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THE RM PARACONSISTENT REFUTATION SYSTEM

Abstract. The aim of this paper is to study the refutation system consisting of the refutation axiom \( p \land \neg p \rightarrow q \) and the refutation rules: reverse substitution and reverse modus ponens \((B/A, \text{if } A \rightarrow B \in \text{RM})\). It is shown that the refutation system is characteristic for the logic of the 3-element \(\text{RM}\) algebra.

Keywords: refutation systems, paraconsistent logic, relevance logic.

1. Introduction

A refutation system is an inference system consisting of some refutation axioms (which are non-valid formulas) and some refutation rules (which are inference rules preserving non-validity) (see [2]). Refutation systems can be regarded as alternative axiom systems capturing some intuitions about non-valid formulas as well as valid ones. It seems worth investigating such systems in paraconsistent logics, which are defined as non-classical logics rejecting the explosive law \((E) := p \land \neg p \rightarrow q\) (cf. [3]). In this paper we study the refutation system consisting of the refutation axiom \((E)\) and the refutation rules: reverse substitution and reverse modus ponens \((B/A, \text{where } A \rightarrow B \in \text{RM})\). It is shown that this refutation system generates the set of formulas non-valid in the 3-element \(\text{RM}\) algebra. The resulting paraconsistent logic (that is, the set of formulas non-refutable in this system) is simple (3-valued), natural (i.e. \((E)\) is rejected and refutability is justified by derivability in \(\text{RM}\); a useful standard relevance logic), and maximal.
2. Preliminaries

Let \( \text{FOR} \) be the set of formulas generated from a set \( \text{VAR} = \{p, q, \ldots\} \) of propositional variables by the connectives: \( \neg, \land, \lor, \rightarrow \). We define

\[
A \equiv B := (A \rightarrow B) \land (B \rightarrow A).
\]

\( \text{RM} \) is the set of formulas provable in the following axiom system.

**Axioms:**

\[
\begin{align*}
A & \rightarrow A \\
(A \rightarrow B) & \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \\
A & \rightarrow ((A \rightarrow B) \rightarrow B) \\
(A \rightarrow (A \rightarrow B)) & \rightarrow (A \rightarrow B) \\
A & \rightarrow (A \rightarrow A) \\
A \land B & \rightarrow A \\
A \land B & \rightarrow B \\
((A \rightarrow B) \land (A \rightarrow C)) & \rightarrow (A \rightarrow B \land C) \\
A & \rightarrow A \lor B \\
B & \rightarrow A \lor B \\
((A \rightarrow C) \land (B \rightarrow C)) & \rightarrow (A \lor B \rightarrow C) \\
(A \land (B \lor C)) & \rightarrow ((A \land B) \lor C) \\
(A \rightarrow \neg B) & \rightarrow (B \rightarrow \neg A) \\
\neg\neg A & \rightarrow A
\end{align*}
\]

**Rules:**

\[
\begin{align*}
\text{(modus ponens)} & \quad \frac{A}{B} \\
\text{(adjunction)} & \quad \frac{A \quad B}{A \land B}
\end{align*}
\]

\( \text{RM} \) can be characterized by the matrix \( \mathbf{M} = \langle \mathbb{Q}, \mathcal{D}, -, \land, \lor, \rightarrow \rangle \) (see [1]), where \( \mathbb{Q} \) is the set of rational numbers, \( \mathcal{D} := \{x \in \mathbb{Q} : x \geq 0\} \), and

\[
\begin{align*}
x \land y & := \min(x, y), \\
x \lor y & := \max(x, y), \\
x \rightarrow y & := \begin{cases} 
\max(-x, y) & \text{if } x \leq y, \\
\min(-x, y) & \text{otherwise.}
\end{cases}
\end{align*}
\]
Thus \textbf{RM} is the set of formulas valid in \textbf{M}, that is, \( A \in \textbf{RM} \) iff \( v(A) \in \textbf{D} \) for every valuation \( v \) in \textbf{M}.

We take for granted the following \textbf{RM} laws:

(1) \((A \to (B \to C)) \to (B \to (A \to C))\) \(\)  
\((A \to (B \to C)) \to ((A \to B) \to (A \to C))\) \(\)  
\(A \land B \equiv B \land A\) \(\)  
\(A \lor B \equiv B \lor A\) \(\)

(2) \((A \to (B \equiv C)) \to ((A \to (C \equiv D)) \to (A \to (B \equiv D)))\) \(\)
\((B \equiv C) \to (D \equiv D(B/C))\) \(\)

(3) \((A \to (B \equiv C)) \to (A \to (D \equiv D(B/C)))\) \(\)

where \(D(B/C)\) results from \(D\) by replacing some occurrences of \(B\) by \(C\).

3. Validity

Let \(P := p \land \neg p\) and \(Q := q \land \neg q\).

\textbf{Lemma 1.} The following formulas are in \textbf{RM}:

\[
\begin{align*}
P & \to (\neg Q \equiv \neg Q) \\
P & \to (\neg \neg Q \equiv Q) \\
P & \to (\neg P \equiv P) \\
P & \to (Q \land \neg Q \equiv Q) \\
P & \to (P \land Q \equiv Q) \\
P & \to (P \land \neg Q \equiv P) \\
P & \to (Q \lor \neg Q \equiv \neg Q) \\
P & \to (P \lor Q \equiv P) \\
P & \to (P \lor \neg Q \equiv \neg Q) \\
P & \to ((Q \to Q) \equiv \neg Q) \\
P & \to ((P \to P) \equiv P) \\
P & \to ((\neg Q \to \neg Q) \equiv \neg Q) \\
P & \to ((Q \to \neg Q) \equiv \neg Q) \\
P & \to ((\neg Q \to Q) \equiv Q) \\
P & \to ((P \to Q) \equiv Q)
\end{align*}
\]
\[ P \to ((Q \to P) \equiv \neg Q) \]
\[ P \to ((P \to \neg Q) \equiv \neg Q) \]
\[ P \to ((\neg Q \to P) \equiv Q) \]

**Proof.** First we note the following simple facts. Let \( x, y \in Q \). We put

\[
X := x \land \neg x, \quad Y := y \land \neg y, \quad \text{and} \quad Z := \{X, \neg X, Y, \neg Y\}.
\]

Then we have:

(I) \( X \leq 0 \) and \( Y \leq 0 \).

(II) If \( a, b \in Z \) then \(-a, a \land b, a \lor b, a \to b \in Z\).

Next we consider the above formulas. They are of the form

\[ P \to A(P, Q) \]

Now let \( v \) be any valuation in \( M \). Then, by (II), we have

\[ (\ast) \quad v(A(P, Q)) \in \{v(P), -v(P), v(Q), -v(Q)\}. \]

For \( v \) we consider two cases.

**Case 1.** \( v(P) \leq v(Q) \). Then, by (I) and (\( \ast \)), we get \( v(P) \leq v(A(P, Q)) \).

Hence \( v(P \to A(P, Q)) = \max(-v(P), v(A(P, Q))) \geq 0 \).

**Case 2.** \( v(P) > v(Q) \). Then it is easy to check that

\[ (\ast\ast) \quad v(A(P, Q)) \in \{v(P), -v(P), -v(Q)\}. \]

We give details only for the cases eighth, fourteenth, and eighteenth; the other ones being similar.

\[
v(P \lor Q \equiv P) = v(P \lor Q \to P) \land v(P \to P \lor Q) = v(P \to P) \land v(P \to P) = \max(-v(P), v(P)) = -v(P), \text{ because } v(P \lor Q) = v(P).
\]

\[
v((-Q \to Q) \equiv Q) = v((-Q \to Q) \to Q) \land v(Q \to (-Q \to Q)) = v(Q \to Q) \land v(Q \to Q) = -v(Q), \text{ because } -v(Q) > v(Q).
\]

\[
v((-Q \to P) \equiv Q) = v((-Q \to P) \to Q) \land v(Q \to (-Q \to P)) = v(Q \to Q) \land v(Q \to Q) = -v(Q), \text{ because } -v(Q) > v(P).
\]

Therefore, by (I) and (\( \ast\ast \)), \( v(P) \leq v(A(P, Q)) \), and so \( v(P \to A(P, Q)) = \max(v(-P), v(A(P, Q))) \geq 0 \).

Thus, for any valuation \( v \) in \( M \) we have \( v(P) \leq v(A(P, Q)) \), and so \( v(P \to A(P, Q)) \geq 0 \) which gives the result. \( \square \)
4. Refutability

Let $\mathbf{3}$ be the submatrix $\langle \{-1, 0, 1\}, \{0, 1\}, -, \land, \lor, \rightarrow \rangle$ of $\mathbf{M}$. We put: $G_{-1} := Q$, $G_{0} := P$, and $G_{1} := \neg Q$. For any valuation $v$ in $\mathbf{3}$, let $s_v$ be the following substitution:

$$s_v(A) = G_{v(A)} \quad \text{(for any } A \in \text{VAR}).$$

**Lemma 2.** For any $B \in \text{FOR}$ we have $P \rightarrow (s_v(B) \equiv G_{v(B)}) \in \text{RM}.$

**Proof.** By induction on the complexity of $B$.

Let $B \in \text{VAR}$. Then this is true, because $s_v(B) = G_{v(B)}$ and $v(s_v(B) \equiv G_{v(B)}) \geq 0$.

Let $B \notin \text{VAR}$. We assume that the lemma holds for formulas simpler than $B$. Then

$$B \in \{-C, C \land D, C \lor D, C \rightarrow D\}$$

and by the induction hypothesis we have

$$P \rightarrow (s_v(C) \equiv G_{v(C)}) \in \text{RM},$$

$$P \rightarrow (s_v(D) \equiv G_{v(D)}) \in \text{RM}.$$  

Hence, by (3) and *modus ponens*, we get

$$P \rightarrow (\neg s_v(C) \equiv \neg G_{v(C)}) \in \text{RM},$$

$$P \rightarrow ((s_v(C) \otimes s_v(D)) \equiv (G_{v(C)} \otimes G_{v(D)})) \in \text{RM},$$

where $\otimes \in \{\land, \lor, \rightarrow\}$. Since by Lemma 1 we have

$$P \rightarrow (\neg G_{v(C)} \equiv G_{v(\neg C)}) \in \text{RM},$$

$$P \rightarrow ((G_{v(C)} \otimes G_{v(D)}) \equiv (G_{v(C \otimes D)})) \in \text{RM},$$

by (2) and *modus ponens* we obtain

$$P \rightarrow (s_v(B) \equiv G_{v(B)}) \in \text{RM}$$

as required.

We say that a formula is *refutable* iff it is derivable in the following refutation system.

\[\square\]
Refutation axiom:
(E) \( p \land \neg p \rightarrow q \)

Refutation rules:
(reverse substitution) \( B/A \), if \( B \) is a substitution instance of \( A \).
(reverse modus ponens) \( B/A \), if \( A \rightarrow B \in RM \).

THEOREM. A formula is refutable if and only if it is not valid in 3.

PROOF. \((\Rightarrow)\) This follows from the fact that (E) is not valid in 3 and the refutation rules preserve non-validity in 3.

\((\Leftarrow)\) Assume that \( A \) is not valid in 3. Then \( v(A) = -1 \) for some valuation \( v \) in 3, so \( G_{v(A)} = G_{-1} := q \land \neg q \). By Lemma 2 we have
\[
P \rightarrow (s_v(A) \equiv q \land \neg q) \in RM.
\]
Hence
\[
P \rightarrow (s_v(A) \rightarrow q) \in RM,
\]
so, by (1) and modus ponens, we obtain
\[
s_v(A) \rightarrow (p \land \neg p \rightarrow q) \in RM.
\]
Therefore \( s_v(A) \) is refutable, by reverse modus ponens and (E), and so \( A \) is refutable, by reverse substitution, which was to be shown.

References

