which is analogous to the proof of the Krasnoselski's theorem ([14], [18]) with the Leray-Schauder degree replaced by the $S^1$-degree. Before we formulate our main result, which is an $S^1$-degree analogue of the Rabinowitz Global Bifurcation Theorem ([10] [18] [20]), we need some notation. Suppose that $(x_0, a_0)$ is an isolated center for $(*)$ and $(x_0, a_0, p_0)$ is a bifurcation point in the Fuller space. Let $S(\varphi)$ denote the closure of the set of all nontrivial periodic points in the

Fuller space. Thus $(x_0, a_0, p_0)$ belongs to $S(\varphi)$. Let $C(x_0, a_0, p_0)$ denote the connected component of $(x_0, a_0, p_0)$ in $S(\varphi)$.

DEFINITION 1.4. For a given center $(x_0, a_0)$ we let

$$\varepsilon(x_0, a_0) = \operatorname{sgn}\det D_x\varphi(x_0, a_0).$$

We define *the oriented $k^{th}$ crossing number at* $(x_0, a_0)$ by

$$\omega_k(x_0, a_0, p_0) = \varepsilon(x_0, a_0) r_k(x_0, a_0, p_0).$$

THEOREM 1.5. *Assume that all stationary points of* $(*)$ *are nonsingular and all centers of* $(*)$ *are isolated. If* $(x_0, a_0)$ *is a center for* $(*)$, *if* $(x_0, a_0, p_0)$ *is a bifurcation point and if* $C(x_0, a_0, p_0)$ *is bounded and* $C \cap \partial U = \emptyset$ *then:*

a) *the number of bifurcation points belonging to* $C(x_0, a_0, p_0)$ *is finite,*
b) *if* $\{(x_0, a_0, p_0), (x_1, a_1, p_1), \ldots, (x_q, a_q, p_q)\}$ *denotes the set of all bifurcation points of* $(*)$ *in* $C(x_0, a_0, p_0)$ *then for every* $k \in \mathbb{N}$

$$\omega_k(x_0, a_0, p_0) + \omega_k(x_1, a_1, p_1) + \cdots + \omega_k(x_q, a_q, p_q) = 0.$$

Assume now that $U_0$ is an open subset of $\mathbb{R}^m$ and $\psi : U_0 \to \mathbb{R}^m$ is a $C^1$-map. Consider the equation

$$u'(t) = \psi(u(t)). \tag{$**$}$$

Assume further that $G : U_0 \to \mathbb{R}$ is a first integral for $(**)$ of class $C^2$. We use the following terminology: $x_0 \in U_0$ is a *stationary point* if $\psi(x_0) = 0$, it is a *nonsingular stationary point* if $\psi(x_0) = 0$ and $D\psi(x_0)$ is an isomorphism. We say that $x_0$ is a *center* if it is nonsingular stationary point and $\sigma(D\psi(x_0)) \cap i\mathbb{R} \neq \emptyset$, it is an *isolated center* if there exists a neighbourhood of $x_0$ in $U_0$ in which $x_0$ is the only center. As before, together with the phase space $U_0$ we consider the Fuller space $U_0 \times (0, \infty)$. We repeat verbatim definitions of Fuller center, nontrivial periodic point and bifurcation point for $(**)$. Embedding $U_0 = U_0 \times \{0\}$ into $U = U_0 \times \mathbb{R}$ we can consider the system $(**)$ as a system defined on $U$ with $\phi(x, a) = \psi(x)$. We leave to the reader a proof of the following lemma.

LEMMA 1.6. *If* $x_0$ *is a nonsingular stationary point of* $(**)$ *then* $x_0$ *is a critical point of* $G$.

Suppose that $x_0$ is a center of $(**)$, $L = D_x\psi(x_0)$. Let $L_k \subset \mathbb{R}^m$ be the generalized eigenspace of $L$ corresponding to $ik\beta$ ($L_k = \{0\}$ if $ik\beta \notin \sigma(L)$) and $S : \mathbb{R}^m \to \mathbb{R}^m$ a selfadjoint linear operator associated with the Hessian $G_2 = D^2G(x_0) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. Proposition 1 of [17] states that $S(L_k) \subset L_k$, *the signature* $\operatorname{sign} G_2|_{L_k}$ is an even integer and if $S|_{L_k}$ is nondegenerate then

$$2^{-1} \operatorname{sign} G_2|_{L_k} = r_k(x_0, 0, 2\pi\beta^{-1}),$$

where $r_k$ is the $k^{th}$ crossing number of the system

$$u'(t) = \psi(u(t)) + a \ \text{grad} \ G(u(t)) \tag{1.7}$$

The last leads to a local bifurcation theorem ([17] Th. 1) which global extension is the following.

THEOREM 1.8. *Assume that all stationary points of* $(**)$ *are nonsingular and all centers of it are isolated. Assume next that a* $C^2$*-first integral* $G$ *of* $(**)$ *is a Morse function with all critical points being stationary points of* $\psi$. *Let* $S$ *be the closure of all nontrivial periodic points of* $(**)$ *in the Fuller space* $U_0 \times (0, \infty)$.

*Then each bounded component* $C$ *of* $S$ *such that* $C \cap \partial U_0 \times (0, \infty) = \emptyset$ *contains only a finite number of distinct centers* $(x_0, p_0), \ldots, (x_q, p_q)$ *and for every* $k \in \mathbb{N}$

$$\sum_{i=0}^{q} \varepsilon(x_j) \ \text{sign} \ G_2(x_j)|_{L_{k,j}} = 0,$$

*where* $\varepsilon(x_j) = \operatorname{sgn} \det D_x\psi(x_j)$, $G_2 = D^2G(x_j)$, *and* $L_{k,j}$ *is the generalized eigenspace corresponding to* $2\pi k p_j^{-1} i$ *at* $x_j$.

Finally we have to emphasize that Theorem 1.8 extends easily to the case of an autonomous system on an open subset of $C^2$-manifold. This allows us to formulate a version of Theorem 1.8 for the Hamiltonian system.

Suppose that $M^{2m}$ is a symplectic manifold with the structure operator $J : TM^* \to TM$. Assume that $H : M \to \mathbb{R}$ is a Morse function. We consider the Hamiltonian system

$$u'(t) = J \ dH(t)$$

PROPOSITION 1.9. *Suppose we are given a Hamiltonian system as above. Then each bounded component* $C$ *of the closure of all nontrivial periodic points of this system contains only a finite number of distinct centers* $(x_0, p_0), \ldots, (x_q, p_q)$. *Moreover, for every* $k \in \mathbb{N}$, *we have*

$$\sum_{0}^{q} \operatorname{sgn} \det H_2(x_j) \cdot \ \text{sign} \ H_2(x_j)|_{L_{k,j}} = 0.$$

*where* $L_{k,j}$ *is the generalized eigenspace of* $JH_2(x_j)$ *corresponding to* $ik\beta_j$) *and* $p_j = 2\pi\beta_j^{-1}$.

## 2. Bifurcations in finite-dimensional representations of $S^1$

If $X, Y$ are topological spaces we let $[X, Y]$ denote the set of all homotopy classes of continuous maps from $X$ into $Y$. If $f : X \to Y$ is a continuous map then $[f]$ denotes its homotopy class.

Suppose now that $\alpha : S^1 \to \mathrm{GL}(m, \mathbb{C})$ is continuous and let $\alpha^* : S^1 \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ be defined by $\alpha^*(z) = \det \alpha(z)$. Since $\mathbb{C}^*$ is homotopy equivalent to $S^1$ we have natural identification

$$[S^1, \mathbb{C}^*] = [S^1, S^1] = \mathbb{Z}.$$

The map $[\alpha] \mapsto [\alpha^*]$ defines a bijection

$$\nabla : [S^1, \mathrm{GL}(m, \mathbb{C})] \to \mathbb{Z}.$$

Note also that $[S^1, \mathrm{GL}(m, \mathbb{C})]$ may be identified in a natural way with the fundamental group $\pi_1(\mathrm{GL}(m, \mathbb{C}))$. In this sense $\nabla$ is a group isomorphism.

Throughout the paper we let

$$G = S^1 = \{z \in \mathbb{C}; |z| = 1\}.$$

For $\nu \in \mathbb{N}$ we identify the group $\mathbb{Z}_\nu = \mathbb{Z}/\nu\mathbb{Z}$ with the subgroup of $G$ consisting of $\nu^\theta$ roots of unity.

First we recall some basic properties of linear representations of $G$. Suppose that $V$ is a finite-dimensional, nontrivial, isotypical real representation of $G$; i.e. there exists $\nu \in \mathbb{N}$ such that $G_x = \mathbb{Z}_\nu$ for all $x \in V\backslash\{0\}$ (here and in what follows $G_x$ denotes the isotropy group of $x$). Set $\xi = \exp(2^{-1}\nu^{-1}\pi i)$. The formula $i * x = \xi x$ defines a multiplication of elements of $V$ by complex numbers, therefore $V$ becomes a linear space over $\mathbb{C}$. Moreover, an $\mathbb{R}$-linear map of $V$ into $V$ is $G$-equivariant if and only if it is $\mathbb{C}$-linear with respect to this structure. Therefore the group of all $G$-equivariant $\mathbb{R}$-linear automorphisms of $V$, denoted by $\mathrm{GL}_G(V)$, coincides with $\mathrm{GL}_{\mathbb{C}}(V)$, the group of all $\mathbb{C}$-linear automorphisms of $V$. The following well-known fact plays a crucial role in our considerations.

PROPOSITION 2.1. *Assume that $V$ is a finite-dimensional nontrivial isotypical representation of $G = S^1$. Then there exists a canonical group isomorphism*

$$\nabla : [S^1, \mathrm{GL}_G(V)] \to \mathbb{Z}$$

*such that*

(*a*) *if* $\dim_{\mathbb{R}} V = 2$ *and* $\psi(g)(v) = g * v,\ v \in V,\ g \in G,$ *then* $\nabla([\psi]) = 1$;

(*b*) *if* $V = V_1 \oplus V_2$ *and* $\psi_j : G \to \mathrm{GL}_G(V_j),\ j = 1, 2,$ *is a continuous map, then* $\nabla([\psi_1 \oplus \psi_2]) = \nabla([\psi_1])\nabla([\psi_2])$. □

Throughout the rest of this section we fix $V$, a finite-dimensional real orthogonal representation of $G$. Recall that there is an orthogonal direct sum decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where $G$ acts trivially on $V_0$ and $G_x = \mathbb{Z}_j$ for $x \in V_j\backslash\{0\},\ i = 1, \ldots, k$, (i.e. the $V_j$ are isotypical factors of $V$). Note that with this convention it may happen that $V_j = \{0\}$ for some $j$. In what follows we denote points of $V \oplus \mathbb{R}^2$ by $(x, \lambda)$ and define an action of $G$ on $V \oplus \mathbb{R}^2$ by $g(x, \lambda) = (gx, \lambda)$.

Let $f : V \oplus \mathbb{R}^2 \to V$ be a $G$-equivariant $C^1$-map. We investigate the equation

$$(*) \qquad f(x, \lambda) = 0$$

In what follows we assume that there exists a closed 2-dimensional submanifold $N \subset V_0 \oplus \mathbb{R}^2$ satisfying the following conditions:

(A) $N \subset f^{-1}(0)$;

(B) if $(x_0, \lambda_0) \in N$ then there exists an open neighbourhood $U_\lambda$ of $\lambda_0$ in $\mathbb{R}^2$, an open neighbourhood $U_0$ of $x_0$ in $V_0$ and a $C^1$-map $\mu : U_\lambda \to V_0$ such that

$$N \cap (U_0 \times U_\lambda) = \{((\mu(\lambda), \lambda) : \lambda \in U_\lambda\}.$$

In the sequel we refer to $N$ (resp., $f^{-1}(0) \backslash N$) as the family of *trivial* (resp., *nontrivial*) solutions of $(*)$. A point $(x, \lambda) \in N$ is called a *bifurcation point* of $(*)$ if it belongs to the closure of the set of all nontrivial solutions. For $(x_0, \lambda_0) \in V \oplus \mathbb{R}^2$ we denote by $D_x f(x_0, \lambda_0) : V \to V$ the derivative of $f$ with respect to $x$. We say that $(x_0, \lambda_0)$ is $V$-regular if $D_x f(x_0, \lambda_0)$ is an isomorphism; otherwise $(x_0, \lambda_0)$ is *$V$-singular*. We say that $(x_0, \lambda_0) \in N$ is an *isolated $V$-singular point* if it is $V$-singular and in some neighbourhood of $(x_0, \lambda_0)$ in $N$ there are no other $V$-singular points.

Assume now that $(x_0, \lambda_0)$ is an isolated $V$-singular point in $N$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ and taking a sufficiently small $\rho > 0$ we define $\alpha : S^1 \to N$ by setting

$$\alpha(z) = (\mu(\lambda_0 + \rho z), \lambda_0 + \rho z),$$

where $\mu$ denotes the map of condition (B). The formula $\psi(z) = D_x f(\alpha(z))$ defines a continuous mapping $\psi : S^1 \to \mathrm{GL}_G(V)$. Therefore

$$\psi = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_k,$$

where $\psi_j : S^1 \to \mathrm{GL}_G(V_j)$, $j = 0, 1, \ldots, k$.

Let $\varepsilon = \operatorname{sgn} \det \psi_0(z)$ (this definition makes sense, since $\operatorname{sgn} \det \psi_0(z)$ does not depend on $z$).

Finally, we let for $j = 1, \ldots, k$,

$$\gamma_j(x_0, \lambda_0) = \varepsilon \nabla([\psi_j]).$$

THEOREM 2.2. *Suppose that $f : V \oplus \mathbb{R}^2 \to V$ is an equivariant $C^1$-map with $N$ satisfying (A) and (B). If $(x_0, \lambda_0) \in N$ is an isolated $V$-singular point and there exists $j$ such that $\gamma_j(x_0, \lambda_0) \neq 0$ then $(x_0, \lambda_0)$ is a bifurcation point. Moreover, there exists a sequence $(x_n, \lambda_n) \to (x_0, \lambda_0)$ of nontrivial solutions of $(*)$ such that the isotropy group of $x_n$ contains $\mathbb{Z}_j$.*

THEOREM 2.3. *Suppose that $f : V \oplus \mathbb{R}^2 \to V$ is an equivariant $C^1$-map with $N$ satisfying (A) and (B). Suppose further that every $V$-singular point in $N$ is isolated in $N$. Let $S(f)$ denote the closure of the set of all nontrivial solutions of $(*)$. Then for each bounded connected component $C$ of $S(f)$ the set $C \cap N$ is finite. Moreover, if $C \cap N = \{(x_1, \lambda_1), \ldots, (x_q, \lambda_q)\}$ then for $j = 1, \ldots, k$,*

$$\gamma_j(x_1, \lambda_1) + \cdots + \gamma_j(x_q, \lambda_q) = 0.$$

Before starting the proof of Theorems 2.2 and 2.3 we will need some technical results. Our proof uses the complementing function method (developed by Ize in [10]) and the $S^1$-equivariant degree introduced in [5]. Recall that if $U$ is an open, bounded, invariant subset of $V \oplus \mathbb{R}^2$ and $F : (\overline{U}, \partial U) \to (V \oplus \mathbb{R}, V \oplus \mathbb{R}\backslash\{0\})$ an equivariant map then there is defined the $G$-equivariant degree of $F$ with respect to $U$, $\mathrm{Deg}(F, U) = \{\deg_H(F, U)\}$, where $H$ runs through the family of all closed subgroups of $G$, $\deg_H(F, U) \in \mathbb{Z}_2$ for $H = G$ and $\deg_H(F, U) \in \mathbb{Z}$ for $H \neq G$. The basic property of Deg is that $\mathrm{Deg}(F, U) \neq 0$ implies that the equation $F(x, \lambda) = 0$ has a solution in $U$; more precisely, $\deg_H(F, U) \neq 0$ implies $F^{-1}(0) \cap U^H \neq \emptyset$. Further, the $S^1$-equivariant degree has the following standard properties:(see [5]for details)

(i) additivity with respect to $U$;

(ii) homotopy invariance with respect to equivariant homotopies;

(iii) contraction property.

Moreover, if $F$ is of class $C^1$ and if 0 is a regular value of $F$, we have some formulas which express Deg in terms of the derivative of $F$ ([5], Theorem 4.9). Since one particular formula plays a crucial role in our considerations, we will give more details. We start with some notation. Suppose that $W$ is a finite dimensional linear space and $A : W \oplus \mathbb{R} \to W$ is a linear map such that $A(W \oplus \mathbb{R}) = W$.

Suppose further that $v \in \ker A$, $v \neq 0$. Let $B : W \oplus \mathbb{R} \to W \oplus \mathbb{R}$ denote a linear isomorphism such that $B(w, t) = (A(w, t), \xi(w, t))$, where $\xi : W \oplus \mathbb{R} \to \mathbb{R}$ is linear and $\xi(v) = 1$. Set

$$\mathrm{sgn}(A, v) = \begin{cases} +1 & \text{if } B \text{ preserves the orientation;} \\ -1 & \text{if } B \text{ reverses the orientation.} \end{cases}$$

PROPOSITION 2.4. *Assume that $U$ is an open, bounded, invariant subset of $V \oplus \mathbb{R}^2$ and $F : (\overline{U}, \partial U) \to (V \oplus \mathbb{R}, V \oplus \mathbb{R}\backslash\{0\})$ an equivariant $C^1$-map such that 0 is a regular value of $F, M = F^{-1}(0)$ is connected and $M \subset V_0 \oplus \mathbb{R}^2$. Assume further that $\eta : S^1 \to M$ is a $C^1$-diffeomorphism and let $v = \eta'(1)$. Define $\alpha_j : S^1 \to \mathrm{GL}_G(V_j)$, $1 \leq j \leq k$ by $(\alpha_j(z))(w) = DF(\eta(z))(w)$, $w \in V_j$. Then for $H = \mathbb{Z}_j$,*

$$\deg_H(F, U) = -\,\mathrm{sgn}(DF^G(\eta(1), \eta'(1))\nabla([\alpha_j]).$$

For $(x_0, \lambda_0) \in N$ and $r, \rho > 0$ we let

$$B_N(x_0, \lambda_0; r, \rho) = \{(x, \lambda) \in V \oplus \mathbb{R}^2; |\lambda - \lambda_0| < \rho, |x - \mu(\lambda)| < r\}.$$

DEFINITION 2.5. Suppose $f, N$ are as in Theorem 2.2. Let $r, \rho > 0$. We say that $U \subset V \oplus \mathbb{R}^2$ is a *special neighbourhood of* $(x_0, \lambda_0)$ *determined by* $r, \rho$ if

(a) $U = B_N(x_0, \lambda_0; r, \rho)$, $\overline{U} \cap N \in \operatorname{int} N$;

(b) $(x, \lambda) \in \overline{U}$, $x \neq \mu(\lambda), |\lambda - \lambda_0| = \rho$ imply $f(x, \lambda) \neq 0$;

(c) $(x_0, \lambda_0)$ is the only $V$-singular point in $U$.

The existence of a special neighbourhood follows from the Implicit Function Theorem.

DEFINITION 2.6. Suppose that $U = B_N(x_0, \lambda_0; r, \rho)$ is a special neighbourhood of an isolated $V$-singular point $(x_0, \lambda_0)$ of $f$. We say that a continuous $G$-invariant function $\theta : \overline{U} \to \mathbb{R}$ is a *complementing function* if

(a) $\theta(\mu(\lambda), \lambda) = -|\lambda - \lambda_0|$ for all $\lambda \in U_l a$;

(b) $\theta(x, \lambda) = r$ if $|x - \mu(\lambda)| = r$.

Note that if $\theta$ is a complementing function then the map $\Phi = (f, \theta)$ maps $(\overline{U}, \partial U)$ into $(V \oplus \mathbb{R}, V \oplus \mathbb{R} \backslash \{0\})$, therefore $\operatorname{Deg}(\Phi, U)$ is well defined.

LEMMA 2.7. *Assume* $\Phi_1 = (f, \theta_1), \Phi_2 = (f, \theta_2)$, *where* $\theta_1, \theta_2$ *are complementing functions. Then*

$$\operatorname{Deg}(\Phi_1, U) = \operatorname{Deg}(\Phi_2, U).$$

PROOF. The homotopy

$$H((x, \lambda), t) = (f(x, \lambda), (1 - t)\theta_1(x, \lambda) + t\theta_2(x, \lambda))$$

does not vanish on $\partial U$ and the statement follows from the homotopy invariance of Deg. □

As before we assume that $U = B_N(x_0, \lambda_0; r, \rho)$ is a special neighbourhood of $(x_0, \lambda_0)$. Define $\theta_0 : \overline{U} \to \mathbb{R}$ by $\theta_0(x, \lambda) = (2^{-1}\rho)^2 - |\lambda - \lambda_0|^2$; we call $\theta_0$ the *Ize function.*

LEMMA 2.8. *Assume that* $U = B_N(x_0, \lambda_0; r, \rho)$ *is a special neighbourhood of* $(x_0, \lambda_0)$, $\theta$ *is an arbitrary complementing function and* $\theta_0$ *is the Ize function. Set* $\Phi = (f, \theta)$, $\Phi_0 = (f, \theta_0)$. *Then there exists* $r_0$, $0 < r_0 < r$, *such that* $\operatorname{Deg}(\Phi_0, U_0)$ *is defined*

$$\operatorname{Deg}(\Phi, U) = \operatorname{Deg}(\Phi_0, U_0),$$

*where* $U_0 = B_N(x_0, \lambda_0; r_0, \rho)$.

PROOF. Using once more the Implicit Function Theorem, we can choose $r_0 < r$ such that $(x,\lambda) \in \overline{U}_0$, $x \neq \mu(\lambda)$ and $|\lambda - \lambda_0| \geq 2^{-1}\rho$ imply $f(x,\lambda) \neq 0$. We now choose a complementing function $\theta : \overline{U} \to \mathbb{R}$ such that

$$\theta(x,\lambda) = r \qquad \text{if } r_0 \leq |x - \mu(\lambda)| \leq r.$$

For $((x,\lambda),t) \in U_0 \times [0,1]$ set

$$H((x,\lambda),t) = (f(x,\lambda), t\theta(x,\lambda) + (1-t)\theta_0(x,\lambda)).$$

Clearly $H$ is a homotopy between the restrictions of $\Phi$ and $\Phi_0$ to $U_0$. Thus $\operatorname{Deg}(\Phi, U_0) = \operatorname{Deg}(\Phi_0, U_0)$. Since $\operatorname{Deg}(\Phi, U) = \operatorname{Deg}(\Phi, U_0)$ the proof is completed. □

Our next result is a natural analog of the Ize lemma ([10], see also [18]).

LEMMA 2.9. *Assume that* $U = B_N(x_0, \lambda_0; r, \rho)$ *is a special neighbourhood of* $(x_0, \lambda_0)$, $\theta$ *is a complementing function and let* $\Phi = (f, \theta)$. *Then*

$$\deg_H(\Phi, U) = \gamma_j(x_0, \lambda_0),$$

*where* $H = \mathbb{Z}_j$.

PROOF. The statement follows from Proposition 2.4 applied to $\Phi_0$ from Lemma 2.8. The only new point is that we need to compute the sign of the function $(D\Phi_0^G(\eta(1)),\ \eta'(1))$ with $\eta = \mu : S^1 \to N$, a $C^1$-parametrization of the zero set of $\Phi_0$. It is easily seen that

$$\operatorname{sgn}(D\Phi_0^G(\eta(1)), \eta'(1)) = -\operatorname{sgn} D_x f^G(\mu(1), \mu'(1)),$$

which yields

$$\deg_H(\Phi_0, U) = \operatorname{sgn} D_x f(\mu(1)) \nabla([\psi_j]) = \gamma_j(x_0, \lambda_0), \qquad \text{for } H = \mathbb{Z}_j.$$

□

PROOF OF THEOREM 2.2. Suppose that the conclusion is not valid. Then there exists a special neighbourhood $U$ of $(x_0, \lambda_0)$ such that $\overline{U}$ contains only trivial solutions of $(*)$. Let $\theta : \overline{U} \to \mathbb{R}$ be a complementing function. Define $\chi : \overline{U} \times [0,1] \to \mathbb{R}$ by

$$\chi(x,\lambda,t) = (1-t)\theta(x,\lambda) - t\rho$$

and $H : (\overline{U} \times [0,1]), \partial U \times [0,1]) \to (V \oplus \mathbb{R}, V \oplus \mathbb{R} \backslash \{0\})$ by

$$H(x,\lambda,t) = (f(x,\lambda), \chi(x,\lambda,t)).$$

Clearly $H(x,\lambda,0) = \Phi(x,\lambda)$ and $H(x,\lambda,1) \neq 0$ for all $(x,\lambda) \in \overline{U}$. Therefore, by the homotopy invariance of Deg, $\deg_H(\Phi, U) = 0$ for $H = \mathbb{Z}_j$, $j = 1,2,\ldots,k$.

This contradiction proves the first part of our theorem. To obtain the second part we apply the first part to the restricted map

$$\Phi^H : (\overline{U}^H, \partial U^H) \to (V^H \oplus \mathbb{R}, V^H \oplus \mathbb{R}\backslash\{0\}).$$

Thus the proof of Theorem 2.2 is complete. □

LEMMA 2.10. *Let $X$ be a closed invariant subset of $V \oplus \mathbb{R}^2$ and $Y$ a bounded connected component of $X$. If $U$ is an open subset of $V \oplus \mathbb{R}^2$ such that $Y \subset U$ then there exists an open invariant bounded subset $U_0$ of $V \oplus \mathbb{R}^2$ such that $Y \subset U_0 \subset U$, $\overline{U}_0$ is compact and $\partial U_0 \cap X = \emptyset$.*

PROOF. The lemma is an equivariant version of the classical result and can be easily deduced from its nonequivariant counterpart. □

PROOF OF THEOREM 2.3. Choose $r, \rho$ such that

(a.1) for $i = 1, \ldots, q$, $B_i = B_N(x_i, \lambda_i; r, \rho)$ is a special neighbourhood of $(x_i, \lambda_i)$;

(a.2) $B_i \cap B_j = \emptyset$ for $i \neq j$.

Let $B = B_1 \cup \cdots \cup B_q$. Choose a bounded open subset $\Omega_1 \subset V \oplus \mathbb{R}^2$ such that $C\backslash B \subset \Omega_1$ and $\overline{\Omega}_1 \cap N = \emptyset$. Let $\Omega_2 = B \cup \Omega_1$; then $C \subset \Omega_2$. Applying Lemma 2.10 we find an open invariant subset $\Omega$ of $V \oplus \mathbb{R}^2$ such that $C \subset \Omega \subset \Omega_2$ and $\partial\Omega \cap S(f) = \emptyset$. Clearly $\Omega$ is bounded. Applying the Implicit Function Theorem we may choose $r_0$ and $\rho_0$ so small that for $i = 1, 2, \ldots, q$,

(b.1) $0 < r_0 \leq r$, $0 < \rho_0 \leq \rho$;

(b.2) $B_N(x_i, \lambda_i; r_0, \rho_0) \subset \Omega$;

(b.3) $S(f) \cap B_N(x_i, \lambda_i; r_0, \rho) \subset B_N(x_i, \lambda_i; r_0, \rho_0)$;

(b.4) $U_i = B_N(x_i, \lambda_i; r_0, \rho)$ is a special neighbourhood of $(x_i, \lambda_i)$.

Set $U = U_1 \cup \cdots \cup U_q$. Let $\theta : \overline{\Omega} \cup \overline{U} \to \mathbb{R}$ be a continuous equivariant (i.e. constant on orbits) function such that

(c.1) $\theta(x, \lambda) = -|\lambda - \lambda_i|$ if $(x, \lambda) \in N \cap U_i$;

(c.2) $\theta(x, \lambda) = r_0$ if $(x, \lambda) \in \overline{\Omega}\backslash U$ (cf. Def. 2.6).

Define $\Phi : \overline{\Omega} \cup \overline{U} \to V \oplus \mathbb{R}$ by $\Phi(x, \lambda) = (f(x, \lambda), \theta(x, \lambda))$. Note first that (c.1) implies $\Phi^{-1}(0) = f^{-1}(0) \cap \theta^{-1}(0) \subset S(f)$. Since $\partial\Omega \cap S(f) = \emptyset$, $\mathrm{Deg}(\Phi, \Omega)$ is defined. Define a homotopy $H : (\overline{\Omega} \times [0,1], \partial\Omega \times [0,1]) \to (V \oplus \mathbb{R}, V \oplus \mathbb{R}\backslash\{0\})$ by

$$H(x, \lambda, t) = (f(x, \lambda), \chi(x, \lambda, t)),$$

where $\chi(x, \lambda, t) = (1 - t)\theta(x, \lambda) - t\rho$. Clearly

$$H(x, \lambda, 0) = \Phi(x, \lambda) \quad \text{and} \quad H(x, \lambda, 1) \neq 0, \qquad \text{for all } (x, \lambda) \in \overline{\Omega}.$$

Therefore, by the homotopy invariance of Deg, $\mathrm{Deg}(\Phi,\Omega)=0$. On the other hand, (b.2) and (b.3) imply $\Phi^{-1}(0)\subset S(f)\cap U\subset S(f)\cap\Omega$. Therefore

$$\mathrm{Deg}(\Phi,\Omega)=\mathrm{Deg}(\Phi,\Omega\cap U)=\mathrm{Deg}(\Phi,U)=0.$$

By the definition of $U$,

$$\mathrm{Deg}(\Phi,U)=\mathrm{Deg}(\Phi,U_1)+\cdots+\mathrm{Deg}(\Phi,U_q)=0.$$

Thus, taking $H=\mathbb{Z}_j$, and using (b.4) together with Lemma 2.9, we obtain

$$\gamma_j(x_1,\lambda_1)+\cdots+\gamma_j(x_q,\lambda_q)=\deg_H(\Phi,U_1)+\cdots+\deg_H(\Phi,U_q)=0$$

and the proof is complete. □

## 3. Bifurcations in infinite-dimensional representations of $S^1$

In this section we give an infinite-dimensional extension of the results of the preceding section. First we introduce notation and recall some basic facts concerning Hilbert representations of $S^1$.

We say that a real (resp. complex) Hilbert space $E$ is a *real* (resp. *complex*) *representation of* $G=S^1$ if there is given a continuous map $\mu:G\times E\to E$ such that:

(a) $gx=\mu(g,x)$ defines a linear automorphism for every $g\in G$;

(b) $1x=x$ for all $g\in G$;

(c) $g_1(g_2x)=(g_1g_2)x$ for all $x\in E$, $g_1,g_2\in G$.

If $E$ is a Hilbert space (real or complex) we let $\mathrm{GL}_c(E)$ denote the group of all linear automorphisms of $E$ which are of the form $I+A$, where $I$ denotes the identity and $A$ is compact. Let $V$ denote a finite-dimensional linear subspace of $E$. For $A\in\mathrm{GL}(V)$ (= the group of all linear automorphisms of $V$) define $\hat{A}$ by $\hat{A}(x+y)=A(x)+y$, where $x\in V$ and $y\in V^{\perp}$. Clearly $A\in\mathrm{GL}_c(E)$. The following fact is a direct consequence of the work of Palais [19].

Proposition 3.1. *For a complex Hilbert space $E$, the assignment $A\to\hat{A}$ induces a bijection*

$$[S^1,\mathrm{GL}(V)]\to[S^1,\mathrm{GL}_c(E)].$$

This result, together with the classical description of $[S^1,\mathrm{GL}(V)]$ (discussed at the beginning of Section 2) yields

Proposition 3.2. *For a complex Hilbert space $E$, there exists a bijection*

$$\nabla:[S^1,\mathrm{GL}_c(E)]\to\mathbb{Z}.$$

Suppose now that $E$ is a real orthogonal Hilbert representation of $S^1$. Suppose further that $E$ is nontrivial and isotypic, i.e. there exists $n \in \mathbb{N}$ such that $G_x = \mathbb{Z}/n\mathbb{Z}$ for all $x \in E\backslash\{0\}$. Set

$$\mathrm{GL}_c^G(E) = \{A \in \mathrm{GL}_c(E); A(gx) = gA(x);\ \text{for all } x \in E,\ g \in G\}.$$

Then as in the preceding section, we define the $*$-multiplication, $z * x$, for $z \in G$, $x \in E$. Let $E_{\mathbb{C}}$ denote the resulting complex space. As in the finite-dimensional case, we have

$$\mathrm{GL}_c^G(E) = \mathrm{GL}_c(E_{\mathbb{C}}).$$

PROPOSITION 3.3. *For an isotypic nontrivial representation there exists a canonical bijection*

$$\nabla : [S^1, \mathrm{GL}_c^G(E)] \to \mathbb{Z}.$$

In the remainder of this section we assume that $E$ is a real orthogonal representation of $G = S^1$. Moreover, we assume that $E^G$ is finite-dimensional. We have the $G$-invariant decomposition

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_n \oplus \ldots$$

where $E_0 = E^G$ (i.e. $G$ acts trivially on $E_0$) and for $k > 0$, $E_k$ denotes the $\mathbb{Z}_k$-isotypic factor of $E$.

We also assume that $\Omega$ is an open invariant subset of $E \oplus \mathbb{R}^2$ and $f : \Omega \to E$ is an equivariant $C^1$-map such that $f(x, \lambda) = x + \varphi(x, \lambda)$, where $\varphi : \Omega \to E$ is completely continuous ( i.e. maps bounded subsets of $E \oplus \mathbb{R}^2$ into relatively compact subsets of $E$ ). We investigate the equation

$$(*) \qquad f(x, \lambda) = 0.$$

In what follows we assume two conditions:

(A) there exists a 2-dimensional closed submanifold $N \subset E^G \oplus \mathbb{R}^2$ such that $N \subset f^{-1}(0)$;

(B) if $(x_0, \lambda_0) \in N$ then there exists an open neighbourhood $U_0$ of $\lambda_0$ in $\mathbb{R}^2$, an open neighbourhood $U$ of $(x_0, \lambda_0)$ in $E^G \oplus \mathbb{R}^2$ and a $C^1$-map $\mu : U_0 \to E^G \oplus \mathbb{R}^2$ such that

$$N \cap U = \{(x, \lambda) \in E^G \oplus \mathbb{R}^2;\ x = \mu(\lambda), \lambda \in U_0\}.$$

We repeat verbatim the definitions of trivial solution, nontrivial solution and bifurcation point. Similarly, following the pattern of Section 2, we say that $(x_0, \lambda_0)$ is *$E$-regular* if $D_x(x_0, \lambda_0)$ is an isomorphism; otherwise it is *$E$-singular.*

Assume now that $(x_0, \lambda_0)$ is an $E$-singular point isolated in $N$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ and taking a sufficiently small $\rho$ define $\alpha : S^1 \to N$ by

$$\alpha(z) = (\mu(\lambda_0 + \rho z), \lambda_0 + \rho z),$$

where $\mu$ denotes that map appearing in (B). Define

$$\psi : S^1 \to \mathrm{GL}_c^G(E)$$

by $\psi(z) = D_x f(\alpha(z))$. Therefore

$$\psi = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_n \oplus \cdots,$$

where $\psi_j : S^1 \to \mathrm{GL}_c^G(E_j)$ for $j = 0, 1, \ldots, n, \ldots$. Let $\varepsilon = \operatorname{sgn}\det\psi_0(z)$. Finally, for $k > 0$ let

$$\gamma_k(x_0, \lambda_0) = \varepsilon\nabla([\psi_k]).$$

THEOREM 3.4. *Suppose that $f : \Omega \to E$ is an equivariant $C^1$-map such that $f(x, \lambda) = x + \varphi(x, \lambda)$, where $\varphi$ is completely continuous, with $N$ satisfying $(A)$ and $(B)$. If $(x_0, \lambda_0)$ is an $E$-singular point isolated in $N$ and there exists $k$ such that $\gamma_k(x_0, \lambda_0) \neq 0$, then $(x_0, \lambda_0)$ is a bifurcation point. Moreover, there exists a sequence $(x_n, \lambda_n) \to (x_0, \lambda_0)$ of nontrivial solutions of $(*)$ such that the isotropy group of $x_n$ contains $\mathbb{Z}_k$ for all $n$.*

THEOREM 3.5. *Suppose that $f : \Omega \to E$ is an equivariant $C^1$-map such that $f(x, \lambda) = x + \varphi(x, \lambda)$, where $\varphi$ is completely continuous, with $N$ satisfying $(A)$ and $(B)$. Suppose further that every $E$-singular point in $N$ is isolated. Let $S(f)$ denote the closure of the set of all nontrivial solutions of $(*)$. If $C$ is a bounded connected component of $S(f)$, then $C \cap N$ is finite. Moreover, if $C \cap \partial\Omega = \emptyset$ and $C \cap N = \{(x_1, \lambda_1), \ldots, (x_q, \lambda_q)\}$, then for every $k \in N$*

$$\gamma_k(x_1, \lambda_1) + \gamma_k(x_2, \lambda_2) + \cdots + \gamma_k(x_q, \lambda_q) = 0.$$

It is evident that one can obtain the proofs of Theorems 3.4 and 3.5 generalizing to the case of Hilbert spaces the proofs of Theorem 2.2 and 2.3, provided we have an infinite-dimensional version of Proposition 2.9 which represents the main geometric ingredient of these proofs. Since Proposition 2.9 is a direct consequence of Proposition 2.4, we need an infinite-dimensional generalization of this theorem. We start with an elementary but useful observation.

LEMMA 3.6. *Suppose $X$ is a compact metric space and $\xi : X \to \mathrm{GL}_c^G(E)$ is a continuous map. Then there exists a finite-dimensional $G$-invariant linear subspace $V \subset E$ such that $E_0 \subset V$ and a continuous map $\psi : X \to \mathrm{GL}_c^G(E)$ such that*

$$\psi(x)(v + w) = \psi(x)(v) + w, \qquad \textit{for all } x \in X, v \in V, w \in V^\perp,$$

*and $\psi$ is homotopic to $\xi$.*

PROOF. Set $\xi(x) = I - A_x$, with $A_x$ completely continuous. Since $X$ is compact, there exists $\epsilon > 0$ such that $\|B - A_x\| < \epsilon$ implies $I - B \in \mathrm{GL}_c^G(E)$ provided $B$ is a completely continuous equivariant linear operator.

Fix $\epsilon > 0$ and $x \in X$. Since $A_x$ is equivariant and completely continuous, there exists a finite dimensional $G$–invariant linear subspace $V_x$ of $E$ and an open neighbourhood $U_x$ of $x$ in $X$ such that

$$\|A_y - P_{V_x} A_y\| < \epsilon,$$

for every $y \in U_x$, where $P_V$ is the orthogonal (equivariant) projection onto an invariant subspace $V$ (cf. [5]).

Observe that for every $v \in E$ we have $\|v - P_W(v)\| < \|v - P(v)\|$ if $V \subset W$.

Using the above and the compactness of $X$ we choose $\{U_1, U_2, \dots, U_k\}$—an open covering of $X$, and $\{V_1, V_2, \dots, V_k\}$—a collection of finite dimensional linear subspaces of $E$ such that for every $y \in U_i$,

$$\|A_y - P_{V_i} A_y\| < \epsilon.$$

Take $W = V_1 + \dots + V_k$ and define an equivariant homotopy

$$\xi_t(x) = I - [(1-t)A_x + P_W A_x].$$

Since $\|\xi_t(x) - \xi(x)\| < \epsilon$, $\xi_t(x) \in \mathrm{GL}^G_c(E)$ for every $t \in [0,1]$. Denote $\xi_0$ by $\psi_0$.

Define another homotopy $\chi$ by $\chi(x,t) = I - [(1-t)PA_x + tPA_xP]$ and set $\psi(x) = I - PA_xP$. Then $\chi : \mathrm{GL}^G_c(E) \times [0,1] \to \mathrm{GL}^G_c(E)$ is a homotopy connecting $\psi_0$ to $\phi = \chi(x,1)$, and $\phi$ has the required property.

In what follows we use an infinite-dimensional version of the $S^1$-degree. This extends the finite-dimensional degree in the same way as the Leray-Schauder degree extends the Brouwer degree (cf. [5]).

PROPOSITION 3.7. *Assume that $U$ is an open, bounded, invariant subset of $E \oplus \mathbb{R}^2$ and $F; (\overline{U}, \partial U) \to (E \oplus \mathbb{R}, E \oplus \mathbb{R} \setminus \{0\})$ an equivariant $C^1$ map such that*

(*a*) *$F(x,\lambda) = x - \Phi(x,\lambda)$, where $\Phi$ is completely continuous;*

(*b*) *$0$ is a regular value of $F$;*

(*c*) *$M = F^{-1}(0)$ is connected and $M \subset E_0 \oplus \mathbb{R}^2$.*

*Assume further that $\eta : S^1 \to M$ is a $C^1$-diffeomorphism. For $j \in \mathbb{N}$ define $\alpha_j : S^1 \to \mathrm{GL}_G(V_j)$, by*

$$(\alpha_j(z))(v) = DF(\eta(z))(v), \qquad v \in V_j.$$

*Let $F_0 : U \cap (E_0 \oplus \mathbb{R}^2) \to E_0 \oplus \mathbb{R}$ denote the restriction of $F$. Then for $H = \mathbb{Z}_j$*

$$\deg_H(F, U) = -\operatorname{sgn}(DF_0(\eta(1)), \eta'(1)) \nabla([\alpha_j]).$$

PROOF. For $(x,\lambda) \in U \cap (E_0 \oplus \mathbb{R}^2)$ let $B(x,\lambda) : (E_0)^\perp \to (E_0)^\perp$ denote the restriction of $D_x F(x,\lambda)$. Choose an open invariant subset $U_0$ of $E_0 \oplus \mathbb{R}^2$ such that

$$X = \overline{U}_0 \subset U \cap (E_0 \oplus \mathbb{R}^2)$$

and $B(x, \lambda)$ is an isomorphism for all $(x, \lambda) \in X$. For $\varepsilon > 0$ set

$$U(\varepsilon) = \{(x, \lambda) \in E \oplus \mathbb{R}^2; x = v + w,\ v \in U_0,\ w \in (E_0)^\perp, \|w\| < \varepsilon\}.$$

Choose $\varepsilon_0$ such that $Y = \overline{U(\varepsilon_0)} \subset U$. Define $H : Y \times [0, 1] \to E \oplus \mathbb{R}$ by

$$H(x, \lambda, t) = (1 - t)F(x, \lambda) + t(F(v, \lambda) + B(v, \lambda)(w)),$$

where $x = v + w,\ v \in X,\ w \in (E_0)^\perp$. Choose $\varepsilon_1 > 0$ such that

$$H^{-1}(0) \cap \overline{U(\varepsilon_1)} \times [0, 1] = M \times [0, 1]$$

and define $F_1 : \overline{U(\varepsilon_1)} \to E \oplus \mathbb{R}$ by $F_1(x, \lambda) = H(x, \lambda, 1)$. Then

$$(\alpha) \qquad \mathrm{Deg}(F, U) = \mathrm{Deg}(F_1, U(\varepsilon_1)).$$

Set $Y_1 = \overline{U(\varepsilon_1)} \cap (E \oplus \mathbb{R}^2) = U = X$ and let $B_1 : Y_1 \to \mathrm{GL}_c^G((E_0)^\perp)$ denote the restriction of $B$. By Lemma 3.6 there exists a finite dimensional equivariant subspace $V \subset (E_0)^\perp$ and a homotopy $K : Y_1 \times [0, 1] \to \mathrm{GL}_c^G((E_0)^\perp)$ such that

$$K(x, \lambda, 0) = B_1(x, \lambda), \qquad \text{for all } (x, \lambda) \in Y_1,$$
$$K(x, \lambda, 1)(v + w) = K(x, \lambda, 1)(v) + w, \qquad \text{for all } (x, \lambda) \in Y_1,\ v \in V,\ w \in V^\perp.$$

Define $F_2 : \overline{U(\varepsilon_1)} \to E \oplus \mathbb{R}$ by

$$F_2(x, \lambda) = F(v, \lambda) + K(v, \lambda, 1)(w),$$

where $x = v + w,\ v \in E_0 \oplus \mathbb{R}^2,\ w \in (E_0)^\perp$. Then

$$(\beta) \qquad \mathrm{Deg}(F_1, U(\varepsilon_1)) = \mathrm{Deg}(F_2, U(\varepsilon_1)).$$

## 4. Characteristic maps

We now define a characteristic map of an isolated center. Since this notion is fundamental for our considerations, we collect here the necessary definitions and some basic properties. We say that a continuous map

$$A : [-1, 1] \to \mathrm{GL}(m, \mathbb{R})$$

is a *characteristic map* if $\sigma(A(a)) \cap i\mathbb{R} = \emptyset$ for $a \in [-1, 0) \cup (0, 1]$. We denote by $\mathbb{A}(m, \mathbb{R})$ the space of all characteristic maps. We say that a continuous map

$$H : [-1, 1] \times [0, 1] \to \mathrm{GL}(m, \mathbb{R})$$

is a *homotopy in* $\mathbb{A}(m, \mathbb{R})$ if $H(\cdot, t) \in \mathbb{A}(m, \mathbb{R})$ for all $t \in [0, 1]$ and there exists a finite subset $\Lambda \subset i\mathbb{R}$ such that $\sigma(H(0, t)) \cap i\mathbb{R} \subset \Lambda$ for all $t \in [0, 1]$. Evidently this defines an equivalence relation in $\mathbb{A}(m, \mathbb{R})$ : $A_0, A_1 \in \mathbb{A}(m, \mathbb{R})$ are *homotopic in* $\mathbb{A}(m, \mathbb{R})$ if there exists $H$, a homotopy in $\mathbb{A}(m, \mathbb{R})$, such that $A_j = H(\cdot, j),\ j = 0, 1$. We say that $A \in \mathbb{A}(m, \mathbb{R})$ is a *trivial characteristic map* if $\sigma(A(0)) \cap i\mathbb{R} = \emptyset$.

For given $A \in \mathbb{A}(m_1, \mathbb{R})$, $B \in \mathbb{A}(m_2, \mathbb{R})$ we define $A \oplus B \in \mathbb{A}(m_1 + m_2, \mathbb{R})$ by $(A \oplus B)(a) = A(a) \oplus B(a)$.

Analogously, we define the space $\mathbb{A}(m, \mathbb{C})$ of continuous maps $A : [-1, 1] \to \mathbb{A}(m, \mathbb{C})$ and the homotopy in $\mathbb{A}(m, \mathbb{C})$. Note that the complexification embeds $\mathbb{A}(m, \mathbb{R})$ into $\mathbb{A}(m, \mathbb{C})$. More precisely, for a given $A \in \mathbb{A}(m, \mathbb{R})$ we define $A^c \in \mathbb{A}(m, \mathbb{C})$ by $A^c(a)$ equal the complexification of $A(a)$. Moreover, if $A$ is homotopic to $B$ in $\mathbb{A}(m, \mathbb{R})$ then $A^c$ is homotopic to $B^c$ in $\mathbb{A}(m, \mathbb{C})$ and the complexification of a trivial characteristic map is a trivial characteristic map.

In the remainder of this section we use the following notation: for a linear map $L : \mathbb{R}^m \to \mathbb{R}^m$ and $\lambda \in \sigma(L)$ we let $\mu(\lambda, L)$ denote the algebraic multiplicity of $\lambda$: if $\lambda \notin \sigma(L)$ then we put $\mu(\lambda, L) = 0$. For a subset $\Lambda \subset \mathbb{C}$ we let $\mu(\Lambda, L)$, be the sum of $\mu(\lambda, L)$ over all $\lambda \in L$. For $r > 0$ and $\beta > 0$ we let

$$\begin{aligned} D_\rho(i\beta) &= \{z \in \mathbb{C}; |z - i\beta| \le \rho\}, \\ S_\rho(i\beta) &= \{z \in \mathbb{C}; |z - i\beta| = \rho\}, \\ B^+_\rho(i\beta) &= \{z \in \mathbb{C}; |z - i\beta| \le \rho,\ \operatorname{Re} z > 0\}. \end{aligned}$$

Assume now that $A \in \mathbb{A}(m, \mathbb{R})$, $\beta > 0$ and $i\beta \in \sigma(A(0))$. Choose $\varepsilon > 0$ such that $\sigma(A(0)) \cap D_\varepsilon(i\beta) = \{i\beta\}$. Then there exists $\delta > 0$ such that $\sigma(A(a)) \cap S_\varepsilon(i\beta) = \emptyset$ for all $a \in [-\delta, \delta]$.

DEFINITION 4.1. Set

$$c(A, i\beta) = \mu(B^+_\varepsilon(i\beta), A(\delta)) - \mu(B^+_\varepsilon(i\beta), A(-\delta)).$$

We call $c(A, i\beta)$ the *crossing number of $A$ through $i\beta$*.

THEOREM 4.2. *Assume that $A \in \mathbb{A}(m, \mathbb{R})$ and*

$$\sigma(A(0)) \cap \{i\mathbb{R}\} = \{\pm i\beta_1, \pm i\beta_2, \dots, \pm i\beta_k\},$$

*where $0 < \beta_1 < \cdots \beta_k$. Let $r_j = c(A, i\beta_j)$, $j = 1, \dots, k$. Then $A^c$ is homotopic in $\mathbb{A}(m, \mathbb{C})$ to $A_0 \oplus A_1 \oplus \cdots \oplus A_k$, where*

(a) *$A_0$ is a trivial characteristic map;*

(b) *for $j = 1, \dots, k$, $A_j \in \mathbb{A}(2|r_j|, \mathbb{C})$,*

$$\begin{aligned} A_j &= A_{j,1} \oplus A_{j,2}, \qquad \text{where } A_{j,1}, A_{j,2} \in \mathbb{A}(|r_j|, \mathbb{C}), \\ A_{j,1}(a) &= (i\beta_j + (\operatorname{sgn}\ r_j)a)I \\ A_{j,2}(a) &= (-i\beta_j + (\operatorname{sgn}\ r_j)a)I. \end{aligned}$$

(*$I$ denotes the identity in* $\mathrm{GL}(|r_j|, \mathbb{C})$).

We postpone the proof of this theorem to the Appendix.

DEFINITION 4.3. Assume that $A \in \mathbb{A}(m,\mathbb{C})$ and $i\beta \in \sigma(A(0)), \beta > 0$. Define $\xi_{\beta,A} : S^1 \to \mathrm{GL}(m,\mathbb{C})$ by $\xi_{\beta,A}(a,p) = 2\pi i I - (\rho p + p_0)A(\rho a)$, where $(a,p) \in S^1$, $p_0 = 2\pi\beta^{-1}$ and $\rho$ is sufficiently small. For $A \in \mathbb{A}(m,\mathbb{R})$ and $i\beta \in \sigma(A(0))$ we set $\xi_{\beta,A} = \xi_{\beta,A^c}$.

Recall, that we have defined in Section 2 the homomorphism

$$\nabla : [S^1, \mathrm{GL}(m,\mathbb{C})] \to \mathbb{Z}.$$

As a direct consequence of the definition of $\xi_{\beta,A}$ we have

REMARK 4.4.

(a) Suppose that $A, B \in \mathbb{A}(m,\mathbb{R})$ (resp. $A, B \in \mathbb{A}(m,\mathbb{C})$) are homotopic in $\mathbb{A}(m,\mathbb{R})$ (resp. $\mathbb{A}(m,\mathbb{C})$) and $i\beta \in \sigma(A(0)), \beta > 0$. Then

$$\nabla\big([\xi_{\beta,A}]\big) = \nabla\big([\xi_{\beta,B}]\big).$$

(b) Suppose that $A \in \mathbb{A}(m_1,\mathbb{R})$, $B \in \mathbb{A}(m_2,\mathbb{R})$ (respectively, $A \in \mathbb{A}(m_1,\mathbb{C})$, $B \in \mathbb{A}(m_2,\mathbb{C})$) and $\beta > 0$. Then

$$\nabla\big([\xi_{\beta,A\oplus B}]\big) = \nabla\big([\xi_{\beta,A}]\big) + \nabla\big([\xi_{\beta,B}]\big).$$

The following proposition establishes an important link between the crossing number and $[S^1, \mathrm{GL}(m,\mathbb{C})]$.

PROPOSITION 4.5. *Assume that $A \in \mathbb{A}(m,\mathbb{R})$ and $i\beta \in \sigma(A(0))$, $\beta > 0$. Then $\nabla([\xi_{\beta,A}]) = c(A, i\beta)$.*

PROOF. From Theorem 4.2 and Remark 4.4 it follows that it is sufficient to prove the following two cases of 4.5:

*Case 1*: $A^c \in \mathbb{A}(2,\mathbb{C}),\ A^c = A_1 \oplus A_2,\ A_1(a) = (i\beta + a)I,\ A_2(a) = (-i\beta + a)I.$

*Case 2*: $A^c \in \mathbb{A}(2,\mathbb{C}),\ A^c = A_1 \oplus A_2,\ A_1(a) = (i\beta - a)I,\ A_2(a) = (-i\beta - a)I.$

Consider Case 1. First observe that $i\beta \notin \sigma(A_2(0))$ implies

$$\nabla([\xi_{\beta,A^c}]) = \nabla([\xi_{\beta,A_1}]).$$

Next, set $D_\rho = D_\rho(0,p_0) = \{(a,p) \in \mathbb{R}^2; a^2 + |p - p_0|^2 = \rho^2\}$, $S_\rho = \partial D_\rho$ and consider the map

$$\xi : (D_\rho, S_\rho) \to (L(\mathbb{C}), \mathrm{GL}(1,\mathbb{C})) = (\mathbb{C}, \mathbb{C}\backslash\{0\}) = (\mathbb{R}^2, \mathbb{R}^2\backslash\{0\})$$

defined by $\xi(a,p) = 2\pi i I - pA_1(a)$. Clearly $\nabla([\xi_{\beta,A^c}])$ equals to the degree of $\xi$. Since 0 is a regular value of $\xi$, we have

$$\text{degree of } \xi = \operatorname{sgn}\det D\xi(0,p_0) = \operatorname{sgn}\det\begin{bmatrix} p_0 & 0 \\ 0 & \beta \end{bmatrix} = 1.$$

This completes the proof of Case 1. The proof of Case 2 is analogous. □

## 5. Proofs of the main theorem

Throughout this section we assume:

(a) $U$ is an open subset of $\mathbb{R}^m \times \mathbb{R}$ and $\varphi : U \to \mathbb{R}^m$ is a $C^1$-map;

(b) every stationary point of $\varphi$ is nonsingular.

Recall that we investigate periodic solutions of the equation

$$v'(t) = \varphi(v(t), a). \tag{$*$}$$

Substituting $u(t) = v(pt)$ we get the equation

$$u'(t) = p\varphi(u(t), a). \tag{$**$}$$

Evidently $u(t)$ is a 1-periodic solution of $(**)$ if and only if $v(t)$ is a $p$-periodic solution of $(*)$. Set

$$E = W^{1,2}(S^1, \mathbb{R}^m), \ F = W^{0,2}(S^1, \mathbb{R}^m).$$

We think of elements of these Sobolev spaces as classes of periodic functions $u : [0,1] \to \mathbb{R}^m$. The argument shift defines an action of $G = S^1$ on $E$ and $F$. Moreover, the map

$$T : E \to F$$

defined by $T(u) = u' + u$, is an equivariant isomorphism. We denote by $E_0$ the subspace of $E$ consisting of constant functions. For a natural $k$ we denote by $E_k$ the subspace of $E$ spanned by functions

$$(\cos 2\pi kt)e_1, (\sin 2\pi kt)e_1, \dots, (\cos 2\pi kt)e_m, \ (\sin 2\pi kt)e_m,$$

where $e_1, \dots, e_m$ denotes the standard basis of $\mathbb{R}^m$. Clearly $E_k$ is the $\mathbb{Z}_k$-isogenic factor of $E$ and $F$. Moreover, $T(E_k) = E_k$.

We have the following direct sum decompositions of Hilbert representations of $G = S^1$

$$E = E_0 \oplus E_1 \oplus \dots \oplus E_k \oplus \cdots,$$
$$F = E_0 \oplus E_1 \oplus \dots \oplus E_k \oplus \cdots.$$

We let

$$\Omega_0 = \{(u, a) \in E \oplus \mathbb{R}; (u(t), a) \in U; \text{ for all } t \in [0,1]\}.$$

Obviously $\Omega_0$ is an open subset of $E \oplus \mathbb{R}$. We set

$$\Omega = \Omega_0 \times (0, \infty) \subset E \oplus \mathbb{R}^2.$$

Define

$$f : \Omega \to F \quad \text{by} \quad f(u, a, p) = u' - p\varphi(u(\cdot), a),$$
$$N_0 = \{(x, a) \in U; \varphi(x, a) = 0\}, \quad N = N_0 \times (0, \infty).$$

In what follows we identify elements of $E_0$ (i.e. constant functions) with points of $\mathbb{R}^m$. Evidently $N$ and $f$ satisfy conditions (A) and (B) of Section 3.

Note that if $(u,a,p) \in f^{-1}(0)$ then $u$ is of class $C^1$ (by the regularization theorem). By definition $(x_0, a_0, \cdot p_0)$ is a Fuller center if and only if there exists $i\beta \in \sigma(D_x\varphi(x_0,a_0))$ and $k \in \mathbb{N}$ such that $p_0 = 2\pi k\beta^{-1}$. This is equivalent to the fact that the equation

$$u'(t) - p_0 D_x\varphi(x_0,a_0)(u(t)) = 0$$

has a nontrivial solution of period 1. Thus $(x_0, a_0 \cdot p_0)$ is a Fuller center if and only if it is an $E$-singular point for $f$. We summarize these and other observations as follows:

REMARK 5.1.

(1) If a sequence $\{(u_n, a_n, p_n)\}$ of points in $f^{-1}(0)$ converges to $(u_0, a_0, p_0)$ then the sequence $\{(u_n(0), a_n, p_n)\}$ converges to $(u_0, a_0, p_0)$ in the Fuller space $U \times \mathbb{R} \times (0,\infty)$.

(2) A stationary point $(x_0, a_0 \cdot p_0)$ of $(*)$ is a center (resp. an isolated center) if and only if $(u_0, a_0 \cdot p_0)$, where $u_0(t) = x_0$ for all $t \in [0,1]$, is an $E$-singular point (resp. an isolated $E$-singular point) for $f$.

(3) Let $S(f)$ denote the closure of the set of all nontrivial zeros of $f$. If $C$ is a connected component of $S(f)$, then the assignment $(u,a,p) \to (u(0),a,p)$ maps $S(f)$ (resp. $C$) onto the closure of the set of all nontrivial periodic points in the Fuller space (resp. onto a component of it). Moreover, $C$ is bounded if and only if its image in the Fuller space is bounded.

(4) If $(u,a,p) \in E^H \cap f^{-1}(0)$, where $H = \mathbb{Z}_k$, then $(u(0),a)$ is periodic with period $pk^{-1}$ and $pk^{-1}$ is a multiple of the minimal period of $(u(0),a)$.

Assume now that $(x_0,a_0)$ is an isolated center for $\varphi$. Since we have assumed (b), there exists $\delta > 0$ and a $C^1$-map

$$\eta : [a_0 - \delta, a_0 + \delta] \to \mathbb{R}^m$$

such that $\eta(x_0) = x_0$, $\varphi(\eta(a),a) = 0$ for all $a \in [a_0 - \delta, a_0 + \delta]$ and the map $A : [-1,1] \to \mathrm{GL}(m,\mathbb{R})$, defined b y $A(a) = D_x\varphi(a_0 + \delta a)$ belongs to $\mathbb{A}(m,\mathbb{R})$. In what follows we call $A$ a *characteristic map for* $(x_0,a_0)$.

PROPOSITION 5.2. *Suppose that* $(x_0,a_0)$ *is an isolated center for* $\varphi$, *and*

$$i\beta \in \sigma(D_x\varphi(x_0,a_0)), \quad \beta > 0.$$

*Then* $(x_0, a_0, 2\pi\beta^{-1})$ *is an* $E$*-singular point of* $f$ *and for every* $k \in \mathbb{N}$

$$\gamma_k(f, x_0, a_0, 2\pi\beta^{-1}) = (-1)^m \omega_k(x_0, a_0, 2\pi\beta^{-1})$$

(*cf. Definition 4.1 and the definition of* $\gamma_k$ *in Section 3*).

Proof. First recall that the complex structure in $E_k$ is given by $i * c_j = -s_j$, $i * s_j = c_j$, where $c_j(t) = (\cos 2\pi kt)e_j$, $s_j(t) = (\sin 2\pi kt)e_j$, $j = 1, \ldots, m$, and $e_1, \ldots, e_m$ denotes the standard base of $\mathbb{R}^m$.

By the definition of $\gamma_k(f, x_0, a_0, p_0) = \varepsilon\nabla([\psi_k])$, where $\varepsilon = \operatorname{sgn}(L)$, $L : E_0 \to E_0$, $L =$ the restriction of $T^{-1}D_x f(x_0, a_0, p_0)$, $\psi_k : S^1 \to \mathrm{GL}_G(E_k)$ is defined by $\psi_k(a,p) =$ the restriction of $T^{-1}D_x f(\eta(a), a, p)$ to $E_k$ and we make the identification

$$S^1 = \{(a,p) \in \mathbb{R}^2 : |a - a_0|^2 + |p - p_0|^2 = \rho^2\}$$

with $\rho$ sufficiently small.

Since $T : E_k \to E_k$ is a $G$-isomorphism, thus a $\mathbb{C}$-linear isomorphism, the homotopy class of $\psi_k$ equals the homotopy class of the map

$$(a,p) \to D_x f(\eta(a), a, p)|_{E_k}.$$

Thus without loss of generality we may assume that

$$\psi_k(a,p)(u) = u' - pD_x\varphi(\eta(a), a)(u),$$

when restricted to $E_k$. It is easy to see that

$$\psi_k(a,p) = 2\pi kiI - pA^c(a).$$

Thus $\psi_k$ is homotopic to $\mu_k$, where

$$\mu_k(a,p) = 2\pi iI - p(k^{-1}A^c(a)).$$

Since $\sigma(k^{-1}A(0)) = k^{-1}\sigma(A(0))$, the homotopy class of $\mu_k$ is trivial if $ik\beta \notin \sigma(A(0))$ and equals the homotopy class of the map $\xi_{\beta,k^{-1}A}$ if $ik\beta \in \sigma(A(0))$ (cf. Def. 4.3). Finally, $\varepsilon = \operatorname{sgn}(\det(-p_0D_x\varphi(x_0, a_0))) = (-1)^m \operatorname{sgn}\det D_x\varphi(x_0, a_0)$.

On the other hand

$$\omega_k(x_0, a_0, p_0) = \operatorname{sgn}\det D_x\varphi(x_0, a_0) r_k(x_0, a_0, p_0),$$

where

$$r_k(x_0, a_0, p_0) = c(x_0, a_0, 2\pi k(p_0)^{-1}), \quad p_0 = 2\pi\beta^{-1},$$

by the definition. Note that $c(A, ik\beta) = c(k^{-1}A, i\beta)$. Thus the statement follows from Proposition 4.5 applied to the characteristic map $k^{-1}A$. □

Remark 5.3. The statement of Proposition 5.2 is equivalent to

$$\gamma_k(f, x_0, a_0, p_0) = (-1)^m\omega_1(x_0, a_0, p_0k^{-1}).$$

Proof of Theorems 1.3 and 1.5. In view of Remark 5.1 the statements follow from Proposition 5.2 and Theorems 3.4 and 3.5. □

Proof of Theorems 1.8 and 1.9. Let us consider the perturbed system (1.7)

$$u'(t) = \psi(u(t)) + a \operatorname{grad} G(u(t)).$$

Differentiating $G$ along trajectories of (1.7) and assuming that $\operatorname{grad} G(x) \neq 0$ outside the zero set of $\psi$, we get the following fact (cf. [17]). $\square$

LEMMA 5.4. *$(u(t), a)$ is a nontrivial periodic solution of* (1.7) *only if $a = 0$ and $u(t)$ is a periodic solution of* $(**)$. *Consequently, $(x, a, p)$ is a periodic point of* (1.7) *only if $a = 0$ and $(x, p)$ is a nontrivial periodic point of* $(**)$.

. This shows that to prove Theorem 1.8 it is sufficient to derive $\omega_k(x_0, 0, p_0)$ at every center $(x_0, p_0)$, $p_0 = 2\pi\beta^{-1}$, $i\beta \in \sigma(D_x\psi(x_0))$ of the system (1.7).

Let $S$ be a selfadjoint linear operator corresponding to $G_2 = D_x^2 G(x_0)$ in some scalar product "associated" with $D_x\psi(x_0)$ (cf. [17]). In [17]it is proved that under these assumptions $S$ preserves the generalized eigenspace $L_k$ of $D_x\psi(x_0)$ at $ik\beta$, and

$$r_k(x_0, 0, p_0) = \tfrac{1}{2} \operatorname{sign} G_2(x_0)|_{L_k} .$$

The statement of Theorem 1.8 follows from the above and Theorem 1.5. $\square$

Finally we claim that Theorems 1.5 and 1.8 extend naturally to the case of system $u'(t) = \phi(u(t), a)$, (or $u'(t) = \psi(u(t))$ with a first integral) where $\phi$ is a tangent vector field on an open subset of a manifold. This can be done by embedding the manifold into the Euclidean space and by extending naturally the field into the normal direction (cf. [8]). By this argument, the statement of Theorem 1.9 follows from Theorem 1.5.

REMARK 5.5. Theorem 1.8 is still valid if we assume only that the matrix $G_2 = D_x^2 G(x_0)$ is nondegenerate only at those zeros of $\psi$ which are centers of the system $(**)$.

## 6. Appendix

We are left with the task of proving Theorem 4.2. The proof is based on an approach given in [9].

We start with two simple but useful observations. First assume that $A \in \mathrm{GL}(m, \mathbb{C})$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Thus there exists a direct sum decomposition $\mathbb{C}^m = V_1 \oplus V_2$ such that

$$\sigma(A_1) = \{z \in \mathbb{C};\ \operatorname{Re} z < 0\},\ \sigma(A_2) = \{z \in \mathbb{C};\ \operatorname{Re} z > 0\},$$

where $A_1$ (resp. $A_2$) denotes the restriction of $A$ to $V_1$ (resp. $V_2$). Let $k = \dim_{\mathbb{C}} V_1$.

LEMMA 6.1. *Suppose that $A$ satisfies the above assumpt ions. Then there exists a continuous map $\eta : [0, 1] \to \mathrm{GL}(m, \mathbb{C})$ such that:*

*a)* $\eta(0) = A$;

*b)* $\sigma(\eta(t) \cap i\mathbb{R}) = \emptyset$ *for* $t \in [0, 1]$;

*c)* $\eta(1) = B_1 \oplus B_2$, *where* $B_1 = -I \in \mathrm{GL}(k, \mathbb{C})$, $B_2 = I \in \mathrm{GL}(m - k, \mathbb{C})$.

Note that $k$ is uniquely determined by $A$.

Proof. Choose $C \in \mathrm{GL}(m, \mathbb{C})$ such that

$$C(V_1) = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^m, \ C(V_2) = \{0\} \times \mathbb{C}^{m-k}.$$

Since $\mathrm{GL}(m, \mathbb{C})$ is connected, there exists a continuous map

$$\mu : [0,1] \to \mathrm{GL}(m, \mathbb{C})$$

such that $\mu(0) = I$ and $\mu(1) = C$. Define $\eta_1 : [0,1] \to \mathrm{GL}(m, \mathbb{C})$ by

$$\eta_1(t) = \mu(t)A(\mu(t))^{-1}.$$

Clearly

$$\eta_1(0) = A \quad \text{and} \quad \eta_1(1) = C_1 \oplus C_2,$$

where $C_1 \in \mathrm{GL}(k, \mathbb{C})$, $C_2 \in \mathrm{GL}(m-k, \mathbb{C}), \sigma(C_1) = \sigma(A_1)$ and $\sigma(C_2) = \sigma(A_2)$. Define $\eta_2 : [0,1] \to \mathrm{GL}(m, \mathbb{C})$ by $\eta_2(t) = (1-t)\eta_1(1) + t(B_1 \oplus B_2)$. Set

$$\eta(t) = \begin{cases} \eta_1(2t), & \text{for } 0 \leq t \leq 1/2; \\ \eta_2(2t-1), & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Since $B_1$ and $B_2$ are the scalar matrices, they commute with $C_1$ and $C_2$ . Therefore $\eta$ has all the desired properties. □

Suppose now that $A \in \mathbb{A}(m, \mathbb{C})$ and there exists a direct sum decomposition $\mathbb{C}^m = V_1 \oplus V_2$ such that for all $a \in [-1,1], V_1, V_2$ are linear invariant subspaces for $A(a)$. For $j = 1,2$ let $A_j : [-1,1] \to \mathrm{GL}(V_j)$ denote the restriction of $A$.

Lemma 6.2. *Suppose that $A$ satisfies the above assumptions. Then there exists $B \in \mathbb{A}(m, \mathbb{C})$ such that*

(1) *$A$ is homotopic in $\mathbb{A}(m, \mathbb{C})$ to $B$;*
(2) *$B = B_1 \oplus B_2$ where $B_j \in \mathbb{A}(m_j, \mathbb{C})$, $m_j = \dim V_j$, $j = 1,2$;*
(3) *$\sigma(B_j(0)) = \sigma(A_j(0))$ for $j = 1,2$.*

Proof. Choose $C \in \mathrm{GL}(m, \mathbb{C})$ such that

$$C(V_1) = \mathbb{C}^{m_1} \times \{0\} \subset \mathbb{C}^m, \ C(V_2) = \{0\} \times \mathbb{C}^{m_2}.$$

Since $\mathrm{GL}(m, \mathbb{C})$ is connected there exists a continuous map $\Gamma : [0,1] \to \mathrm{GL}(m, \mathbb{C})$ such that $\Gamma(0) = I$ and $\Gamma(1) = C$. Define

$$H : [-1,1] \times [0,1] \to \mathrm{GL}(m, \mathbb{C})$$

by $H(a,t) = \Gamma(t)A(a)(\Gamma(t))^{-1}$. Evidently $H$ is a homotopy in $\mathbb{A}(m, \mathbb{C})$ and $B = H(\,\cdot\,, 1)$ is the desired map. □

Remark. Note that if $A_1, \ A_2 \in \mathbb{A}(2m, \mathbb{C})$ then $A_1 \oplus A_2$ is homotopic to $A_2 \oplus A_1$ in $\mathbb{A}(m, \mathbb{C})$.

PROOF OF THEOREM 4.2. We prove Theorem 4.2 in a few steps, replacing in each step one characteristic map by another characteristic map which is homotopic in $\mathbb{A}(m, \mathbb{C})$.

*Step 1.* We assume that $A \in \mathbb{A}(m, \mathbb{C})$ and

$$\sigma(A) \cap i\mathbb{R} = \{i\beta_1, \dots, i\beta_k\}.$$

Since the spectral decomposition is upper semicontinuous there exist $\delta > 0$ and continuous mappings

$$P_0, P_1, P_2, \dots, P_k : [-\delta, \delta] \to L(\mathbb{C}^m)$$

such that for all $a \in [-\delta, \delta]$ and $j = 0, 1, \dots, k$.

($\sigma$1) $P_j(a)$ is a linear projection commuting with $A(a)$;

($\sigma$2) $\mathbb{C}^m = V_0(a) \oplus V_1(a) \oplus \cdots \oplus V_k(a)$, where $V_j(a) = P_j(a)(\mathbb{C}^m)$;

($\sigma$3) $\sigma(A_0(0)) \cap i\mathbb{R} = \emptyset$ and $\sigma(A_j(0)) \cap i\mathbb{R} = \{i\beta_j\}$ for $j > 1$;

($\sigma$4) $\sigma(A_j(a)) \cap i\mathbb{R} = \emptyset$ for $a \neq 0$, where $A_j(a) = A(a)_{|V_j}$.

Define $\Gamma : [-\delta, \delta] \to \mathrm{GL}(m, \mathbb{C})$ by

$$\Gamma(a) = P_0(0)P_0(a) + P_1(0)P_1(a) + \cdots + P_k(0)P_k(a).$$

Define $H, K : [-1, 1] \times [0, 1] \to \mathrm{GL}(m, \mathbb{C})$ by

$$H(a, t) = A((1-t)a + t\delta a),$$
$$K(a, t) = \Gamma(t\delta a)A(\delta a)(\Gamma(t\delta a))^{-1}.$$

It is easy to check that $H$ and $K$ are homotopies in $\mathbb{A}(m, \mathbb{C})$ and $H(\cdot, 0) = A$. Let $B = K(\cdot, 1)$. Then the linear subspaces $V_0(0), V_1(0), \dots, V_k(0)$ are invariant for $B(a)$ for all $a \in [-1, 1]$. Then Lemma 6.2 yields that there exists $C \in \mathbb{A}(m, \mathbb{C})$ such that $C$ is homotopic in $\mathbb{A}(m, \mathbb{C})$ to $B$ and $C = C_0 \oplus C_1 \oplus \cdots \oplus C_k$, $C_j \in \mathbb{A}(m_j, \mathbb{C})$, $m_j = \dim V_j(0)$, $j = 0, 1, \dots, k$. Moreover $C_0$ is a trivial element of $\mathbb{A}(m_0, \mathbb{C})$ and $\sigma(C_j(0)) = \{i\beta_j\}$ for $j = 1, \dots, k$.

*Step 2.* In view of Step 1 we may assume that $A \in \mathbb{A}(m, \mathbb{C})$ and

$$\sigma(A(0)) \cap i\mathbb{R} = \{\lambda\}, \qquad \lambda = i\beta,\ \beta > 0.$$

Define $B : [-1, 1] \to L(\mathbb{C}^m)$ by

$$B(a) = A(a) - \lambda I.$$

Note that $\sigma(B(a)) \cap i\mathbb{R} = \emptyset$ for $a \neq 0$. Define $H : [-1, 1] \times [0, 1] \to \mathrm{GL}(m, \mathbb{C})$ by

$$H(a, t) = \lambda I + [1 - t(1 - |a|)]B(a).$$

Therefore,

$$\sigma(H(a, t)) = \lambda + [1 - t(1 - |a|)\sigma(B(a)).$$

Thus $H$ is a homotopy in $\mathbb{A}(m,\mathbb{C})$, $H(\,\cdot\,,0) = A$, and $H(a,1) = \lambda I + |a|B(a)$. Next define $K : [-1,1] \times [0,1] \to \mathrm{GL}(m,\mathbb{C})$ by

$$K(a,t) = \begin{cases} \lambda I + |a|B(a) & \text{for } |a| \geq t; \\ \lambda I + |a|B(-t) & \text{for } -t < a < 0; \\ \lambda I + |a|B(t) & \text{for } 0 \leq a < t. \end{cases}$$

Clearly $K$ is a homotopy in $\mathbb{A}(m,\mathbb{C})$ and $K(\,\cdot\,,0) = H(\,\cdot\,,1)$. Set $C = K(\,\cdot\,,1)$.

$$C(a) = \begin{cases} \lambda I + |a|B(1) & \text{for } a \geq 0; \\ \lambda I + |a|B(-1) & \text{for } a < 0. \end{cases}$$

*Step 3.* In view of Step 2 we may assume that $A \in \mathbb{A}(m,\mathbb{C})$ and

$$A(a) = \begin{cases} \lambda I + |a|A^+ & \text{for } a \geq 0; \\ \lambda I + |a|A^- & \text{for } a < 0, \end{cases}$$

where $A^+, A^- \in \mathrm{GL}(m,\mathbb{C})$ and $\sigma(A^+) \cap i\mathbb{R} = \emptyset$, $\sigma(A^-) \cap i\mathbb{R} = \emptyset$.

With respect to Lemma 6.2 we may assume that $A^+$ and $A^-$ in Step 3 are of the form as $B = \eta(1)$ of Lemma 6.1. The statement of Theorem 4.2 follows by an easy computation of dimensions. $\square$

REMARK 6.3. The conjugated map appears in the statement of Theorem 4.2, since we started from the complexification $A^c$ of an $\mathbb{R}$-linear map $A$. Thus in Step 1 we have to split out at both eigenvalues $i\beta$ and $-i\beta$.

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