Krystyna Mruczek-Nasieniewska

ON SOME EXTENSIONS OF THE CLASS OF MV-ALGEBRAS

Abstract. In the present paper we will ask for the lattice \( L(\text{MV}_{\text{Ex}}) \) of subvarieties of the variety defined by the set \( \text{Ex}(\text{MV}) \) of all externally compatible identities valid in the variety \( \text{MV} \) of all MV-algebras. In particular, we will find all subdirectly irreducible algebras from the classes in the lattice \( L(\text{MV}_{\text{Ex}}) \) and give syntactical and semantical characterization of the class of algebras defined by \( P \)-compatible identities of MV-algebras.

Keywords: MV-algebra; variety; identity; \( P \)-compatible identity; equational base; subdirectly irreducible algebras

1. Introduction

As it is known J. Łukasiewicz (see [9]) introduced a 3-valued propositional calculus with one designated truth-value. Łukasiewicz and Tarski [10] generalized this construction to an \( m \)-valued propositional calculus (where \( m \) is a natural number or it equals \( \aleph_0 \)) using matrices again with one designated truth-value. While giving an algebraic proof of the completeness of the Łukasiewicz infinite-valued sentential calculus, C. C. Chang introduced MV-algebras. As it is known Boolean algebras being used to semantically formulate the classical logic are in particular MV-algebras. Of course, the converse statement is not true, i.e. it is not the case that each MV-algebra is a Boolean algebra. Chang’s aim was to adopt a method of prime ideal that had been used for Boolean algebras to the case of MV-algebras.

Let us recall that the above mentioned theorem states that for any Boolean algebra \( \mathfrak{A} \) and disjoint an ideal \( I \) and a filter \( F \) in \( \mathfrak{A} \), there is a prime ideal containing \( I \), that is disjoint with \( F \). This theorem
being formulated in various versions (for example as a relative Lindenbaum lemma known as Łoś-Asser lemma) plays the key role in proofs of completeness theorems. Chang shows that as regards symbols of $+, \cdot$ and $-$ a difference between MV-algebras understood as ordered 6-toples $\langle A, +, \cdot, -, 0, 1 \rangle$ and Boolean algebras relies on the lack of the itempotence low for $+$, while the low of excluded middle has not to be fulfilled in a given MV-algebra.

An axiomatisation of the 3-valued logic was given by M. Wajsberg [18]. An axiomatisation of the $m$-valued, where $m \neq \aleph_0$, with arbitrary number of designated values had been proposed by J.B. Rosser and A.R. Turquette [16]. In [10] a hypothesis that $\aleph_0$-valued calculus is axiomatised by a system with modus ponens and substitution as sole rules of inference was given. Suggested axioms had the following form:

1. $p \rightarrow (q \rightarrow p)$
2. $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$
3. $((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)$
4. $((p \rightarrow q) \rightarrow (q \rightarrow p)) \rightarrow (q \rightarrow p)$
5. $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$.

A. Tarski [17, s. 51] in a footnote indicates Wajsberg [19] as one who confirmed this hypothesis. Rose and Rosser gave its proof in [15]. An algebraic proof of the appropriate theorem was given be Chang [1, 2]. In [7] a description of pure implication logics containing implicational fragment of infinitely many valued Łukasiewicz logic, while in [8], overlogics of this logic where described.

In the below definition, axioms are treated as a formulation of properties of particular operations on the set $A$:

**DEFINITION 1.1.** An MV-algebra is a system $\langle A, +, \cdot, \neg, 0, 1 \rangle$, where $A$ is a nonempty set, 0 and 1 are constants in the set $A$, $+$ and $\cdot$ are operations of arity two in the set $A$ and $\neg$ is a unarny operation on the set $A$, where the following equations are fulfilled:

\[
\begin{align*}
\text{Ax.1} & \quad x + y \approx y + x & \text{Ax.1'} & \quad x \cdot y \approx y \cdot x \\
\text{Ax.2} & \quad x + (y + z) \approx (x + y) + z & \text{Ax.2'} & \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\
\text{Ax.3} & \quad x + \overline{0} \approx 1 & \text{Ax.3'} & \quad x \cdot \overline{0} \approx 0 \\
\text{Ax.4} & \quad x + 1 \approx 1 & \text{Ax.4'} & \quad x \cdot 1 \approx 0 \\
\text{Ax.5} & \quad x + 0 \approx x & \text{Ax.5'} & \quad x \cdot 1 \approx x \\
\text{Ax.6} & \quad \overline{x + y} \approx \overline{x} \cdot \overline{y} & \text{Ax.6'} & \quad \overline{x \cdot y} \approx \overline{x} + \overline{y} \\
\text{Ax.7} & \quad x \approx (\overline{x}) & \text{Ax.8.} & \quad \overline{0} \approx 1
\end{align*}
\]
On some extensions of the class of MV-algebras

Ax.9 \( x \lor y \approx y \lor x \) \hspace{1cm} Ax.9' \( x \land y \approx y \land x \)
Ax.10 \( x \lor (y \lor z) \approx (x \lor y) \lor z \) \hspace{1cm} Ax.10' \( x \land (y \land z) \approx (x \land y) \land z \)
Ax.11 \( x + (y \land z) \approx (x + y) \land (x + y) \) \hspace{1cm} Ax.11' \( x \cdot (y \lor z) \approx (x \cdot y) \lor (x \cdot y) \),

where operations \( \lor \) and \( \land \) are given for any \( x, y \in A \) as follows:
\[
\begin{align*}
x \lor y & \approx (x \cdot \overline{y}) + y \\
x \land y & \approx (x + \overline{y}) \cdot y
\end{align*}
\]

Besides we recall:

**Definition 1.2.** Let \( \text{MV} \) denote the class of all MV-algebras while \( \text{Id}(\text{MV}) \) — the set of all identities valid in \( \text{MV} \).

Chang mentioned that the above axiomatisation is not very “economic”. He stressed however, that it is very intuitive and it way we recall it. It is obvious that elements 0 and 1, as well as operations +, \( \cdot \), and \( \lor \) and \( \land \) are respectively dual. Beside, one assumes that the operation \( \cdot \), similarly as in arithmetics bides stronger than +.

This fact that this axiomatisation is not “non-economic”, caused a search for more elegant axiomatisations. In [3] by an MV-algebra one understands any algebra \( \mathfrak{A} = \langle A, 0, 1, *, \odot, \oplus \rangle \) fulfilling the following conditions:

Ax.12 \( x \odot (y \odot z) \approx (x \odot y) \odot z \)
Ax.13 \( x \odot y \approx y \odot x \)
Ax.14 \( x \odot 0 \approx 0 \)
Ax.15 \( x \odot 1 \approx x \)
Ax.16 \( 0^* \approx 1 \)
Ax.17 \( 1^* \approx 0 \)
Ax.18 \( (x^* \odot y)^* \odot \approx (y^* \odot x)^* \odot x \)
Ax.19 \( x \oplus y \approx (x^* \odot y^*)^* \).

It is known, that the set \( \text{Id}(\text{MV}) \) determines a variety (a nonempty class of algebras that is closed under any subalgebras, arbitrary products and homomorphic images) and this variety is \( \text{MV} \).

When considering MV-algebras as structures in the type \( \langle 2, 2, 1, 0, 0 \rangle \) with operations +, \( \cdot \), \( \neg \), 0, 1 one can formulate a notion of externally compatible identities by stipulating that:

**Definition 1.3.** An identity is **externally compatible** iff it is of any of the below form:

\[
\begin{align*}
\varphi_1 & \approx \varphi_1 \quad (1.1) \\
\varphi_1 + \varphi_2 & \approx \psi_1 + \psi_2 \quad (1.2)
\end{align*}
\]
\[ \varphi_1 \cdot \varphi_2 \approx \psi_1 \cdot \psi_2 \quad (1.3) \]
\[ \overline{\varphi_1} \approx \overline{\psi_1}, \quad (1.4) \]

where \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) are any terms in the type \( \langle 2, 2, 1, 0, 0 \rangle \).

Let us notice that some identities valid in the class of MV-algebras are externally compatible, but some are not. For example the commutative low \( x + y \approx y + x \) is an externally compatible identity, while de Morgana low \( (x \cdot y) \approx \overline{x} + \overline{y} \) is not.

## 2. Syntax and semantics

While searching for an equational basis of the class MV_{Ex}, it is convenient to consider this class in the type \( \langle 2, 2, 1 \rangle \). Thus, we assume that the constant 0 can be defined for example as \( x \cdot \overline{x} \). The constant 1 can be defined as well, for example as \( x + \overline{x} \).

Let \( V \) a variety in the type \( \tau \) fulfilling the following conditions:

(2.1) There is a non-trivial unary term \( q(x) \), such that for any \( f \in F \), the identity \( q(f(x_0, \ldots, x_{\tau(f)-1})) \approx q(f(q(x_0), \ldots, q(x_{\tau(f)-1}))) \) belongs to \( Id(V) \).

(2.2) If \([f]_P \) is a nullary block (i.e., a block with only nullary operations) and \( g, h \in [f]_P \), then there is a non-trivializing, unary term \( q_{g,h}(x) \), such that the most external operational symbol in the term \( q_{g,h}(x) \) belongs to \([f]_P \) and moreover the following identities:

\[
\begin{align*}
g(x_0, \ldots, x_{\tau(g)-1}) &= q_{g,h}(q(g(x_0, \ldots, x_{\tau(g)-1}))), \\
h(x_0, \ldots, x_{\tau(h)-1}) &= q_{g,h}(q(h(x_0, \ldots, x_{\tau(h)-1})))
\end{align*}
\]

belong to \( Id(V) \).

(2.3) If \([f]_P \) is a nullary block of the partition \( P \), then for any \( g \in [f]_P \) identity \( f = g \) belongs to \( Id(V) \).

Let \( B \) be an equational basis of a variety \( V \). We define a set \( B^* \) of identities of the type \( \tau \) with the help of the following three conditions:

(2.4) Identities (2.1), (2.2) and (2.3) belong to \( B^* \).

(2.5) If \( \phi = \psi \) belong to \( B \), then the identity \( q(\phi) = q(\psi) \) belongs to \( B^* \).

(2.6) \( B^* \) includes only identities described in conditions (2.4) and (2.5).
It has been shown in [13] that the following theorem holds:

**Theorem 2.1.** If \( B \) is an equational basis of a variety \( V \) fulfilling the conditions (2.1), (2.2) and (2.3), then the set \( B^* \) defined by the conditions (2.4), (2.5) and (2.6) is an equational basis of the variety \( V^p \).

Besides, we have:

**Theorem 2.2 ([11]).** For any nontrivial variety \( V \in \mathcal{L} (\text{MOL}) \) there is a lattice embedding of the lattice \( B \) into \( V \), where \( B \) is a class of Boolean algebras.

The theorem below theorem holds:

**Theorem 2.3.** The following identities:

\[
\begin{align*}
\text{Ax.1. } & x + y \approx y + x & \text{Ax.1’. } & x \cdot y \approx y \cdot x \\
\text{Ax.2. } & x + (y + z) \approx (x + y) + z & \text{Ax.2’. } & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\
\text{Ax.3. } & x + \overline{x} \approx y + \overline{y} & \text{Ax.3’. } & x \cdot \overline{x} \approx y \cdot \overline{y} \\
\text{Ax.4. } & x + 1 \approx 1 & \text{Ax.4’. } & x \cdot 0 \approx 0 \\
\text{Ax.5. } & x + y + 0 \approx x + y & \text{Ax.5’. } & x \cdot y \cdot 1 \approx x \cdot y \\
& (x + 0) \cdot y \approx x \cdot y & & (x \cdot 1) + y \approx x + y \\
& x + 0 \approx \overline{x} & & x \cdot 1 \approx \overline{x} \\
\text{Ax.6. } & x + y + z \approx \overline{x} \cdot \overline{y} + z & \text{Ax.6’. } & \overline{x} \cdot \overline{y} + z \approx (\overline{x} + \overline{y}) + z \\
& (x + y) \cdot z \approx (\overline{x} \cdot \overline{y}) \cdot z & & (x \cdot y) \cdot z \approx (\overline{x} + \overline{y}) \cdot z \\
& x + y \cdot 0 \approx \overline{x} \cdot \overline{y} & & x \cdot \overline{y} \approx \overline{x} \cdot \overline{y} \\
\text{Ax.7. } & \overline{x} \approx \overline{x} & \text{Ax.8. } & \overline{0} + x \approx 1 + x \\
& \overline{x} + y \approx x + y & & \overline{0} \cdot x \approx 1 \cdot x \\
& \overline{x} \cdot y \approx x \cdot y & & \overline{0} \approx 1 \\
\text{Ax.9. } & x \lor y \approx y \lor x & \text{Ax.9’. } & x \land y \approx y \land x \\
\text{Ax.10. } & x \lor (y \lor z) \approx (x \lor y) \lor z & \text{Ax.10’. } & x \land (y \land z) \approx (x \land y) \land z \\
\text{Ax.11. } & (x + (y \land z)) + t \approx ((x + y) \land (x + y)) + t & \text{Ax.11’. } & (x \cdot (y \lor z)) + t \approx (x \cdot y) \lor (x \cdot z) + t \\
& (x + (y \land z)) \cdot t \approx ((x + y) \land (x + y)) \cdot t & & (x \cdot (y \lor z)) \cdot t \approx (x \cdot y) \lor (x \cdot z) \cdot t \\
& x + (y \land z) \approx (x + y) \land (x + y) & & x \cdot (y \lor z) \approx (x \cdot y) \lor (x \cdot z) \\
& (x \cdot (y \lor z)) + t \approx (x \cdot y) \lor (x \cdot z) + t & & x \cdot (y \lor z) \approx (x \cdot y) \lor (x \cdot z) \\
& x \cdot (y \lor z) \approx (x \cdot y) \lor (x \cdot z) & & x \cdot (y \lor z) \approx (x \cdot y) \lor (x \cdot z) \\
\end{align*}
\]

constitute an equational basis of the class \( \text{MV}_{\text{Ex}} \).
Schetch of the proof. Let us notice that the class $\text{MV}_{Ex}$ fulfills assumptions of Theorem 2.1. The set composed of identities $\text{Ax.1}–\text{Ax.11}$ and $\text{Ax.1'}–\text{Ax.11'}$ is denoted by $B_1$. Let $B_2$ denote the set of identities given by Theorem 2.1 when applied to the class $\text{MV}_{Ex}$. We skip details of the proof since it comes down to showing that $\text{Cn}(B_1) = \text{Cn}(B_2)$ and goes in the standard way.

Let us consider algebras $\mathfrak{A} = (A; F^\mathfrak{A})$ and $\mathfrak{J} = (I; F^\mathfrak{J})$ of type $\tau$ and a partition $P$ of the set $F$. The algebra $\mathfrak{A}$ is a $P$-dispersion of $\mathfrak{J}$ (see [6], [13]) iff there exists a partition $\{A_i\}_{i \in I}$ of $A$ and there exists a family $\{c_{[f]}_P\}_{f \in F}$ of mappings $c_{[f]}_P : I \to A$ satisfying the following conditions:

\begin{align}
(2.7) & \text{ For each } i \in I: c_{[f]}_P(i) \in A_i. \\
(2.8) & \text{ For each } f \in F \text{ and for each } a_i \in A_{k_i}, i = 0, \ldots, \tau(f) - 1, f^\mathfrak{A}(a_0, \\
& \ldots, a_{\tau(f) - 1}) = c_{[f]}_P(f^\mathfrak{J}(k_0, \ldots, k_{\tau(f) - 1})). \\
(2.9) & \text{ If } f \in [g]_P, \text{ then for each } i \in I: c_{[f]}_P(i) = c_{[g]}_P(i). 
\end{align}

The following theorem holds:

**Theorem 2.4 ([13]).** If $P$ is a partition of a set $F$ and $V$ is a variety of the type $\tau$ fulfilling conditions (2.1), (2.2) and (2.3), then $\mathfrak{A}$ belongs to the class $V_P$ iff $\mathfrak{A}$ is a $P$-dispersion of a certain algebra belonging to $V$. 

The following theorem is obvious:

**Theorem 2.5 ([6]).** The lattice $\mathcal{L}(Ex(\tau))$ is isomorphic with the lattice $\Pi_{F + 1}$ of all partitions of the set $F$ with the unit element $1$.

**Theorem 2.6 ([4]).** Let $V$ be a variety of the type $\tau$, such that for a certain unary term $\phi(x)$, which is not a variable, then the identity $\phi(x) \approx x$ belongs to the set $\text{Id}(V)$. Let moreover a partition $P$ of the set $F$ fulfills the condition:

$$V_P = D_P(V).$$

(VP)

Thus, lattices $\mathcal{L}(V)$ and $P(V)$ are isomorphic.

Let us consider the following example.

**Example 2.1.** Let an algebra $\mathcal{A} = \langle\{0, \frac{1}{2}^+, \frac{1}{2}^-; 1\}; +, \cdot, -\rangle$ be a dispersion of the following algebra $\mathcal{B} = \langle\{0, \frac{1}{2}, 1\}; +, \cdot, -\rangle$ (see Diagram 1). Then: $c_+(k) = c_-(k) = c\cdot(k) = k$, for $k \in \{0, 1\}$, $c_+(\frac{1}{2}) = c\cdot(\frac{1}{2}) = \frac{1}{2}^+$, and $c_-(\frac{1}{2}) = \frac{1}{2}^-$. Moreover, one can see that $\frac{1}{2} = \frac{1}{2}^+$. Thus, the identity $\overline{x} \approx x$ is not fulfilled in the algebra $\mathcal{A}$.
It can be shown that this algebra verifies all identities externally compatible valid in the class $\text{MV}_{Ex}$. It is the case since this class is fulfills assumption of Theorem 2.4. So, the next theorem follows:

**Theorem 2.7.** The class $\text{MV}_{Ex}$ equals the class all dispersions of all MV-algebras.

We have of course also a more general theorem:

**Theorem 2.8 (Characterisation of the class $\text{MV}_{Ex}$).** For any partition $P$ the class $\text{MV}_P$ equals the class of all dispersions of all $P$-dispersions of algebras from the class $\text{MV}$.

### 3. Subdirectly irreducible algebras from the variety of $\text{MV}_n$-algebras

In the present section we describe all subdirectly irreducible algebras from the class of $\text{MV}_n$-algebras.

#### 3.1. Variety of $\text{MV}_n$-algebras

In [5] R. Grigolia indicated algebras being semantical counterparts of $n$-valued logics for any $2 < n < \aleph_0$. The class $\text{MV}_n$ of all $\text{MV}_n$-algebras is a subclass of the class of all MV-algebras. It is determined by the set of all identities valid in the class of all MV-algebras extended by the following identities:

Ax.12. $(n - 1)x + x \approx (n - 1)x$

Ax.12’. $x^{n-1} \cdot x \approx x^{n-1}$

and for $n > 3$, additionally the following axioms are added:
Ax.13. \((jx) \cdot (\bar{x} + ((j-1) \cdot x)^{-}))^{(n-1)} \approx 0\)

Ax.13'. \((n-1)(x^j + (\bar{x} \cdot (x^{j-1}))^{-}) \approx 1\),

where \(1 < j < n - 1\) and \(n - 1\) is divided by \(j\).

We obtain \(MV_n\) – a class of MV\(_n\)-algebras. Thus, each Boolean algebra is a MV\(_n\)-algebra for every \(2 < n < \aleph_0\) and each MV\(_n\)-algebra for every \(2 < n < \aleph_0\) is a MV-algebra.

Let \(L_n = \langle L_n, +, \cdot, -, 1, 0 \rangle\), where \(L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}\) and for any \(x, y \in L_n\):

- \(x + y = \min(1, x + y)\),
- \(x \cdot y = \max(0, x + y - 1)\),
- \(\bar{x} = 1 - x\).

Let us recall:

**Theorem 3.1 ([5]).** Each MV\(_n\)-algebra \(A\) is isomorphic to a subdirect product of algebras \(L_m\), where \(m \leq n\) and \(m - 1\) divides \(n - 1\).

Let an algebra \(A\) belong to the class \(MV_{nEx}\). It is known that \(A\) is a dispersion of a certain algebras \(I\) from the variety \(MV_n\).

The following cases can occur (cf [14]):

1. If \(|A_i| = 1\) for every \(i \in I\), then \(A\) belongs to the variety \(MV_n\), since each function \(c_f\) determines an isomorphism of algebras \(I\) and \(A\). Thus, \(A\) is subdirectly-irreducible iff it fulfils the condition of Theorem 3.1 concerning subdirectly-irreducible MV\(_n\)-algebras.

2. If \(|I| = 1\) (i.e., \(A\) is a trivial algebra), then \(A\) belongs to the class determined by the externally compatible identities in the type \(\langle 2, 2, 1, 0, 0 \rangle\). One can easily prove that in this case the algebra \(A\) is subdirectly irreducible iff it is a 2-element algebra defined by all externally compatible identities in the type \(\langle 2, 2, 1, 0, 0 \rangle\).
3. Let $|I| > 1$ and there is $i \in I$, such that $|A_i| > 1$ (see the above figure). For any such $i$ we define a relation $R_i$ in $A$ stipulating for $a, b \in A$ as follows:

$$aR_ib \text{ iff } a = b \text{ or } a, b \in A_i.$$ 

The relation $R_i$ is a congruence that differs from $\Delta$. Now, for any $i, j \in I$, such that $i \neq j$ and $|A_i| \neq 1 \neq |A_j|$, $A$ is subdirectly irreducible. It is so since $R_i \cap R_j = \Delta$.

![Diagram](image)

4. There is exactly one element $i \in I$, such that the cardinality of the set $A_i$ bigger than 1. Without the loss of generality we can assume that is bigger than 2 (see the above diagram). Then, for every $a \in A_{i_0}$ one can define a congruence relation $R(a)$ stipulating for any $x, y$:

$$xR(a)y \text{ iff } x = y \text{ or } x, y \in A \setminus \{a\}.$$ 

Each of relations $R(a)$ is a congruence relation different from $\Delta$ and

$$\bigcap_{a \in A_{i_0}} R(a) = \Delta.$$ 

Thus $A$ is subdirectly irreducible (see Diagram 2).

5. There is exactly one element $i \in I$, for which $A_i = \{0_1, 0_2\}$, where $0_1$ is different from $0_2$ and is a function $c_f$ that is defined as follows (again see the above picture):

$$C_+(i_0) = C_1(i_0) = C_-(i_0) = O_2.$$ 

In this case we consider a congruence $R''$ defined in the following way:

$$aR''b \text{ iff } a = b \text{ or } a, b \in A \setminus \{O_1\}.$$
One can easily check that:

\[ R_{i_0} \cap R'' = \Delta. \]

Thus, \( \mathcal{A} \) is subdirectly irreducible.

Obviously, among dispersions only these described below can be subdirectly irreducible algebras: there is exactly one element \( i_0 \in I \), taki że \( |A_{i_0}| = 2 \), say \( A_{i_0} = \{O_1, O_2\} \) and there is a partition \( \{F_1, F_2\} \) of the set \( \{+, \cdot, -\} \) with blocks \( F_1, F_2 \neq \emptyset \) such that \( c_f(i_0) = O_k \) for \( f \in F_k \) where \( k = 1, 2 \).

It appears that the above mentioned dispersions are indeed subdirectly irreducible.

Thus, we have the following, main result of this part:

**Theorem 3.2.** Let \( \mathcal{A} \) be an algebra from the class \( \text{MV}_{n_{Ex}} \). The algebra \( \mathcal{A} \) is subdirectly irreducible iff at least one of the following three conditions holds:

1. \( \mathcal{A} \) belongs to the variety of MV\(_n\)-algebras and is subdirectly irreducible,
2. \( \mathcal{A} \) is a 2-element algebra from the class defined by all externally compatible identities in the type \( (2, 2, 1, 0, 0) \),
3. \( \mathcal{A} \) is a dispersion of an algebra \( \mathcal{I} \) from the class of MV\(_n\)-algebras and there is exactly one element \( i_0 \in I \) such that \( |A_{i_0}| = 2 \), say \( A_{i_0} = \{O_1, O_2\} \), and there is a partition \( \{F_1, F_2\} \) of the set \( \{+, \cdot, -\} \), where \( F_1, F_2 \neq \emptyset \) and \( c_f(i_0) = O_k \) for \( f \in F_k \) (\( k = 1, 2 \)).
4. The lattice of varieties generated by $Ex(MV)$

One can see that $Ex(MV)$ is a proper subset of the set $Id(MV)$. We conclude that the variety of MV-algebra is a proper subvariety of the variety $MV_{Ex}$. Obviously, each subvariety of the class $MV$ is also a proper subvariety of the variety $MV_{Ex}$.

Let us start with an analysis of the variety $MV$-algebra. For any variety $V$ in the type $\tau$ we put:

$$P^{(V)} = \{ K \in \mathcal{L}(V_P) : Id(K) = P(K) \}.$$  

We use the following notation (see [4]):

$$P^{(MV)} = \{ K \in \mathcal{L}(MOL_P) : Id(K) = P(MV) \}.$$  

The set $P^{(MV)}$ with the inclusion as an order is a lattice. One can say referring to the class $MV$, that it is $F$-normal and considering it in the type $\langle 2, 2, 1 \rangle$ we see that there are five partitions of the set of symbol of basic operations. Applying theorems 2.8, 2.5, and 2.6 we get:

**Theorem 4.1.** For any partition $P$ of the set $\{+, \cdot, -\}$ the lattice $P^{(MV)}$ is isomorphic to $\mathcal{L}(MV)$.  

In the below diagram we present mutual positions of lattices $P^{(MV)}$ in the lattice $\mathcal{L}(MV_{Ex})$.

[Diagram showing the lattice structure of $MV_{Ex}$, $MV_{P_1}$, $MV_{P_2}$, $MV_{P_3}$, $MV_{N}$, $MV$, $T_{Ex}$, $T_{P_1}$, $T_{P_2}$, $T_{P_3}$, $T_N$, and $T$.]

Subvariety of MV-algebras were examined by R. Grigolia, Y. Komori, A. Di Nola, and A. Lettieri. Lettieri and Di Nola [3] have given an equational basis for all $MV$-varieties, while Komori determined the lattice of subvarieties of the variety of MV-algebras (see [8]).
Following [3] we define for any natural $i > 1$ a set $\delta(i)$ as follows:

$$\delta(i) = \{n \in \mathbb{Z} : 1 \leq n \text{ and } n \text{ dzieli } i\}.$$  

On the other hand, we any finite, nonempty set $J$ of positive numbers, we put:

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j \in J} \delta(j)\}$$

In the case that $J = \emptyset$, we stipulate:

$$\Delta(i, J) = \delta(i).$$

We recall the following result:

**Theorem 4.2 ([3]).** Let $V$ be a proper subvariety of the variety $\text{MV}$. Then there are finite sets $I$ and $J$ of natural numbers bigger than 1, such that $I \cap J \neq \emptyset$ and for any $\text{MV}$-algebra $\mathfrak{A}$, $\mathfrak{A}$ belongs to $V$ iff $\mathfrak{A}$ fulfils the following identities:

$$((n + 1)x^n)^2 \approx 2x^{n+1}, \text{ gdzie } n = \max\{I \cup J\};$$  

$$px^{p-1}x^{n-1} \approx (n + 1)x^p$$

and for any positive number $p$, such that $1 < p < n$ which does not divide any number from $I \cup J$;

$$(n + 1)x^q \approx (n + 2)x^q, \text{ for any } q \in \bigcup_{j \in J} \Delta(i, J).$$  

Let us recall that the smallest proper subvariety of the variety of Boolean algebras is the class of Boolean algebras. This class is characterised be a single identity $x + x \approx x$ (i.e., in this context, to determine the class of Boolean algebras it is enough to consider the identity $x + x \approx x$ and all identities fulfilled in the class $\text{MV}$ and the obtained set close under the operator $C_n$).

Let us recall:

**Theorem 4.3 ([11]).** The lattice of all nontrivial subvarieties of the variety $\text{MOL}_{Ex}$, that are generated be the sum of the set $Ex(\text{MOL})$ and the set of all identities of one variable in the type $\langle 2, 2, 1 \rangle$, is isomorphic to the lattice $(\mathcal{L}(\text{MOL}) \setminus T) \times \overline{B}$. 
For any class \( V \) from the lattice \( \mathcal{L}(\text{MV}) \) we consider a set \( \{ K \in \mathcal{L}(\text{V}_{\text{Ex}}) : V \subseteq K \subseteq \text{V}_{\text{Ex}} \} \). Of course, this set is a lattice which is denoted by \( \overline{V} \).

The following two theorems are true. We skip proofs since they are similar to proofs of theorems 2.2 and 4.3.

**Theorem 4.4.** For every nontrivial variety \( V \in \mathcal{L}(\text{MV}) \) there is a lattice embedding of the lattice \( \overline{B} \) into \( \overline{V} \), where \( B \) is a class of Boolean algebras.

This theorem has been illustrated on Diagram 3.

![Diagram 3. The lattice of subvarieties of the variety \( \text{MV}_{\text{Ex}} \)](image)

Although we do not know the full description of the whole lattice \( \mathcal{L}(\text{MV}_{\text{Ex}}) \), we do know how the sublattice of this lattice generated by identities of one variable looks like. Strictly speaking the following theorem holds:

**Theorem 4.5.** The lattice of all subvarieties of the variety \( \text{MV}_{\text{Ex}} \) that are generated by identities of one variable is isomorphic to the lattice \( \overline{T} \cup ((\mathcal{L}(\text{MV}) \setminus T) \times \overline{B}) \).

Having analysed structures of subdirectly irreducible algebras in the class determined be externally compatible identities of \( \text{MV}_n \)-algebras we see that there is quite a lot of them — if I may say so — of specific “types of algebras”. It is connected to the fact, that the lattice \( \mathcal{L}(\text{MV}_{\text{Ex}}) \) is also quite big and — is some sense — rather complicated. A “horizontal” analysis — selecting varieties described by Komori, Di Nola, and Lettieri,
as well as a “vertical” analysis—stressing a correlation with the class of Boolean algebras, can be treated as a partial solution of the problem mentioned at the very beginning of the paper.

Finally, we have the following:

**Hypothesis.** In the lattice $\mathcal{L}({\text{MV}}_{Ex})$ there is no other elements than those predicted by Theorem 4.5.

**References**


Krystyna Mruczek-Nasieniewska
Department of Logic
Nicolaus Copernicus University in Toruń
Toruń, Poland
mruczek@uni.torun.pl