ON A POWER-TYPE COUPLED SYSTEM OF MONGE–AMPÈRE EQUATIONS

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ABSTRACT. We study an elliptic system coupled by Monge–Ampère equations:

\[
\begin{align*}
    \det D^2 u_1 &= (-u_2)^\alpha \quad \text{in } \Omega, \\
    \det D^2 u_2 &= (-u_1)^\beta \quad \text{in } \Omega, \\
    u_1 &< 0, \ u_2 < 0 \quad \text{in } \Omega, \\
    u_1 = u_2 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

here \( \Omega \) is a smooth, bounded and strictly convex domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( \alpha > 0, \ \beta > 0 \). When \( \Omega \) is the unit ball in \( \mathbb{R}^N \), we use index theory of fixed points for completely continuous operators to get existence, uniqueness results and nonexistence of radial convex solutions under some corresponding assumptions on \( \alpha, \beta \). When \( \alpha > 0, \beta > 0 \) and \( \alpha\beta = N^2 \) we also study a corresponding eigenvalue problem in more general domains.

1. Introduction

Consider the following system coupled by Monge–Ampère equations:

\[
\begin{align*}
    \det D^2 u_1 &= (-u_2)^\alpha \quad \text{in } \Omega, \\
    \det D^2 u_2 &= (-u_1)^\beta \quad \text{in } \Omega, \\
    u_1 &< 0, \ u_2 < 0 \quad \text{in } \Omega, \\
    u_1 = u_2 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

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Here $\Omega$ is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$; $\det D^2 u$ stands for the determinant of Hessian matrix $\left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)$ of $u$.

Monge–Ampère equations are fully nonlinear second order PDEs, and there are important applications in geometry and other scientific fields. Monge–Ampère equations have been studied in the past years [1], [6], [9], [12], [16]. However, to our best knowledge, only a few works have been devoted to coupled systems. We refer the reader to [10] where the author established a symmetry result for a system, which arises in studying the relationship between two noncompact convex surfaces in $\mathbb{R}^3$. It seems to be H. Wang [13], [14] who first considered systems for Monge–Ampère equations. He investigated the following system of equations:

\[
\begin{cases}
\det D^2 u_1 = f(-u_2) & \text{in } B, \\
\det D^2 u_2 = g(-u_1) & \text{in } B, \\
u_1 = u_2 = 0 & \text{on } \partial B.
\end{cases}
\]

Here and in the following $B := \{ x \in \mathbb{R}^N : |x| < 1 \}$. By reducing it to a system coupled by ODEs and using the fixed point index, the author obtained the following results:

**Theorem 1.1** ([13, Theorem 1.1]). Suppose $f, g: [0, \infty) \to [0, \infty)$ are continuous.

(a) If $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, then (1.2) has at least one nontrivial radial convex solution.

(b) If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then (1.2) has at least one nontrivial radial convex solution.

The notations were

\[ f_0 := \lim_{x \to 0^+} \frac{f(x)}{x^N}, \quad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x^N}. \]

The above theorem implies the solvability of (1.2) is related to the asymptotic behavior of $f, g$ at zero and at infinity. Obviously, it asserts the existence of a radial convex solution for system (1.1) if $\Omega = B$ and one of the following cases holds:

1. $\alpha > N$, $\beta > N$,
2. $\alpha < N$, $\beta < N$.

What we are curious about is, for the sublinear-superlinear case, i.e. $\alpha < N$, $\beta > N$, does system (1.1) admits a radial convex solution when $\Omega = B$?

We obtain that:

**Theorem 1.2.** Let $\Omega = B$, then (1.1) has a radial convex solution if $\alpha > 0$, $\beta > 0$ and $\alpha \beta \neq N^2$. 