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ERRATA CORRIGE to "Pragmatic and dialogic interpretation of bi-intuitionism. Part I"

Abstract. The goal of [3] is to sketch the construction of a *syntactic categorical model* of the *bi-intuitionistic logic of assertions and hypotheses* **AH**, axiomatized in a sequent calculus **AH-G1**, and to show that such a model has a *chirality-like* structure inspired by the notion of *dialogue chirality* by P-A. Melliès [8]. A chirality consists of a pair of adjoint functors $L \dashv R$, with $L\colon \mathcal{A} \to \mathcal{B}$, $R\colon \mathcal{B} \to \mathcal{A}$, and of a functor $(\ )^*\colon \mathcal{A} \to \mathcal{B}^{op}$ satisfying certain conditions. The definition of the logic **AH** in [3] needs to be modified so that our categories $\mathcal{A}$ and $\mathcal{B}$ are actually dual. With this modification, a more complex structure emerges.

Keywords: bi-intuitionism; categorical proof theory; justificationism; meaning-as-use; speech-acts theory.

In the paper [3] (Bellin *et al*, "Pragmatic and dialogic interpretations of bi-intuitionism. Part I") a *bi-intuitionistic logic for pragmatics of assertions and conjectures* **AH** is given, extending both the intuitionistic logic of assertions (essentially, intuitionistic propositional logic **Int**) and the co-intuitionistic logic of hypotheses (**co-Int**). A modal translation into **S4** is given, see (3.2) in Section 3 for intuitionistic logic and (3.4) in Section 3.1 for co-intuitionism. The logic **AH** is axiomatized by the sequent

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calculus **AH-G1** given in Section 4, Tables 4.1–4.5.[1] The fragment of the language $\mathcal{L}^{AH}$ relevant here is given by the following grammar:[2]

$$\mathcal{L}^{AH}: \begin{array}{lccl} \mathcal{L}^{A}: & A, B & := & \vdash p \mid \curlyvee \mid A \cap B \mid \sim A \mid [\,C^{\perp}\,] \\ \mathcal{L}^{H}: & C, D & := & \mathcal{H} p \mid \curlywedge \mid C \curlyvee D \mid \frown C \mid [\,A^{\perp}\,] \end{array}$$

where $C^{\perp} \notin \mathcal{L}^{A}$, $A^{\perp} \notin \mathcal{L}^{H}$ and $C^{\perp}, A^{\perp} \in \mathcal{L}^{AH}$.

Symmetry and chiralities

The main idea is to study a fundamental property of negations in the logic **AH** in a more abstract framework. Let us use the following abbreviations:

$$\boxdot C := \sim(C^{\perp}) \quad \text{and} \quad \diamond A := \frown(A^{\perp}) \tag{1}$$

Then in **AG-G1** we can prove the following facts:[3]

$$A\ ;\ \Rightarrow\ \boxdot \diamond A; \qquad \text{and} \qquad ;\ \diamond \boxdot C\ \Rightarrow\ ;\ C \tag{2}$$

We aim at characterizing the property (2) through Melliès' notion of *dialogue chirality*. A dialogue chirality requires the following data (see [8, Section 3, Definition 2]):

1. two monoidal categories $(\mathcal{A}, \wedge, \mathrm{true})$ and $(\mathcal{B}, \vee, \mathrm{false})$;
2. an adjunction $L \dashv R$ between functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$.
3. a monoidal functor $(\ )^{*}\colon \mathcal{A} \to \mathcal{B}^{op}$ satisfying additional conditions that make it possible to define a notion of implication in $\mathcal{A}$ using disjunction in $\mathcal{B}$ and the functors $(\ )^{*}$ and R:

$$\mathcal{A}(m \wedge a, R(b)) \equiv \mathcal{A}(a, R(m^{*} \vee b)).$$

Remark 1. We may assume that the functor $(\ )^{*}$ is invertible and therefore determines a *monoidal equivalence* between $\mathcal{A}$ and $\mathcal{B}^{op}$ (see [7, Definition 6, Section 6]).

[1] Essential feature of intuitionistic elementary formulas in **AL** is that they consist of a sign of illocutionary force of assertion ($\vdash$) or hypothesis ($\mathcal{H}$) applied to an *atomic* proposition p; here a case is made for allowing also elementary formulas of the form $\vdash \neg p$ and $\mathcal{H} \neg p$, where '$\neg$' is classical negation.

[2] Here intuitionistic negation is definable as $\sim A := A \supset \mathbf{u}$ if we have implication $A \supset B$ and an expression $\mathbf{u}$ (*unjustified*) in $\mathcal{L}^{A}$; also co-intuitionistic supplement can be defined as $\frown C := \mathbf{j} \smallsetminus C$ if we have subtraction $C \smallsetminus D$ and $\mathbf{j}$ (*justified*) in $\mathcal{L}^{H}$.

[3] Expanding the definitions, we see that $\boxdot \diamond A \equiv \sim\sim A$ and $\diamond \boxdot C \equiv \frown\frown C$.

In our context we have the following structures.

1. Define the logic **A** as the purely intuitionistic part of **AH** on the language $\mathcal{L}^A$. Let $\mathcal{A}$ be the free cartesian category on the syntax of **A**, i.e., with formulas $\mathcal{L}^A$ as objects and (equivalence classes of) intuitionistic sequent calculus derivations on **A** as morphisms, with additional structure to model intuitionistic negation ($\sim$).
2. Similarly, define the logic **H** as the purely co-intuitionistic part of **AH** on the language $\mathcal{L}^H$ and let $\mathcal{H}$ be the free co-cartesian category on the syntax of **H**, with additional structure to model co-intuitionistic supplement ($\frown$).
3. We claimed that both a contravariant functor $(\ )^* \colon \mathcal{A} \to \mathcal{H}^{op}$ and its inverse can be defined from the action of the two connectives $(\ )^\perp$ of **AH** on the formulas and proofs of **A** and of **H**. Thus we assumed that the functor $(\ )^*$ represents a notion of duality between the models of **A** and of **H** and that its definition on proofs can be given through the sequent calculus **AH-G1**.
4. The functors $L = \Diamond$ and $R = \Box$ are defined on objects as in (1). The **AH-G1** proofs of (2) can be interpreted as the unit and the co-unit of the adjunction, i.e., proofs η of $A; \Rightarrow \Box\Diamond A;$ and ϵ of $; \Diamond\Box\ C \Rightarrow ; C$.

Remark 2. (i) In our definition, $R(C) = \Box C = \sim(C^\perp)$ and $L(A) = \Diamond A = \frown(A^\perp)$ express "notions of double negations" and are covariant, so that a proof of $A; \Rightarrow B$ is mapped to $; \Diamond A \to ; \Diamond B$ and similarly $; C \Rightarrow ; D$ is mapped to $\Box C; \Rightarrow \Box D;$. In fact we are trying to characterize properties of the interaction of the connectives $(\ )^\perp$ with intuitionistic negation and co-intuitionistic supplement. Simpler notions of chirality, such as *cartesian closed chiralities* (see [7, Section 1]), may also be explored in bi-intuitionism.

(ii) In this note we only address the definition of the duality functor $(\ )^*$, assuming that it represents a notion of duality between $\mathcal{A}$ and $\mathcal{H}$, which is based on a duality of the logics **A** and **H**, and that the duality of logics corresponds to a duality in the **S4** translation.

Logic and dualities

There is an obvious oversight in the interpretation of duality in *"polarized" bi-intuitionism* **AH** that undermines the main claim (Proposition 4.4), i.e., that *the free categorical model built from the syntax of* **AH** *can be given a chirality-like structure.* Once the error is removed, a more complex structure emerges.

Indeed the logics **A** and **H** do not represent a duality, as we can see from an informal argument and from notion of duality in the **S4** translation. Informally, the dual of an assertion that p is the *hypothesis of the negation of* p; the dual of a hypothesis that p is the *assertion of the negation of* p.

Consider an *elementary assertion* $\vdash p$ in $\mathcal{L}^A$. In **S4** the dual of $(\vdash p)^M = \Box p$ is $\neg\Box p = \Diamond\neg p$. Although *in the logic* **AH** $((\vdash p)^\perp)^M = \neg\Box p$, in the language $\mathcal{L}^H$ we could only have $\Diamond\neg p = (\mathcal{H} H)^M$ and the only formula H such that $(\mathcal{H} H)^M = \Diamond\neg p$ is $\neg p$; but $\mathcal{H}\neg p \notin \mathcal{L}^H$. Thus $(\vdash p)^* = \mathcal{H}\neg p$ is the only possible choice for a duality map $(\ )^*$ compatible with the **S4** translation. *Notice that here '*$\neg$*' represent* classical *negation, not intuitionistic negation nor co-intuitionistic supplement.*

Symmetrically, the dual in **S4** of $(\mathcal{H} p)^M = \Diamond p$ is $\neg\Diamond p = \Box\neg p = (\vdash\neg p)^M$; in **AH** $((\mathcal{H} p)^\perp)^M = \neg\Diamond p$ but $\vdash\neg p \notin \mathcal{L}^A$; also $(\mathcal{H} p)^* = \vdash\neg p$ is the only possible choice for a duality map compatible with the **S4** translation. On the other hand, intuitionistic and co-intuitionistic *connectives* are actually dual.

We have the following definition of duality in our *bi-intuitionistic logic of assertions and hypotheses.*

Definition 1. Consider the languages $\mathcal{L}^{H_*}$ and $\mathcal{L}^{A_*}$ generated by the following grammars:

$$\begin{array}{lrcl} \mathcal{L}^{H_*}: & C, D & := & \mathcal{H}\neg p \mid \curlywedge \mid C \curlyvee D \mid \frown C \\ \mathcal{L}^{A_*}: & A, B & := & \vdash\neg p \mid \curlyvee \mid A \cap B \mid \sim A. \end{array}$$

Now we define the languages $\mathcal{L}^{AH_*}$ and $\mathcal{L}^{A_* H}$:

$$\mathcal{L}^{AH_*}: \begin{array}{lrcl} \mathcal{L}^{A}: & A, B & := & \vdash p \mid \curlyvee \mid A \cap B \mid \sim A \mid \quad [\, C^\perp \,] \\ \mathcal{L}^{H_*}: & C, D & := & \mathcal{H}\neg p \mid \curlywedge \mid C \curlyvee D \mid \frown C \mid \quad [\, A^\perp \,] \end{array}$$

$$\mathcal{L}^{A_* H}: \begin{array}{lrcl} \mathcal{L}^{A_*}: & A, B & := & \vdash\neg p \mid \curlyvee \mid A \cap B \mid \sim A \mid \quad [\, C^\perp \,] \\ \mathcal{L}^{H}: & C, D & := & \mathcal{H} p \mid \curlywedge \mid C \curlyvee D \mid \frown C \mid \quad [\, A^\perp \,] \end{array}$$

Then we have the following duality maps:[4]

[4] As pointed out by Crolard [5, p. 160], in Rauszer's bi-intuitionism (Heyting-Brouwer algebras) there is a *pseudo-duality* between intuitionism and co-intuitionism, since "atoms are unchanged" by the duality. Things are different in a logic of assertions and hypotheses. The correct definition was given in [2, Section 2.3, Definition 5], where the dual of $\vdash p$ is $\mathcal{H}\neg p$. The solution in Section 5 is close to the one suggested

$$\begin{array}{rcl} (\ )^* & := & \mathcal{L}^A \to \mathcal{L}^{H_*}: \\ (\vdash p)^* & = & \mathcal{H}\neg p \\ (\curlyvee)^* & = & \curlywedge \\ (A \cap B)^* & = & A^* \curlyvee B^* \\ (\sim A)^* & = & \frown(A^*) \end{array} \qquad \begin{array}{rcl} (\ )^* & := & \mathcal{L}^H \to \mathcal{L}^{A_*}: \\ (\mathcal{H} p)^* & = & \vdash \neg p \\ (\curlywedge)^* & = & \curlyvee \\ (C \curlyvee D)^* & = & C^* \cap D^* \\ (\frown C)^* & = & \sim C^* \end{array}$$

Proposition 1. *The maps* $(\ )^* \colon \mathcal{L}^A \to \mathcal{L}^{H_*}$ *and* $(\ )^* \colon \mathcal{L}^H \to \mathcal{L}^{A_*}$ *are invertible.*

Then the internal *duality connectives* $A^\perp$ and $C^\perp$ of can be interpreted by the duality maps of $\mathcal{L}^{AH_*}$ and of $\mathcal{L}^{A_*H}$. Namely, for A and C in $\mathcal{L}^{AH_*}$

$$A^\perp = A^* \qquad C^\perp = C^*$$

and similarly for A and C in $\mathcal{L}^{A_*H}$.

The sequent calculus **AH-G1** on the language $\mathcal{L}^{AH_*}$ allows us to extend the duality maps $(\ )^*$ on formulas to maps on proofs

$$A; \Rightarrow B; \quad \mapsto \quad ; B^* \Rightarrow ; A^*$$

Therefore we can define the following data:

1. A functor $(\ )^* \colon \mathcal{A} \to \mathcal{H}_*$ sending $\vdash p$ to $\mathcal{H}\neg p \in \mathcal{L}^{H_*}$; it has an inverse functor $(\ )^* \colon \mathcal{H}_* \to \mathcal{A}$ sending $\mathcal{H}\neg p$ to $\vdash p$.
2. A functor $(\ )^* \colon \mathcal{H} \to \mathcal{A}_*$ sending $\mathcal{H}p$ to $\vdash \neg p \in \mathcal{L}^{A_*}$ with inverse $(\ )^* \colon \mathcal{A}_* \to \mathcal{H}$.
3. A covariant functor $L = \diamonddot \colon \mathcal{A} \to \mathcal{H}_*$, left adjoint of the functor $R = \boxdot \colon \mathcal{H}_* \to \mathcal{A}$.
4. There is another pair of covariant adjoint functors $R' = \boxdot \colon \mathcal{H} \to \mathcal{A}_*$ and $L' = \diamonddot \colon \mathcal{A}_* \to \mathcal{H}$.

Question. From our data can we define *two chiralitiy-like structures* in the logics $\mathbf{AH}_*$ and $\mathbf{A}_*\mathbf{H}$ over the languages $\mathcal{L}^{AH_*}$ and $\mathcal{L}^{A_*H}$?

To answer the question one should show how the sequent calculus **AH-G1** over the new languages could be used to define the categorical structures. Further questions on the present formulation of bi-intuitionism and duality are asked in the conclusion.

here: elementary formulas with non-atomic radical are admitted. Also in [1, Section 2.3 definition 3] the correct definition of duality is considered. A loose usage of the expression"duality between assertions and hypotheses" within a system of bi-intuitionistic logic can be found in those papers and also in [4].

Notice that since the actions of $(\;)^{\perp}$ and $(\;)^{*}$ coincide, we can use the duality $(\;)^{*}$ to eliminate the $(\;)^{\perp}$ connectives, as shown in the following example.

Example 1. Consider the expression

$$;L(\mathtt{a}) \Rightarrow ; \mathtt{m}^{*} \vee L(\mathtt{m} \wedge \mathtt{a}), \tag{3}$$

where both $\mathtt{a} = {\vdash} a$ and $\mathtt{m} = {\vdash} m$ belong to $\mathcal{L}^{A}$. After expanding the definitions the sequent (3) is provable in **AH-G1** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{m}; \Rightarrow \mathtt{m}; \qquad \mathtt{a}; \Rightarrow \mathtt{a};}{\mathtt{m}, \mathtt{a}; \Rightarrow \mathtt{m} \cap \mathtt{a};}\ \cap\,\mathrm{R}}{; (\mathtt{m} \cap \mathtt{a})^{\perp} \Rightarrow ; \mathtt{m}^{\perp}, \mathtt{a}^{\perp}}\ \perp\mathrm{R}, \perp\mathrm{R}, \perp\mathrm{L}}{; \frown(\mathtt{a}^{\perp}) \Rightarrow ; \mathtt{m}^{\perp}, \frown(\mathtt{m} \cap \mathtt{a})^{\perp}}\ \frown\mathrm{R}, \frown\mathrm{L}}{; \frown(\mathtt{a}^{\perp}) \Rightarrow ; \mathtt{m}^{\perp} \curlyvee \frown(\mathtt{m} \cap \mathtt{a})^{\perp}}\ \curlyvee\,\mathrm{R}$$

Applying the map $(\;)^{*} \colon \mathcal{L}^{A} \to \mathcal{L}^{H_{*}}$, only to eliminate the $(\;)^{\perp}$ connectives, the sequent (3) is transformed as follows:

$$; \frown(\mathcal{H}\neg a) \Rightarrow ; (\mathcal{H}\neg m) \curlyvee \frown(\mathcal{H}\neg m \vee \mathcal{H}\neg a)\ .$$

Thus, the proof of (3) is in the language $\mathcal{L}^{AH_{*}}$, but can be transformed into a proof in $\mathbf{H}_{*}$. On the other hand, applying $(\;)^{*}$ to the sequent (3), one obtains a proof in **A** of

$${\vdash} m \ \cap\ \sim({\vdash} m \ \cap\ {\vdash} a); \Rightarrow \sim {\vdash} a\ .$$

However, other cases are not covered by the above definitions.

Example 2. Consider the formal expression

$$\mathtt{m} \wedge R(\mathtt{m}^{*} \vee \mathtt{b}); \Rightarrow R(\mathtt{b}); \tag{4}$$

where $\mathtt{m} = {\vdash} m \in \mathcal{L}^{A}$ and $\mathtt{b} = \mathcal{H} b \in \mathcal{L}^{H}$. After expanding the definitions the sequent (4) becomes

$$\mathtt{m}\ \cap \sim(\mathtt{m}^{\perp} \curlyvee \mathtt{b})^{\perp}; \Rightarrow \sim(\mathtt{b}^{\perp});$$

But applying the map $(\;)^{*} \colon \mathcal{L}^{A} \to \mathcal{L}^{H_{*}}$ we obtain $\mathtt{m}^{\perp} = \mathcal{H}\neg m$ and now $\mathcal{H}\neg m \curlyvee \mathcal{H} b$ does not belong to $\mathcal{L}^{H}$.

Conclusions and further questions

In conclusion, it seems that a grammar for a language formally expressing our notions of duality should be as follows:

$$\mathcal{L}^{AA_*HH_*}: \quad \begin{array}{lll} \mathcal{L}^{AA_*}: & A,B := & \vdash p \mid \vdash\neg p \mid \curlyvee \mid A\cap B \mid \sim A \mid \ [C^{\perp}] \\ \mathcal{L}^{H_*H}: & C,D := & \mathcal{H}p^* \mid \mathcal{H}p \mid \curlywedge \mid C\curlyvee D \mid \frown C \mid \ [A^{\perp}] \end{array}$$

One can define maps $(\ )^*\colon \mathcal{L}^{AA_*} \to \mathcal{L}^{HH_*}$ and $(\ )^*\colon \mathcal{L}^{HH_*} \to \mathcal{L}^{AA_*}$ so that the sequent (4) becomes

$$\vdash m \ \cap \ \sim(\vdash m \cap \vdash\neg b); \Rightarrow \ \sim \vdash\neg b; \ .$$

However, the sequent calculus **AH-G1** over the language $\mathcal{L}^{AA_*HH_*}$ is no longer complete for the **S4** semantics.

Perhaps one can say that a pragmatic interpretation of bi-intuitionistic logic suitable for representing bi-intuitionistic dualities is the logic $\mathbf{AA_*HH_*}$ of *assertions*, *objections*, *hypotheses* and *denials*, where an *objection* to the assertion $\vdash p$ is the hypothesis $\mathcal{H}\neg p$ that p is not true and a *denial* of a hypothesis $\mathcal{H}p$ is the assertion $\vdash\neg p$ that p is false. Thus all elementary formulas of the forms $\vdash p$, $\vdash\neg p$, $\mathcal{H}p$ and $\mathcal{H}\neg p$ must belong to the language of $\mathbf{AA_*HH_*}$. We expect that an axiomatization of $\mathbf{AA_*HH_*}$ can be obtained by the sequent calculus **AH-G1** together with the following *proper axioms* that express logical relations between the elementary formulas according to their intended meaning. We conjecture that such a sequent calculus is sound and complete for the **S4** semantics and enjoys the cut-elimination property.

$$\textbf{Proper axioms of } \mathbf{AA_*HH_*} \qquad \begin{array}{ll} \vdash p;\ \mathcal{H}\neg p \Rightarrow; & ; \Rightarrow \ \vdash p;\ \mathcal{H}\neg p \\ \vdash\neg p;\ \mathcal{H}p \Rightarrow; & ; \Rightarrow \ \vdash\neg p;\ \mathcal{H}p \\ \vdash p,\ \vdash\neg p; \Rightarrow \mathbf{u}; & ; \mathbf{j} \Rightarrow;\ \mathcal{H}p,\ \mathcal{H}\neg p \\ \vdash p,\ \vdash\neg p; \mathbf{j} \Rightarrow; & ; \Rightarrow \mathbf{u};\ \mathcal{H}p,\ \mathcal{H}\neg p \end{array}$$

Remark 3. In the modal translation we have $(\mathcal{H}\neg p)^M = ((\vdash p)^{\perp})^M$ and $(\vdash\neg p)^M = ((\mathcal{H}p)^{\perp})^M$. Notice that if we replace $\mathcal{H}\neg p$ and $\vdash\neg p$ with their counterparts $(\vdash p)^{\perp}$ and $(\mathcal{H}p)^{\perp}$, respectively, then the *Proper Axioms of* $\mathbf{AA_*HH_*}$ become provable in **AH-G1**. The first four are proved trivially; the last four require the proper axioms of assertions and hypotheses

$$\vdash p; \mathbf{j} \Rightarrow;\ \mathcal{H}p \qquad \text{and} \qquad \vdash p; \Rightarrow \mathbf{u};\ \mathcal{H}p \tag{5}$$

The axioms (5) break the symmetry between assertions and hypotheses: here logic prevails over symmetry. But they are needed here to guarantee the coherence of *two systems of duality*.

There are more general questions about the proof-theory of our logics and of the sequent calculus **AH-G1** which we can only mention briefly here.

Remark 4. (i) The expressions '$\vdash\neg p$' for *denial that* p and '$\mathcal{H}\neg p$' for *objection to* p appear to formalize classical notions, given that '$\neg$' is classical negation. Indeed the assertion of a classical negation can be regarded as an intuitionistic statement only under special conditions such as the decidability of p. Is the logic $\mathbf{AA_*HH_*}$ an intermediate logic between intuitionistic and classical logic?[5]

(ii) The connectives $(A)^{\perp}$ and $(C)^{\perp}$ have the meaning of negations. Their main property

$$(A)^{\perp\perp} \equiv A \qquad \text{and} \qquad (C)^{\perp\perp} \equiv C \tag{6}$$

makes it possible to represent the functors $(\ )^*$ within the calculus **AH-G1**. But are these *intuitionistically acceptable connectives*? This is presupposed in our interpretation of bi-intuitionism, but it has not been argued for explicitly.

The form of the *implication right* rule

$$\frac{\Theta, A_1\ ;\ \Rightarrow\ A_2\ ; \Upsilon}{\Theta\ ;\ \Rightarrow\ A_1 \supset A_2\ ; \Upsilon}$$

allowing extra formulas Υ in the sequent premise without restrictions, and similarly of the *subtraction left*

$$\frac{\Theta\ ;\ C_1\ \Rightarrow\ ; C_2, \Upsilon}{\Theta\ ;\ A_1 \supset A_2\ ; \Upsilon}$$

allowing extra formulas Θ in the sequent premise, is equivalent to allowing the connectives $(\ )^{\perp}$ with the properties (6) in a calculus with cut-elimination (see [1, Section 2.4]). This feature is characteristic of the calculus **AH-G1** in opposition to the tradition of Rauszer's bi-intuitionism. However, the interaction between intuitionistic and co-intuitionistic logic may take different forms and be formalized in different ways than through the connectives $(\ )^{\perp}$. A definition of intuitionistic dualities that would be less dependant on duality in the **S4** translation is certainly desirable.

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