PARAINCONSISTENCY OF CREDIBILITY-BASED BELIEF STATES*

Abstract. In our approach credibility of information plays an important role in modeling of both belief state and belief change [4]. It turns out that the credibility-based consequence operators used to define the notion of belief state tolerate inconsistency under some conditions.

Keywords: parainconsistency, credibility of information, belief state.

1. Introduction

In [4] we proposed a general model of belief state and belief change, where both rationality of an agent as well as credibility of information were taken into account. In the present paper we develop some of the ideas, concerning the concept of belief state, presented there and discuss the problem of parainconsistency of such belief states.

It is hardly an exaggeration to say that our notion of belief state is credibility-based, while rationality, if understood along the standard lines, plays an auxiliary role only. It is a simple consequence of the fact that the consequence operators used to define a belief state are “credibility-oriented”. Moreover, these operators tolerate inconsistency of sets of formulae, which is another interesting feature.

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Information and beliefs, viewed as sufficiently credible information, are represented by formulae of the classical propositional logic (PC). Starting with credibility of information, expressed in the form of pairs of real numbers attached to formulae, we arrive at credibility-based preference relations on formulae. Any such preference relation induces a consequence operator in the Tarski sense. Next, information is classified into three classes: highly credible information, lowly credible information, and the rest. Finally, a notion of belief state is defined by means of our consequence operators and the concept of high and low credibility. Belief states are pairs of sets of formulae that are closed under the consequence operators mentioned above and consist of sufficiently credible information. The first element of a belief state defined in such a way is called a belief set. It contains stronger beliefs and is a subset of the second element called a plausibility set. The plausibility set may contain weaker beliefs as well.

When information comes from several sources, it can happen that there is much evidence both for a considered fact as well as for its negation. As a consequence, both a piece of information and its negation may be highly (or at least not-lowly) credible to a given agent. On the other hand, contradictory statements like $p \land \neg p$ are not believed in general. Along the standard lines [17], a set of formulae $X$ of a given language is parainconsistent (or, in other words, paraconsistent) if there is a formula $\alpha$ such that $\alpha$ and $\neg \alpha$ can be derived from it by means of a considered consequence operation but the set of all consequences of $X$ is a proper subset of the set of all formulae. In our case we are faced with a credibility-based form of parainconsistency of an agent’s state of beliefs. In the paper we consider this form of parainconsistency and investigate whether (and under what conditions if answered positively) it may be the case that (i) both a formula and its negation are highly credible; (ii) a formula is highly credible and its negation is not lowly credible; and (iii) neither a formula nor its negation are lowly credible. Similarly, we answer the question whether and under what conditions it may be the case that (i) both a formula and its negation belong to the same belief set; (ii) a formula belongs to a belief set, while its negation is a member of the corresponding plausibility set; (iii) both a formula and its negation are members of the same plausibility set. Finally, we obtain necessary conditions for belief and plausibility sets to be parainconsistent in our sense.

Section 2 contains preliminaries. Among others, we define a consequence operator which is based on a preference relation and which plays a fundamental role in defining of the notion of belief state. In Section 3 we present some notions concerning credibility of information, recall their properties...
given in [4], and also give new results useful or simply interesting from our standpoint. In particular, we recall such notions as pre-degree of credibility, credibility mapping, degree of credibility, credibility-based preference relation, and low and high credibility. In Section 4 we define a notion of belief state based on credibility of information and investigate the question of parainconsistency. Final remarks are given in Section 5.

2. Preliminaries

Information and, in particular, beliefs are represented by formulae of the classical propositional logic (PC). Propositional letters are denoted by \( p, q \) with sub/superscripts if needed, propositional connectives by \( \land, \lor, \rightarrow, \leftrightarrow, \) and \( \neg \), formulae, formed along the standard lines, by \( \alpha, \beta, \gamma \) with sub/superscripts if needed, the set of all formulae by \( \text{FOR} \), and the power set of a set \( X \) by \( \mathcal{P}(X) \).

An operator \( C : \mathcal{P}(\text{FOR}) \rightarrow \mathcal{P}(\text{FOR}) \) is a consequence operator in the Tarski sense if for any \( X, Y \subseteq \text{FOR} \), \( C \) is increasing (i.e., \( X \subseteq C(X) \)), monotonic (\( C(X) \subseteq C(Y) \) if \( X \subseteq Y \)), idempotent (\( C(C(X)) \subseteq C(X) \)), and compact (\( C(X) = \bigcup\{C(X') \mid X' \subseteq X \land \text{card}(X') < \aleph_0\} \)). \( Cn \) denotes the classical consequence operator. Whenever convenient, \( \alpha \in Cn(X) \) and \( X \vdash \alpha \) are used interchangeably, while \( \text{TAUT} \) stands for the set \( Cn(\emptyset) \) of all PC-tautologies. \( \text{co-TAUT} \) denotes the set of all PC-countertautologies, that is:

\[
\text{co-TAUT} = \{\alpha \in \text{FOR} \mid \neg\alpha \in \text{TAUT}\}.
\]

A set \( X \subseteq \text{FOR} \) is \( C \)-closed if \( C(X) = X \). It is absolutely (resp., traditionally) \( C \)-consistent iff \( C(X) \neq \text{FOR} \) (resp., \( -(\exists \alpha)(\alpha, \neg\alpha \in C(X)) \)); otherwise \( X \) is absolutely (traditionally) \( C \)-inconsistent.\(^1\) Along the standard lines [17], sets of formulae traditionally \( C \)-inconsistent and absolutely \( C \)-consistent are called \( C \)-parainconsistent (or paraconsistent when looking from another perspective). An intersection \( C_1 \cap C_2 \) of two consequence operators \( C_1 \) and \( C_2 \), defined as

\[
(C_1 \cap C_2)(X) = C_1(X) \cap C_2(X),
\]

is a consequence operator.

A relation \( \preceq \subseteq X^2 \) is a pre-ordering on \( X \), while \( (X, \preceq) \) is a pre-ordered set if \( \preceq \) is reflexive and transitive. The corresponding relations \( \prec \) (strict

\(^1\) In general, absolute and traditional \( C \)-consistency are different notions. However, they coincide for \( C = Cn \). Clearly, traditional \( C \)-consistency implies the absolute one.
pre-ordering) and \( \approx \) on \( X \) are defined as usual, i.e., for any \( x, y \in X \):

\[
\begin{align*}
    x &< y \iff x \leq y \land y \not< x.
    \\
    x &\approx y \iff x \leq y \land y \leq x.
\end{align*}
\]

Pre-orderings may be viewed as preference relations, where \( x \leq y \) (resp., \( x < y \), \( x \approx y \)) is read as ‘\( y \) is preferred to \( x \)’ (‘\( y \) is strictly preferred to \( x \)’, ‘\( x \) and \( y \) are equally preferred’). Along the standard lines if \( \leq \) is also antisymmetric, it is called partial ordering and \( (X, \leq) \) is partially ordered. Partial orderings which are also connected (i.e., any two elements are comparable) are called total orderings. Note that \( \approx \) is the equality relation for partial orderings. An element \( x \in X \) is the greatest (resp., least) in a partially ordered set \( (X, \leq) \) iff \( (\forall y \in X) y \leq x \) (resp., \( x \leq y \)).

Let \( D = \{(x, y) \in [0, 1]^2 \mid x \leq y\} \), where \([0, 1]\) is the unit interval and \( \leq \) is the natural total ordering of real numbers. Some pre-orderings on \( D \) are defined below.

\[
\begin{align*}
    (x_1, y_1) &\preceq_1 (x_2, y_2) \iff x_1 \leq x_2 \land y_1 \leq y_2. \\
    (x_1, y_1) &\preceq_2 (x_2, y_2) \iff x_1 < x_2 \lor (x_1 = x_2 \land y_1 \leq y_2). \\
    (x_1, y_1) &\preceq_3 (x_2, y_2) \iff y_1 < y_2 \lor (y_1 = y_2 \land x_1 \leq x_2). \\
    (x_1, y_1) &\preceq_4 (x_2, y_2) \iff y_1 \leq x_2 \lor (x_1 = x_2 \land y_1 = y_2).
\end{align*}
\]

Relation \( \preceq_2 \) (resp., \( \preceq_3 \)) is a total ordering known as the lexicographic (anti-lexicographic) ordering. Relations \( \preceq_1 \) and \( \preceq_4 \) are partial orderings. Let us note that \((1, 1)\) (resp., \((0, 0)\)) is the greatest (least) element in \((D, \preceq_i)\) for \( i = 1, \ldots, 4 \).

**Proposition 2.1.** For any \((x_j, y_j) \in D \ (j = 1, 2)\) and \( i = 1, \ldots, 4 \), we have that:

\[
\begin{align*}
    (x_1, y_1) &\prec_1 (x_2, y_2) \iff (x_1 < x_2 \land y_1 \leq y_2) \lor (x_1 \leq x_2 \land y_1 < y_2). \\
    (x_1, y_1) &\prec_2 (x_2, y_2) \iff x_1 < x_2 \lor (x_1 = x_2 \land y_1 < y_2). \\
    (x_1, y_1) &\prec_3 (x_2, y_2) \iff y_1 < y_2 \lor (y_1 = y_2 \land x_1 < x_2). \\
    (x_1, y_1) &\prec_4 (x_2, y_2) \iff x_1 < y_1 = x_2 \lor y_1 < x_2 \lor y_1 = x_2 < y_2.
\end{align*}
\]

If \((x_1, y_1) \neq (0, 0)\) then \((0, 0) \prec_1 (x_1, y_1)\).

For any pre-ordering \( \leq \subseteq X^2 \), we define two operators \( \nabla_{\leq}, \triangle_{\leq} : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) as follows:

\[
\begin{align*}
    \nabla_{\leq} Y &= \{ x \in X \mid (\exists y \in Y) \ y \leq x \}. \\
    \triangle_{\leq} Y &= \{ x \in X \mid (\exists y \in Y) \ x \leq y \}.
\end{align*}
\]
For simplicity, \( \nabla \leq x \) stands for \( \nabla \leq \{x\} \), and similarly for \( \triangle \leq \). Some basic properties of \( \nabla \leq \) and \( \triangle \leq \) are given below. Easy proofs are left as exercises.

**Proposition 2.2.** Let \( O \) stand for \( \nabla \leq \) or \( \triangle \leq \). For any \( X_1, X_2 \subseteq X \), we have that:

\[
O(X_1) = \emptyset \quad \text{iff} \quad X_1 = \emptyset.
\]

If \( X_1 \subseteq X_2 \) then \( O(X_1) \subseteq O(X_2) \).

\[
X_1 \subseteq O(X_1).
\]

\[
O(O(X_1)) \subseteq O(X_1).
\]

\[
O(X_1) = \bigcup \{O(x) \mid x \in X_1\}.
\]

\[
O(X_1 \cup X_2) = O(X_1) \cup O(X_2).
\]

\[
O(X_1 \cap X_2) \subseteq O(X_1) \cap O(X_2).
\]

**Corollary 2.3.** If \( X = \text{FOR} \), then \( \nabla \leq \) and \( \triangle \leq \) are consequence operators.

### 3. Credibility of information

In [4] we defined a **pre-degree of credibility** of information as a quantity (e.g., a number, a pair of numbers, an interval) describing initially how credible a considered piece of information is in a given situation. Such a quantity may be obtained as the result of some measurement, calculation or logical inference. In the next step pre-degrees are refined to satisfy some rationality postulates. In the present paper pre-degrees of credibility are elements of \( D \), i.e., pairs of real numbers \((x, y)\) such that \(0 \leq x \leq y \leq 1\). This is a generalization of the following situation.

**Example 3.1.** In a group of \( k \) \((0 < k)\) experts or sources of information, \( m \) experts state that \( \alpha \) holds, while \( n \) experts state that \( \neg \alpha \) holds in the sense that it is not the case that \( \alpha \) holds. Assume that experts have consistent opinions, i.e., no one states that both \( \alpha \) and its negation hold simultaneously. However, some experts may be indecisive, i.e., \( m + n \leq k \). Credibility of \( \alpha \) is initially assessed by two fractions \( \frac{m}{k} \) and \( \frac{n}{k} \) of positive and negative answers, respectively. Thus, \( (\frac{m}{k}, 1 - \frac{n}{k}) \) may be taken as the pre-degree of credibility of \( \alpha \). Similarly, the pre-degree of credibility of \( \neg \alpha \) is \( (\frac{n}{k}, 1 - \frac{m}{k}) \).

From a rational point of view, degrees of credibility of information should satisfy some postulates. For instance, it is reasonable to require for logically equivalent formulae to be given the same degree of credibility. Therefore pre-degrees of credibility may need an appropriate adjustment. Let us consider the ordered sets \((D_i, \leq_i) (i = 1, \ldots, 4)\) defined in Section 2. Every mapping
$cr_i: \text{FOR} \rightarrow D$ satisfying the following postulates for any formulae $\alpha, \beta$ is called a credibility mapping and its values are referred to as credibility degrees:

(CR1) \quad \text{If } \alpha \vdash \beta, \text{ then } cr_i(\alpha) \preceq_i cr_i(\beta).

(CR2) \quad \text{If } \vdash \alpha, \text{ then } cr_i(\alpha) = (1, 1).

(CR3) \quad \text{If } \vdash \lnot \alpha, \text{ then } cr_i(\alpha) = (0, 0).

The constraints imposed on credibility by (CR1)–(CR3) are natural. Tautologies are credible to the highest degree as opposite to countertautologies. Logically equivalent pieces of information are equally credible by (CR1). Starting with pre-degrees, one can arrive at credibility degrees \cite{4}. Henceforth we will consider such credibility mappings $cr_i$ that $cr_i(\alpha) = (x, y)$ iff $cr_i(\lnot \alpha) = (1 - y, 1 - x)$.

Every $cr_i$ induces a preference (pre-ordering) relation on $\text{FOR}$, $\preceq_{cr_i}$, as follows:

\begin{align*}
\alpha &\preceq_{cr_i} \beta \quad \text{iff} \quad cr_i(\alpha) \preceq_i cr_i(\beta). \\
\alpha &\prec_{cr_i} \beta \quad \text{iff} \quad cr_i(\alpha) \prec_i cr_i(\beta).
\end{align*}

Relations $\approx_{cr_i}$ and $\prec_{cr_i}$ are defined along the standard lines. Henceforth $\nabla_i$ will stand for $\nabla_{\preceq_{cr_i}}$ for the sake of simplicity, and analogously for $\Delta_{\preceq_{cr_i}}$.

**PROPOSITION 3.2.** For any $X \subseteq \text{FOR}$ and $\alpha, \beta \in \text{FOR}$, we have that:

\begin{align*}
\alpha &\approx_{cr_i} \beta \quad \text{iff} \quad cr_i(\alpha) = cr_i(\beta). \\
\alpha &\prec_{cr_i} \beta \quad \text{iff} \quad cr_i(\alpha) \prec_i cr_i(\beta).
\end{align*}

If $\vdash \alpha \leftrightarrow \beta$, then $\alpha \approx_{cr_i} \beta$.

$\alpha \preceq_{cr_i} \beta \quad \text{iff} \quad \lnot \beta \preceq_{cr_i} \lnot \alpha$, for $i = 1, 4$.

$\alpha \preceq_{cr_3} \beta \quad \text{iff} \quad \lnot \beta \preceq_{cr_3} \lnot \alpha$.

If $\vdash \alpha$, then $(\forall \beta)(\beta \preceq_{cr_i} \alpha \land \beta \approx_{cr_i} (\alpha \land \beta))$.

If $\vdash \lnot \alpha$, then $(\forall \beta)(\alpha \preceq_{cr_i} \beta \land \beta \approx_{cr_i} (\alpha \lor \beta))$.

If $\alpha \vdash \beta$, then $(\forall \gamma)\alpha \land \gamma \preceq_{cr_i} \beta \land \gamma$.

If $X \neq \emptyset$, then $\text{TAUT} \subseteq \nabla_i X \land \text{co-TAUT} \subseteq \Delta_i X$.

$\nabla_i \text{co-TAUT} = \text{FOR} = \Delta_i \text{TAUT}$.

If $X \neq \emptyset$, then $\Delta_i \nabla_i X = \text{FOR} = \nabla_i \Delta_i X$.

$\nabla_i X = \text{FOR} \quad \text{iff} \quad \text{co-TAUT} \cap \nabla_i X \neq \emptyset$.

**PROOF.** We only prove the last property. Assume that there is a countertautology $\alpha \in \nabla_i X$. Hence $cr_i(\alpha) = (0, 0)$ and there is $\beta \in X$ such that $\beta \preceq_{cr_i} \alpha$. Thus $cr_i(\beta) = (0, 0)$ and $(\forall \gamma)\beta \preceq_{cr_i} \gamma$. In other words, $\text{FOR} \subseteq \nabla_i X$. The remaining part is obvious. \hfill $\square$
It is easy to see that operators $\nabla_i$ tolerate $\nabla_i$-inconsistency in the sense that there exist $\nabla_i$-parainconsistent sets of formulae.

Example 3.3. Let $X = \{q_1, q_2\}$, $\text{cr}_i(q_1) = (0.5, 0.6)$, $\text{cr}_i(q_2) = (0.4, 0.4)$, and $\text{cr}_i(p) = (0.5, 0.6)$ ($i = 1, \ldots, 4$). Hence $\text{cr}_i(\neg p) = (0.4, 0.5)$. Clearly, $\nabla_i X \neq \text{FOR}$. On the other hand, $p, \neg p \in \nabla_i X$.

Given credibility of information, one may have to decide whether a particular piece of information is highly or lowly credible. The concepts of high and low credibility are subjective and change with agents, situation or domain of application. In our model we distinguish three classes of information: highly credible information, lowly credible information, and the rest. The rules for decision making are simple: a piece of information is highly (resp., lowly) credible if its degree of credibility is higher (lower) than some threshold. A formula is highly (resp., lowly) credible in case the information represented by the formula is highly (lowly) credible.

Let $l_i(\alpha)$ (resp., $h_i(\alpha)$) denote that $\alpha$ is lowly (highly) credible relative to the ordering $\preceq_i$ ($i = 1, \ldots, 4$) and $k_i$ stand for $h_i$ or $l_i$. Given a pair of threshold values $0 \leq t_1 \leq t_2 \leq 1$, low and high credibility of a formula may be defined as follows:

$$l_i(\alpha) \iff \text{cr}_i(\alpha) \preceq_i (t_1, t_2) \text{ and } h_i(\alpha) \iff (t_1, t_2) \preceq_i \text{cr}_i(\alpha).$$

Some basic properties of the notions just introduced are given below.

**Proposition 3.4.** For any formulae $\alpha, \beta$, we have that:

- If $\alpha \preceq_{\text{cr}_i} \beta \land l_i(\beta)$, then $l_i(\alpha)$.
- If $\alpha \vdash \beta \land l_i(\beta)$, then $l_i(\alpha)$.
- If $\alpha \preceq_{\text{cr}_i} \beta \land h_i(\alpha)$, then $h_i(\beta)$.
- If $\alpha \vdash \beta \land h_i(\alpha)$, then $h_i(\beta)$.
- If $\alpha \approx_{\text{cr}_i} \beta$, then $(k_i(\alpha) \iff k_i(\beta))$.
- If $\vdash \alpha \leftrightarrow \beta$, then $(k_i(\alpha) \iff k_i(\beta))$.
- If $l_i(\alpha) \land h_i(\beta)$, then $\alpha \preceq_{\text{cr}_i} \beta$.
- If $i = 2, 3$, then $h_i(\alpha) \iff \neg l_i(\alpha)$.

The proof is easy and hence omitted.

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2 For instance, information about a new physical phenomenon may be highly credible for ordinary people and of low credibility for physicists working in that area.

3 First, note that our present approach is quantitative. In [4] a qualitative version is considered as well. Next, several interesting and non-trivial questions arise, e.g., how to obtain the threshold which is the most relevant in a given situation.
There arises a problem: Whether and under what conditions is there a formula $\alpha$ such that

1. $h_i(\alpha)$ and $h_i(-\alpha)$;
2. $h_i(\alpha)$ and $\neg l_i(-\alpha)$;\(^4\)
3. neither $l_i(\alpha)$ nor $l_i(-\alpha)$?

If (1) (resp., (2), and (3)) holds, then the set $A_i = \{(x, y) \in D \mid (t_1, t_2) \preceq_i (x, y) \wedge (t_1, t_2) \preceq_i (1 - y, 1 - x)\}$ (resp., $B_i = \{(x, y) \in D \mid (t_1, t_2) \preceq_i (x, y) \wedge (1 - y, 1 - x) \not\preceq_i (t_1, t_2)\}$, and $C_i = \{(x, y) \in D \mid (x, y) \not\preceq_i (t_1, t_2) \wedge (1 - y, 1 - x) \not\preceq_i (1 - t_1, t_2)\}$) is non-empty. The converse, however, is not true in general since the co-domain $cr^{-1}_i(FOR)$ is a proper subset of $[0, 1]$. The following equivalences hold.

**Proposition 3.5.**

$$A_1 \neq \emptyset \iff B_1 \neq \emptyset \iff t_2 \leq 1 - t_1.$$  
$$C_1 \neq \emptyset \iff t_1 < 0.5 \lor t_2 < 1 \lor t_2 \leq 1 - t_1.$$  
$$A_2 = B_2 = C_2 \neq \emptyset \iff t_1 < 0.5 \lor t_2 \leq 1 - t_1.$$  
$$A_3 = B_3 = C_3 \neq \emptyset \iff t_2 < 1 \lor t_2 \leq 1 - t_1.$$  
$$A_4 \neq \emptyset \iff t_2 \leq 0.5 \lor t_2 = 1 - t_1.$$  
$$B_4 \neq \emptyset \iff t_1 < 0.5 \lor t_1 = t_2 \leq 0.5.$$  
$$C_4 \neq \emptyset \iff t_1 < 1 \lor t_1 = t_2 \leq 0.5.$$  

**Proof.** Leaving the remaining cases as exercises, we only show that $B_1 \neq \emptyset$ iff $t_2 < 1 - t_1$. Let us note that $(t_1, t_2) \preceq_1 (x, y)$ and $(1 - y, 1 - x) \not\preceq_1 (t_1, t_2)$ iff $t_1 \leq x$, $t_2 \leq y$, and neither $(1 - y < t_1 \land 1 - x \leq t_2)$ nor $(1 - y \leq t_1 \land 1 - x < t_2)$. After necessary transformations we obtain that $(t_1, t_2) \preceq_1 (x, y)$ and $(1 - y, 1 - x) \not\preceq_1 (t_1, t_2)$ iff $t_1 \leq x < 1 - t_2 \land t_2 \leq y$ or $t_1 \leq x \land t_2 \leq y < 1 - t_1$ or $t_1 \leq x = 1 - t_2 \land t_2 \leq y = 1 - t_1$. There are such $(x, y) \in D$ iff $t_2 \leq 1 - t_1$. \[Q.E.D.\]

Now we can give a partial answer\(^5\) to the problem formulated above.

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\(^4\) The case that there is $\alpha$ such that $h_i(-\alpha)$ and $\neg l_i(\alpha)$ can be easily reduced to that one.

\(^5\) The conditions below are necessary but not sufficient.
Corollary 3.6. Let $m_i$ stand for $h_i$ or $\neg l_i$.

If $(\exists \alpha) (h_1(\alpha) \land m_1(\neg \alpha))$, then $t_2 \leq 1 - t_1$.
If $(\exists \alpha) (\neg l_1(\alpha) \land \neg l_1(\neg \alpha))$, then $t_1 < 0.5 \lor t_2 < 1 \lor t_2 \leq 1 - t_1$.
If $(\exists \alpha) (h_2(\alpha) \land m_2(\neg \alpha))$, then $t_1 < 0.5 \lor t_2 \leq 1 - t_1$.
If $(\exists \alpha) (\neg l_2(\alpha) \land \neg l_2(\neg \alpha))$, then $t_1 < 0.5 \lor t_2 \leq 1 - t_1$.
If $(\exists \alpha) (h_3(\alpha) \land m_3(\neg \alpha))$, then $t_2 < 1 \lor t_2 \leq 1 - t_1$.
If $(\exists \alpha) (\neg l_3(\alpha) \land \neg l_3(\neg \alpha))$, then $t_2 < 1 \lor t_2 \leq 1 - t_1$.
If $(\exists \alpha) (h_4(\alpha) \land h_4(\neg \alpha))$, then $t_2 \leq 0.5 \lor t_2 = 1 - t_1$.
If $(\exists \alpha) (\neg l_4(\alpha) \land \neg l_4(\neg \alpha))$, then $t_1 < 0.5 \lor t_1 = t_2 \leq 0.5$.
If $(\exists \alpha) (h_4(\alpha) \land \neg l_4(\neg \alpha))$, then $t_1 < 1 \lor t_1 = t_2 \leq 0.5$.

For any set of formulae $X$, let $l_i(X) = \{\alpha \in X \mid l_i(\alpha)\}$ (the set of lowly credible members of $X$) and $h_i(X) = \{\alpha \in X \mid h_i(\alpha)\}$ (the set of highly credible members of $X$). Note that $X - l_i(X) = X - l_i(\text{FOR})$. Below we relate the notions of low and high credibility to $Cn$, $\nabla_i$, and $\triangle_i$. As earlier $k_i$ stands for $h_i$ or $l_i$.

Proposition 3.7. For any sets of formulae $X, Y$, we have that:

\[ k_i k_j(X) = k_i(X), \]
\[ k_i(X \cap Y) = k_i(X) \cap k_i(Y), \text{ and similarly for } \cup. \]
\[ \triangle_i l_i(\text{FOR}) = l_i(\text{FOR}) \land \nabla_i h_i(\text{FOR}) = h_i(\text{FOR}). \]
\[ l_i \triangle X = \triangle X \subseteq l_i \triangle X. \]
\[ h_i \nabla_i h_i(X) = \nabla_i h_i(X) \subseteq h_i \nabla_i(X). \]
\[ \nabla_i(X - l_i(\text{FOR})) \subseteq \nabla_i X - l_i(\text{FOR}). \]
\[ h_i(Cn(h_i(X))) \subseteq h_i(Cn(X)). \]
\[ (Cn \cap \nabla_i)(X - l_i(\text{FOR})) \subseteq (Cn \cap \nabla_i)(X) - l_i(\text{FOR}). \]
\[ (Cn \cap \nabla_i)h_i(X) \subseteq h_i(Cn \cap \nabla_i)(X). \]
\[ h_i(\text{FOR}) = \text{FOR} \iff l_i(\text{FOR}) = \emptyset \iff t_1 = t_2 = 0. \]

Proof. We only prove the last property.\(^6\) First, let us note that $h_i(\text{FOR}) \cap l_i(\text{FOR}) = \emptyset$. Hence if $h_i(\text{FOR}) = \text{FOR}$, then $l_i(\text{FOR}) = \emptyset$. Now suppose that $l_i(\text{FOR}) = \emptyset$, i.e., $(\forall \alpha) \cr_i(\alpha) \neq_i (t_1, t_2)$. In particular if $\alpha \in \text{co-TAUT}$ and hence $cr_i(\alpha) = (0, 0)$, then $(\ast) (0, 0) \neq_i (t_1, t_2)$. Let $i = 2, 3$. Since $\leq_i$ is total, $(t_1, t_2) \leq_i (0, 0)$. Hence $t_1 = t_2 = 0$. If $i = 1$, then $(\ast)$ is equivalent

\(^6\) A proof of the remaining cases can be found in [4].
to the disjunction: \( t_1 < 0 \) or \( t_2 < 0 \) or \( t_1 = t_2 = 0 \). Hence \( t_1 = t_2 = 0 \). If
\[ i = 4, \] then \((*)\) is equivalent to the disjunction: \( t_1 = t_2 = 0 \) or \( t_1 < 0 \). Hence
\( t_1 = t_2 = 0 \) as earlier. Now if \( t_1 = t_2 = 0 \), then \( h_i(co-TAUT) = co-TAUT \).
Hence \( h_i(FOR) = FOR. \)

4. Belief states

Many various models for representing states of beliefs are known in the areas
of logic, philosophy, and artificial intelligence, depending on the perspective
taken by the author or the intended area of application. For lack of space,
instead of giving an exhaustive survey of the proposals, let us mention belief
sets, belief bases, hypertheories, and pairs of belief sets.

As before it is assumed that beliefs are represented by formulae of a given
language. Along the standard lines, belief sets are \( C \)-closed sets of formulae,
where \( C \) is a logical consequence operator. In the AGM theory [1], it is as-
sumed that the underlying logic contains the classical one. In modal logics
approaches [10, 11], \( C \) is the consequence operator of an epistemic/doxastic
modal logic. In nonmonotonic approaches, \( C \) may be the consequence oper-
ator of PC like in the autoepistemic logic [14, 15], the consequence operator
of the classical first-order logic like in the default logic [18] or the conse-
quence operator of a modal logic like in nonmonotonic modal logics [13, 20].
In nonmonotonic approaches however, being \( C \)-closed is a necessary but not
sufficient condition for a set of formulae to represent beliefs. Indeed, belief
sets based on a set of premises \( I \) called extensions (or expansions) of \( I \) are de-
fined as \( C \)-closed sets of formulae satisfying some groundedness and stability
conditions. Whenever direct or indirect practical applications are consid-
ered, belief states are represented by finite sets of formulae called belief bases
[3, 6, 8, 12, 16]. In this case there arises a problem of independence of the
results of operations on belief bases from the syntax. In [21] belief states of a
rational agent are represented by hypertheories to grasp how doxastic dispo-
sitions of a rational agent change. In [5] a nonmonotonic framework for rep-
resenting belief states in the presence of incomplete knowledge is proposed.
Belief states are represented by \( AE2 \) extensions which are pairs of sets of for-
mulae, where the first (resp., second) set consists of all beliefs (disbeliefs)\(^7\) of a
rational agent in a given situation. The first set is \( Cn \)-closed, while the second
one is closed under a rejection consequence \( Cn' \). Both have to satisfy some
groundedness and stability conditions in the spirit of autoepistemic logic.

\(^7\) In [5] the terms ‘accepted’ and ‘rejected information’ were used.
In the present approach the main stress is laid on credibility of information. From this standpoint beliefs may be stronger or weaker. In low-risk situations we usually use beliefs of both types, while in risky cases we rather trust our stronger beliefs only. It can also be the case that normally we use stronger beliefs, while in problematic situations we also use weaker beliefs to enhance our reasoning capabilities or decision making. Keeping with the common terminology \[19, 22\], the stronger beliefs are simply called beliefs, while the weaker ones are referred to as plausible information. Belief states are represented by pairs of sets of formulae, where the first (resp., second) set called a belief set (plausibility set) consists of all believed (plausible) information.\(^8\) In our approach two aspects are combined: rationality of an agent and credibility of information. In \[4\] we consider three kinds of consequence operators: (1) \(Cn\); (2) \(\nabla_i\); and (3) \(Cn \cap \nabla_i\) \((i = 1, \ldots, 4)\). The first operator yields purely rational consequences of premises, operators of the second kind yield consequences which are at least as credible as some premise, while operators of the last kind take into account both rationality and credibility. As traditional and absolute \(Cn\)-consistency coincide, there are no \(Cn\)-parainconsistent sets of formulae. For \(Cn\)-inconsistent sets of formulae, \(Cn \cap \nabla_i = \nabla_i\). In the present paper we will only take into account operators of the second kind. Given a set of premises \(I\) and \(i = 1, \ldots, 4\), the belief state based on \(I\) is a pair \(B_i(I) = (BS_i(I), PS_i(I))\), where \(PS_i(I)\) (the plausibility set) and \(BS_i(I) \subseteq PS_i(I)\) (the belief set) are defined as follows:

\[
BS_i(I) = \nabla_i h_i(I) \quad \text{and} \quad PS(I) = \nabla_i(I - l_i(FOR))
\]

Informally speaking, beliefs have to be at least as credible as some highly credible premises and plausible information has to be at least as credible as some non-lowly credible premises. In this way, beliefs are highly credible and plausible information is not lowly credible. Given a belief state \(B_i(I)\) as above, we will say that the plausibility set \(PS_i(I)\) is corresponding for \(BS_i(I)\), and similarly for the belief set. Let \(S_i\) stand for \(BS_i(I)\) or \(PS_i(I)\).

**Proposition 4.1.** For any formulae \(\alpha, \beta\), we have that:

1. If \(\alpha \vdash \beta \land \alpha \in S_i\), then \(\beta \in S_i\).
2. If \(\vdash \alpha \leftrightarrow \beta\), then \(\alpha \in S_i\) \(\text{iff} \) \(\beta \in S_i\).
3. \(S_i = \text{FOR} \text{ iff} \ S_i \cap \text{co-TAUT} \neq \emptyset \text{ iff} \ t_1 = t_2 = 0 \text{ and } (\exists \beta \in I) cr_i(\beta) = (0, 0)\).

\(^8\) In \[4\] we also consider the case that belief/plausibility sets are not closed under a logical consequence operator.
PROOF. We only prove that $BS_i(I) = \text{FOR}$ iff $t_1 = t_2 = 0$ and $(\exists \beta \in I) cr_i(\beta) = (0, 0)$, for a given $cr_i$ and a threshold $(t_1, t_2)$. The equivalence

$$BS_i(I) = \text{FOR} \quad \text{iff} \quad BS_i(I) \cap \text{co-TAUT} \neq \emptyset$$

follows from Proposition 3.2. Thus if $BS_i(I) = \text{FOR}$, then $(\exists \alpha) \alpha \land \neg \alpha \in BS_i(I)$. Clearly, $cr_i(\alpha \land \neg \alpha) = (0, 0)$. Hence there is $\beta \in I$ such that $h_i(\beta)$ and $\beta \leq cr_i \alpha \land \neg \alpha$, i.e., $(t_1, t_2) \leq h_i(\beta) = (0, 0)$. Thus $t_1 = t_2 = 0$ and there is $\beta \in I$ such that $cr_i(\beta) = (0, 0)$. If $t_1 = t_2 = 0$ and $(\exists \beta \in I) cr_i(\beta) = (0, 0)$, then $h_i(\text{FOR}) = \text{FOR}$ by Proposition 3.7 and $(\exists \alpha) \alpha \land \neg \alpha \in \bigtriangledown_i I$. Finally $BS_i(I) = \bigtriangledown_i h_i(I) = \bigtriangledown_i I = \text{FOR}$. \hfill \Box

Roughly speaking, property 1 (resp., 2) states that belief and plausibility sets are closed under direct derivation (equivalence) in PC. According to 3, belief and plausibility sets based on a set of premises $I$ are either both absolutely $\bigtriangledown_i$-consistent or both inconsistent in this sense. Moreover, they are inconsistent iff the threshold is $(0, 0)$ and so is the degree of credibility of some premise.

Below we formulate the necessary and sufficient conditions for (1) both a formula and its negation to be in a belief set $BS_i(I)$; (2) a formula to be in $BS_i(I)$ and its negation to be in the corresponding plausibility set $PS_i(I)$; (3) both a formula and its negation to be in $PS_i(I)$.

In other words, the conditions are necessary and sufficient for $BS_i(I)$ (resp., $PS_i(I)$) to be traditionally $\bigtriangledown_i$-inconsistent in the case (1) (resp., (3)). Clearly, (1) and (2) are special cases of (3) since $BS_i(I) \subseteq PS_i(I)$. Needless to say, necessary (resp., sufficient) conditions for $\bigtriangledown_i$-inconsistency of $PS_i(I)$ ($BS_i(I)$) remain necessary (sufficient) for $BS_i(I)$ ($PS_i(I)$) as well.

**Proposition 4.2.** For any formula $\alpha$, we have that:\hfill \(9\)

1. $\alpha, \neg \alpha \in BS_i(I)$ iff $(\exists \beta, \gamma \in I)(h_i(\beta) \land h_i(\gamma) \land \beta \leq cr_i \alpha \land \gamma \leq cr_i \neg \alpha)$ iff $h_i(I) \cap \bigtriangleup_i \alpha \neq \emptyset$ and $h_i(I) \cap \bigtriangleup_i \neg \alpha \neq \emptyset$.

2. $\alpha \in BS_i(I)$ and $\neg \alpha \in PS_i(I)$ iff $(\exists \beta, \gamma \in I)(h_i(\beta) \land \neg h_i(\gamma) \land \beta \leq cr_i \alpha \land \gamma \leq cr_i \neg \alpha)$ iff $h_i(I) \cap \bigtriangleup_i \alpha \neq \emptyset$ and $(I \cap \bigtriangleup_i \neg \alpha) \cap l_i(\text{FOR}) \neq \emptyset$.

3. $\alpha, \neg \alpha \in PS_i(I)$ iff $(\exists \beta, \gamma \in I)(\neg h_i(\beta) \land h_i(\gamma) \land \beta \leq cr_i \alpha \land \gamma \leq cr_i \neg \alpha)$ iff $(I \cap \bigtriangleup_i \alpha) \cap l_i(\text{FOR}) \neq \emptyset$ and $(I \cap \bigtriangleup_i \neg \alpha) \cap l_i(\text{FOR}) \neq \emptyset$.

The proof is left as an exercise.

\(9\) Since $(\exists \alpha)(\alpha \in PS_i(I) \land \neg \alpha \in BS_i(I))$ is equivalent to $(\exists \alpha)(\alpha \in BS_i(I) \land \neg \alpha \in PS_i(I))$, the former case need not to be considered separately.
Despite of the elegant form of the above conditions and their sufficiency, what we would rather like to have is a characterization in terms of the threshold values $t_1$ and $t_2$. It is interesting that the obtained conditions are the same as in Corollary 3.6. They are necessary but not sufficient, unfortunately. As earlier $S_i$ stands for $BS_i(I)$ or $PS_i(I)$.

**Proposition 4.3.** 1. If $(\exists \alpha)(\alpha \in BS_1(I) \land \neg \alpha \in S_1)$, then $t_2 \leq 1 - t_1$.
2. If $(\exists \alpha)(\alpha \in PS_1(I))$, then $t_1 < 0.5 \lor t_2 < 1 \lor t_2 \leq 1 - t_1$.
3. If $(\exists \alpha)(\alpha \in BS_2(I) \land \neg \alpha \in S_2)$, then $t_1 < 0.5 \lor t_2 \leq 1 - t_1$.
4. If $(\exists \alpha)(\alpha \neg \alpha \in PS_2(I))$, then $t_1 < 0.5 \lor t_2 \leq 1 - t_1$.
5. If $(\exists \alpha)(\alpha \in BS_3(I) \land \neg \alpha \in S_3)$, then $t_2 < 1 \lor t_2 \leq 1 - t_1$.
6. If $(\exists \alpha)(\alpha \neg \alpha \in PS_3(I))$, then $t_2 < 1 \lor t_2 \leq 1 - t_1$.
7. If $(\exists \alpha)(\alpha \in BS_4(I))$, then $t_2 \leq 0.5 \lor t_2 = 1 - t_1$.
8. If $(\exists \alpha)(\alpha \in BS_4(I) \land \neg \alpha \in PS_4(I))$, then $t_1 < 0.5 \lor t_1 = t_2 \leq 0.5$.
9. If $(\exists \alpha)(\alpha \neg \alpha \in PS_4(I))$, then $t_1 < 1 \lor t_1 = t_2 \leq 0.5$.

**Proof.** Since the proof is technical and rather long, we show its idea only. Let us consider, e.g., the third condition for $S_2 = BS_2(I)$. Assume there is a formula $\alpha$ such that $\alpha \neg \alpha \in BS_2(I)$. By Proposition 4.2, there are $\beta_1, \beta_2 \in I$ such that $h_2(\beta_j) (j = 1, 2)$, $\beta_1 \leq cr_2(\alpha)$, and $\beta_2 \leq cr_2 \neg \alpha$. Let $cr_2(\alpha) = (x, y)$ (hence $cr_2(\neg \alpha) = (1 - y, 1 - x)$) and $cr_2(\beta_j) = (x_j, y_j)$. One can check that there are $(x, y), (x_j, y_j) \in D (j = 1, 2)$ such that $(t_1, t_2) \approx (x, y)$, $(x_1, y_1) \approx (x, y)$, and $(x_2, y_2) \approx (1 - y, 1 - x)$ iff $t_1 < 0.5$ or $t_2 \leq 1 - t_1$. Hence our condition follows.

From Propositions 4.1 and 4.3 we obtain the necessary conditions for $BS_i(I)$ (or equivalently $PS_i(I)$) to be absolutely $\nabla_i$-consistent and a formula $\alpha$ to exist, where (1) $\alpha, \neg \alpha \in BS_i(I)$; (2) $\alpha \in BS_i(I)$ and $\neg \alpha \in PS_i(I)$; (3) $\alpha, \neg \alpha \in PS_i(I)$. In other words, the conditions are necessary for $\nabla_i$-parainconsistency of $BS_i(I)$ (resp., $PS_i(I)$) in the case (1) (resp., (3)). For simplicity, let $\Psi_i$ be $(\forall \beta \in I)(0, 0) \prec_i cr_i(\beta) (i = 1, \ldots, 4)$ and $\Phi_j (j = 1, \ldots, 9)$ be the necessary conditions from Proposition 4.3, respectively.\(^\text{10}\)

**Corollary 4.4.** If $BS_i(I) \neq \text{FOR} (i = 1, \ldots, 4)$, then the following conditions hold:

1. If $(\exists \alpha)(\alpha \in BS_1(I) \land \neg \alpha \in S_1)$, then $0 = t_1 \prec t_2 \lor t_2 \leq 1 - t_1 \leq 1 \lor t_1 \prec (\Phi_1 \land \Psi_1)$.

\(^{10}\) That is, $\Phi_1 = t_2 \leq 1 - t_1$, $\Phi_2 = (t_1 < 0.5 \lor t_2 < 1 \lor t_2 \leq 1 - t_1)$, etc.
2. If $(\exists \alpha)(\alpha, \neg \alpha \in PS_1(I))$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor 0 < t_2 < 1 \lor (\Phi_2 \land \Psi_1). \]

3. If $(\exists \alpha)(\alpha \in BS_2(I) \land \neg \alpha \in S_2)$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor t_2 \leq 1 - t_1 < 1 \lor (\Phi_3 \land \Psi_2). \]

4. If $(\exists \alpha)(\alpha, \neg \alpha \in PS_2(I))$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor t_2 \leq 1 - t_1 < 1 \lor (\Phi_4 \land \Psi_2). \]

5. If $(\exists \alpha)(\alpha \in BS_3(I) \land \neg \alpha \in S_3)$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor t_2 \leq 1 - t_1 < 1 \lor (\Phi_5 \land \Psi_3). \]

6. If $(\exists \alpha)(\alpha, \neg \alpha \in PS_3(I))$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor t_2 \leq 1 - t_1 < 1 \lor (\Phi_6 \land \Psi_3). \]

7. If $(\exists \alpha)(\alpha, \neg \alpha \in BS_4(I))$, then
   
   \[ 0 = t_1 < t_2 \leq 0.5 \lor (t_1 = 0 \land t_2 = 1) \lor 0 < t_1 \leq t_2 \leq 0.5 \lor \]
   
   \[ t_2 = 1 - t_1 < 1 \lor (\Phi_7 \land \Psi_4). \]

8. If $(\exists \alpha)(\alpha \in BS_4(I) \land \neg \alpha \in PS_4(I))$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 0.5 \lor 0 < t_1 = t_2 \leq 0.5 \lor (\Phi_8 \land \Psi_4). \]

9. If $(\exists \alpha)(\alpha, \neg \alpha \in PS_4(I))$, then
   
   \[ 0 = t_1 < t_2 \lor 0 < t_1 < 1 \lor 0 < t_2\leq 0.5 \lor (\Phi_9 \land \Psi_4). \]

Except for the conditions of the form $\Phi_j \land \Psi_i$ not discussed here for the sake of simplicity, the remaining ones may be classified into several groups. (a) The conditions, where $t_1 = 0$ (i.e., $(t_1 = 0 \land t_2 = 1)$, $0 = t_1 < t_2$, and $0 = t_1 < t_2 \leq 0.5$) seem to be unintuitive from the common-sense standpoint. (b) According to the conditions $0 < t_1 \leq t_2 \leq 0.5$ and $0 < t_1 = t_2 \leq 0.5$ (cases 7–9), the threshold values should be relatively low which is not very promising. In the most optimistic case $t_1 = t_2 = 0.5$. (c) The condition $0 < t_1 < 0.5$ (cases 2–4, and 8) requires that $t_1$ be fairly low but, on the other hand, $t_2$ may be arbitrary (i.e., $t_1 \leq t_2 \leq 1$). (d) The conditions $t_2 \leq 1 - t_1 < 1$ and $t_2 = 1 - t_1 < 1$ (cases 1 and 3–7) form yet another class. In this case $t_1 = 0.5 - a$ and $t_2 \leq 0.5 + a$ for some $0 \leq a < 0.5$. Hence the distance between $t_1$ and $t_2$ should not be greater than $2a$. Intuitively, the situation is realistic, especially for small $a$, and it should be no particular problems with finding paraconsistent belief and plausibility sets. (e) The last class considered here consists of the conditions $0 < t_1 \leq t_2 < 1$ and $0 < t_1 \leq t_2 \leq 1$ (cases 2, 5, 6, and 9). Here it should be easy to find paraconsistent belief states since the terms imposed on $t_1$ and $t_2$ are indeed minimal.
5. Summary

The aim of the paper was to show that the situation, where a belief state is parainconsistent in the sense that its belief and/or plausibility sets are $C$-parainconsistent, is natural if the consequence operator $C$ takes into account credibility of information like the operators $\nabla_i$ ($i = 1, \ldots, 4$) do. Indeed, the necessary conditions for belief and plausibility sets to be $\nabla_i$-parainconsistent obtained above can be satisfied in a fairly large number of cases.

Since $\nabla_i$ tolerates inconsistency (in the sense that there exist $\nabla_i$-parainconsistent sets of formulae) and for any formulae $\alpha, \beta$, we have that $cr_i(\alpha \rightarrow (\beta \rightarrow \alpha)) = (1, 1)$, operator $\nabla_i$ is not closed under the adjunction rule:

$$
\alpha, \beta \Rightarrow \alpha \land \beta
$$

It is easy to give an informal example justifying this statement, however, systematic considerations are postponed to a separate paper.

Example 5.1. Let $p_i$ ($i = 1, 2$) represent the information that 'Skier $S_i$ wins the competition.' Suppose each $p_i$ is highly credible. On the other hand, $p_1 \land p_2$ is almost incredible.

Another problem worthy of consideration is the problem of change of (possibly parainconsistent) belief states and the threshold values.

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