Kristof De Clercq*

TWO NEW STRATEGIES FOR INCONSISTENCY-ADAPTIVE LOGICS

Abstract. In this paper I present two new strategies for inconsistency-adaptive logics: the reliable sufficient information strategy of ACLuN3 and the minimally abnormal sufficient information strategy of ACLuN4. I give proof theory and semantics for both ACLuN3 and ACLuN4. I also compare them with the well-known inconsistency-adaptive logics ACLuN1 and ACLuN2.

1. Introduction

Inconsistency-adaptive logics are a special brand of paraconsistent logics, and were developed by Diderik Batens around 1980.1 The best studied inconsistency-adaptive logics are Batens’ ACLuN1 and ACLuN2 (see especially [5]). Loosely speaking, they ‘oscillate’ between a lower limit logic, the paraconsistent CLuN, and an upper limit logic, Classical Logic (CL): they localize the inconsistencies of a set of premises Γ, safeguard Γ for triviality by preventing specific rules of CL being applied ‘in the neighbourhood of inconsistencies’, but behave exactly like CL for all other derivations from Γ. They allow for inconsistencies but presuppose the consistency of all sentences ‘unless and until proven otherwise’. Interpreting a set of premises

---

* Research Assistant of the Fund for Scientific Research – Flanders (Belgium)(F.W.O.).
1 See [2], although the paper was written much earlier. For a study of the predicative version see [5]. A survey of the domain is presented in Batens [6]. For an informal description and the relation with argumentation, see [4].
‘as consistently as possible’, they adapt themselves to the specific inconsistencies that occur in it. In [6] it is shown that there are two different strategies to do so: the reliability strategy of \textsc{ACLuN1}, and the minimal abnormality strategy of \textsc{ACLuN2}.

For a long time, it seemed that those strategies were the only strategies to devise inconsistency-adaptive logics. An attempt to reconstruct Default Logic by means of an inconsistency-adaptive logic, brought me to develop two new strategies (that are even more cautious than reliability): the reliable sufficient information strategy of \textsc{ACLuN3} and the minimally abnormal sufficient information strategy of \textsc{ACLuN4}. After all, it would have been a logical mystery that there exist exactly two and only two strategies to devise inconsistency-adaptive logics.

\textsc{ACLuN3} and \textsc{ACLuN4} are based on the paraconsistent logic \textsc{CLuN}. \textsc{CLuN} is a poor and basic paraconsistent logic. It is obtained by extending \textsc{CL} (with $\neg$ as the classical negation) with the very poor paraconsistent negation $\sim$, by means of the axiom schema $A \lor \sim A$ (semantically, $\sim$ is characterized by a negation-completeness clause only: if $v_M(A) = 0$, then $v_M(\sim A) = 1$). \textsc{CLuN} maximally isolates inconsistencies in that no contradiction $A \land \sim A$ entails any other contradiction (not even one for any subformula or superformula of $A$). For a detailed presentation of \textsc{CLuN}, as well of \textsc{ACLuN1} and \textsc{ACLuN2}, I refer to [5] and [7]. Albeit the fact that classical negation, $\neg$, is defined in \textsc{CLuN}, we only allow the occurrence of paraconsistent negation, $\sim$, in the premises.

In section 2, I mention a problem concerning the reconstruction of nonmonotonic logics of the default-type by means of inconsistency-adaptive logics. In section 3, I present the sufficient information strategy. In section 4, I present the inconsistency-adaptive logics \textsc{ACLuN3} and \textsc{ACLuN4}. In section 5, I make some comparisons between \textsc{ACLuN1/2} and \textsc{ACLuN3/4}. I mention some open problems in section 6.

2. Inconsistency-adaptive logics as reconstruction tools for “mixed nonmonotonic logics” \footnote{In [3], Diderik Batens introduces the label “mixed nonmonotonic logics” for those (popular) non-monotonic logics in which a deductive and preferential component are blended together (e.g., the Circumscription approach, Default Logics).}
Two new strategies for inconsistency-adaptive logics

A deductive component leads from the premises to a possibly inconsistent consequence set. Several candidates for the deductive component are evaluated, and inconsistency-adaptive logics prove most suitable in this respect. The ensuing preferential component is formulated in terms of models and consists itself of two parts: (i) a purely logical procedure connects a set of consistent models to the set of (possibly inconsistent) models of the premises; (ii) a selection procedure picks out the preferred models by relying on the preferences.

Batens offers a successful reconstruction of circumscription by using the inconsistency-adaptive logic ACLuN2 for the deductive component. Circumscription minimizes (the occurrence of) abnormality predicates in a certain order. This minimization of the abnormality predicates corresponds to a purely logical step in the reconstruction procedure: the restriction to ACLuN2-models of the premises minimizes inconsistencies. The order in which abnormality predicates are minimized corresponds to the step in which the preferences come in: the selection of the preferred models by relying on the preferences. I will not go into details here, the interested reader should consult [3].

In [9] I attempt to reconstruct (fragments of) Default Logic, using the general procedure of [3]. It turned out that using ACLuN1 or ACLuN2 for the deductive component, the yielded consequences were in a sense too strong. Let me illustrate this with an example. Consider the following default theory \( T = \langle W, D \rangle \), where 

\[
W = \{ (\forall x)(Px \supset \sim Fx), (\forall x)(Px \supset Bx), Pt, Ba \} \quad \text{and} \quad D = \{ Bx : Fx \}. 
\]

The default theory \( T \) has one extension 

\[ E = \text{Cn}(W \cup \{Fa\}). \]

So Tweety is a penguin, and hence a bird, that does not fly, and \( a \) is a flying bird.

For a reconstruction of this default theory, we take \( \Gamma = \{ (\forall x)(Px \supset \sim Fx), (\forall x)(Px \supset Bx), Pt, Ba, (\forall x)(Bx \supset Fx) \} \). The CLuN-consequence set of \( \Gamma \) contains the following formulas: 

\[
Pt, Bt, Ft, \sim Ft, Ba, Fa, (\forall x)(\sim Px \lor (Fx \land \sim Fx)).
\]

If we take ACLuN1 or ACLuN2 for the deductive component, the consequence set of \( \Gamma \) contains the following formulas: 

\[
Pt, Bt, Ft, \sim Ft, Ba, Fa, \sim Pa, (\forall x)(x \neq t \lor \sim Px). 
\]

The last formula expresses that there are no penguins other than Tweety. Albeit this is exactly what we get following the circumscription approach of the Tweety-example.

---

3 The fragment of normal defaults and those semi-normal defaults that can also be represented in a Prioritized Default Logic. See [1] and [8].

4 Interpreting \( P, B, F \) and \( t \) as resp. ‘Penguin’, ‘Bird’, ‘Fly’ and ‘Tweety’, we obtain (a version of) the well known Tweety example.
the default theory $T$ does not imply that Tweety is the only non-flying bird (i.e. Tweety is the only abnormal bird) nor that there are no penguins other than Tweety. This difference between the circumscription approach and Default Logic reveals different underlying intuitions behind rules with exceptions. The circumscription approach presupposes that anything that is not bound to be abnormal in view of the premises, is normal: as all penguins are abnormal birds (with respect to being a flyer), all birds not given to be non-flyers are supposed to be flyers (and hence non-penguins). In Default Logic, the intuition behind rules with exceptions is rather different: if there is one exception to a rule, it is plausible there will be others, so it is credulous to assume that the known exceptions to a rule are the only exceptions to that rule. As Tweety is an exception to the rule ‘Birds fly’, it is plausible that there will be other exceptions to the rule (e.g. other penguins), hence we do not want to conclude that there are no penguins other than Tweety.

Why is $(\forall x)(x \neq t \supset \sim Px)$ an ACLuN1/2-consequence of $\Gamma$? As $(\forall x)(\sim Px \lor (Fx \& \sim Fx))$ is true in all CLuN-models of $\Gamma$, and ACLuN1/2 presupposes that all formulas are consistent wherever the premises do not command inconsistency ($Ft \& \sim Ft$ is the only contradiction that is ‘forced’ by the premises), ACLuN1/2 presupposes that $F\alpha \& \sim F\alpha$ is false for all $\alpha$ other than $t$. Hence, $(\forall x)(x \neq t \supset \sim Px)$ is true in all ACLuN1/2-models of $\Gamma$.

If we want to avoid consequences as $(\forall x)(x \neq t \supset \sim Px)$, we will have to use a logic that is ‘weaker’ (or ‘more cautious’) than the inconsistency-adaptive logics ACLuN1 and ACLuN2. At the other hand, we want a logic that is stronger (leads to a richer consequence set) than the paraconsistent logic CLuN (in the example, $\sim Pa$ is not a CLuN-consequence).

3. Sufficient information strategy

3.1. Intuitive formulation

As described in section 2, attempts to reconstruct Default Logic by using an inconsistency-adaptive logic for the deductive component, forced me into a search for other, less powerful strategies for inconsistency-adaptive logics. One of the most viable candidates was the following strategy:

(S) If all CLuN-models of $\Gamma$ verify $A$ (respectively $\sim A$) and some of them falsify $\sim A$ (respectively $A$), then eliminate the CLuN-models that verify $\sim A$ (respectively $A$).
As an illustration, let $\Gamma = \{\neg p \lor q, p\}$. As all $\text{CLuN}$-models of $\Gamma$ verify $p$, and some $\text{CLuN}$-models falsify $\neg p$, all $\text{CLuN}$-models that verify $\neg p$ are eliminated. As a result, all non-eliminated $\text{CLuN}$-models verify $q$ (because all $\text{CLuN}$-models of $\Gamma$ verify $\neg p \lor q$). In the next subsection I will show that this intuitive formulation (S) has two major drawbacks, so it has to be modified.

### 3.2. Two problems

Consider the set of premises $\Gamma = \{p, q, \neg p \lor \neg q\}$. If we apply (S), we get the following ’instructions’:

(i) Because $\Gamma \models_{\text{CLuN}} p$ and $\Gamma \not\models_{\text{CLuN}} \neg p$, all $\text{CLuN}$-models of $\Gamma$ that verify $\neg p$ have to be eliminated. Hence, in all non-eliminated models $v_M(\neg p) = 0$, hence $\neg q$ is true in all of them.

(ii) Because $\Gamma \models_{\text{CLuN}} q$ and $\Gamma \not\models_{\text{CLuN}} \neg q$, all $\text{CLuN}$-models of $\Gamma$ that verify $\neg q$ have to be eliminated. Hence, in all non-eliminated models $v_M(\neg q) = 0$, hence $\neg p$ is true in all of them.

Initially, both (i) as (ii) are applicable. However, as soon as one applies either (i) or (ii), the other becomes inapplicable. From instruction (i) it follows that $\sim p$ is false in all models, hence $p \& \sim p$ is false in all of them, hence $\sim q$ has to be true in all of them. Hence instruction (ii) becomes inapplicable. By analogous reasoning, the same holds if we start by instruction (ii). It turns out that it depends merely on the accidental order in which we apply the instructions whether $\sim p$ is derivable and $\sim q$ is not, or the other way around.

The diagnosis of the trouble is that the premises do not provide sufficient information to decide which of the sentences behaves inconsistently and which consistently. As $\Gamma \models_{\text{CLuN}} (p \& \sim p) \lor (q \& \sim q)$ but $\Gamma \not\models_{\text{CLuN}} p \& \sim p$ and $\Gamma \not\models_{\text{CLuN}} q \& \sim q$, $p$ and $q$ are connected with respect to their consistency (in the terminology of $\text{ACLuN1}$: both $p$ and $q$ are $\Gamma$-unreliable). A remedy is straightforward: (S) should be applied on $\Gamma$-reliable formulas only.

A second problem is that by applying (S) we do not reach a fixed point. Consider the set of premises $\Gamma = \{p, \sim p \lor q, r \lor \sim q\}$. As $\Gamma \models_{\text{CLuN}} p$ and $\Gamma \not\models_{\text{CLuN}} \sim p$, all $\text{CLuN}$-models that verify $\sim p$ are eliminated. This implies that all non-eliminated models verify $q$ (because $\sim p \lor q$ has to be true in all of them). Following (S) as it stands, no more models can be eliminated (hence it can not be deduced that all non-eliminated $\text{CLuN}$-models verify $r$).\footnote{Due to the specific formulation of (S): albeit all non-eliminated models of $\Gamma$ verify $q$, the original $\text{CLuN}$-models of $\Gamma$ do not verify $q$ ($\Gamma \not\models_{\text{CLuN}} q$), and (S) cannot be applied.}
Now we would want to apply (S) to these non-eliminated CLuN-models: as all non-eliminated CLuN-models verify \( q \), and some of them falsify \( \sim q \), we eliminate all (remaining) CLuN-models of \( \Gamma \) that verify \( \sim q \). Hence all remaining CLuN-models of \( \Gamma \) verify \( r \).

### 3.3. Decent characterization of the Sufficient Information Strategy

The improved version of the Sufficient Information Strategy, goes as follows:

(SI) If \( A \) is reliable with respect to \( \Gamma \), and all remaining (i.e. not yet eliminated) CLuN-models of \( \Gamma \) verify \( A \) (respectively \( \sim A \)), then eliminate the CLuN-models that verify \( \sim A \) (respectively \( A \)). This elimination procedure should be iterated as long as it can (until no more models are eliminated).

It is important to notice that (SI) does indeed lead to a fixed point, and that (SI) leads to a unique set of remaining CLuN-models, independent of the order in which the CLuN-models are eliminated. Although this order depends on the arbitrary picking out of \( \Gamma \)-reliable formulas \( A \), this has no impact at all on the final set of remaining (i.e. non-eliminated) CLuN-models of the premises.\(^6\)

By means of (SI) we can formulate two (slightly different) inconsistency-adaptive logics, depending on the way in which \( \Gamma \)-unreliable formulas are interpreted.

### 4. The inconsistency-adaptive logics ACLuN3 and ACLuN4

#### 4.1. Some definitions

In order to formulate the logics ACLuN3 and ACLuN4, we first need some definitions. Let \( DEK\{A_1, \ldots, A_n\} \) refer to \( \exists(A_1 \& \sim A_1) \lor \cdots \lor \exists(A_n \& \sim A_n) \) : a disjunction of (where necessary) existentially quantified contradictions. A \( DEK \)-formula is a formula of the form \( DEK\{A_1, \ldots, A_n\} \), and \( A_1, \ldots, A_n \) are said to be the factors of \( DEK\{A_1, \ldots, A_n\} \). Henceforth, it will be easier to write \( DEK(\Delta) \), recalling that this is a formula and hence that \( \Delta \) is finite.

**Definition.** A \( DEK \)-consequence of \( \Gamma \) is a \( DEK \)-formula which is CLuN-derivable from \( \Gamma \).

**Definition.** \( DEK(\Delta) \) is a minimal \( DEK \)-consequence of \( \Gamma \) iff \( \Gamma \models_{CLuN} DEK(\Delta) \) and, for no \( \Theta \subset \Delta \), \( \Gamma \models_{CLuN} DEK(\Theta) \).

\(^6\) Proofs of these claims will have to be postponed for another paper.
Two new strategies for inconsistency-adaptive logics

Definition. Let $U(\Gamma) = \{ A \mid A \in \Delta \text{ for some minimal DEK-consequence } \Delta \text{ of } \Gamma \}$. $U(\Gamma)$ is the set of formulas that are (semantically) unreliable with respect to $\Gamma$.

To get grip on the definitions I give a simple example. Let $\Gamma = \{ p \lor q, \sim p, \sim q \}$. It is obvious that $\Gamma \vdash \text{CLuN}(p & \sim p) \lor (q & \sim q)$, while neither $p & \sim p$ nor $q & \sim q$ is CLuN-derivable from $\Gamma$. Hence, $\text{U}(\Gamma) = \{ p, q \}$, which means that both $p$ and $q$ are unreliable with respect to the set of premises $\Gamma$.

Definition. Where $M$ is a model, $\text{Ab}(M) = \{ A \mid v_M(\exists (A \& \sim A)) = 1 \}$

Definition. Where $M$ is a model of a set of premises $\Gamma$, $\text{Ab}_U(M) = \text{Ab}(M) \cap U(\Gamma)$.

Definition. Where $M$ is a model of a set of premises $\Gamma$, and $\mathcal{F}$ the set of all formulas, $\text{Ab}_R(M) = \text{Ab}(M) \cap (\mathcal{F} - U(\Gamma))$.

It is easy to see that $\text{Ab}_U(M) \cap \text{Ab}_R(M) = \emptyset$, and $\text{Ab}_U(M) \cup \text{Ab}_R(M) = \text{Ab}(M)$.

Definition. A CLuN-model $M$ of $\Gamma$ is minimally abnormal with respect to the $\Gamma$-unreliable formulas iff there is no CLuN-model $M'$ of $\Gamma$ such that $\text{Ab}_U(M') \subset \text{Ab}_U(M)$.

Definition. A CLuN-model $M$ of $\Gamma$ is minimally abnormal with respect to the $\Gamma$-reliable formulas iff there is no CLuN-model $M'$ of $\Gamma$ such that $\text{Ab}_R(M') \subset \text{Ab}_R(M)$.

4.2. The Reliable Sufficient Information Strategy of ACLuN3

4.2.1. Proof theory

The idea of the proof theory of ACLuN3 is that we apply all rules of (or derivable in) CLuN unconditionally, whereas a conditional rule is applied on a provisional basis and on the condition that certain formulas are reliable (with respect to their consistent behaviour). To keep the matter algorithmic, the consistent behaviour of a formula will be determined by the stage of the proof. As a result (in accordance with ACLuN1/2), proofs will be dynamic in that wffs derived at some stage of the proof may not be derivable at a later stage.

Following [5], ACLuN3-proofs are written in a specific format. Each line in a proof consists of five elements: (i) a line number; (ii) the wff derived;
(iii) the line numbers of the wffs from which it is derived; (iv) the rule by which it is derived; and (v) the set of formulas that should be reliable in order for the wff to be derivable.

DEFINITION. A formula \( A \) occurs unconditionally at some line of a proof iff the fifth element of that line is empty.

DEFINITION. A behaves consistently at a stage of a proof iff \( \exists (A \& \sim A) \) does not occur unconditionally in the proof at that stage.

DEFINITION. The consistent behaviour of \( A_1 \) is connected to the consistent behaviour of \( A_2, \ldots, A_n \) at a stage of a proof iff \( DEK(A_1, \ldots, A_n) \) occurs unconditionally in the proof at that stage whereas \( DEK(\Delta) \) does not occur unconditionally in it for any \( \Delta \subset \{A_1, \ldots, A_n\} \).

DEFINITION. \( A \) is reliable at a stage of a proof iff \( A \) behaves consistently at that stage and its consistent behaviour is not connected to the consistent behaviour of other formulas.

Given these definitions, proofs in ACLuN3 are governed by an unconditional rule, a conditional rule and a deletion rule. An application of RU or RC to a proof at a stage produces the next stage.

RU All derivation rules of CLuN are unconditionally valid in any ACLuN3-proof. The fifth element of the new line is the union of the fifth elements of the lines mentioned in its third element.

RC If \( A \) (resp. \( \sim A \)) occurs as the second element of a line in the proof at depth zero (i.e. not depending on any hypothesis), then you may derive \( \sim A \) (resp. \( \sim A \)) provided that \( A \) is reliable at that stage of the proof. The fifth element of the new line is the union of the fifth element of the line on which \( A \) (resp. \( \sim A \)) occurs and \( \{A\} \).

RD If \( C \) is not (any more) reliable, then delete from the proof all lines the fifth element of which contains \( C \).

Wffs that occur unconditionally are CLuN-derivable from the premises (and cannot possibly be ‘deleted’ later). The unconditional occurrence of \( DEK \)-formulas at a stage determines which formulas are reliable at that stage. Wffs that occur in the proof at a stage are derivable at that stage. Of course, we need a more stable notion, final derivability, that does not depend on the stage of the proof.

DEFINITION. \( A \) is finally derived at some line in an ACLuN3-proof iff, (i) \( A \) is the second element of the line and (ii) where \( \Delta \) (\( \subseteq \emptyset \)) is the fifth element
of the line, any extension of the proof can be further extended in such a way that it contains a line that has \( A \) as its second element and \( \Delta \) as its fifth element.

**Definition.** \( \Gamma \vdash_{\text{ACLuN3}} A \) (\( A \) is an ALCuN3-consequence of \( \Gamma \)) iff \( A \) is finally derived at some line in an ALCuN3-proof.

### 4.2.2. Semantics

The ALCuN3-semantics is obtained from the CLuN-semantics by defining, for each \( \Gamma \), a subset of the CLuN-models of \( \Gamma \). The idea is that any \( \Gamma \) defines a set of (semantically) unreliable formulas, and that the ALCuN3-models of \( \Gamma \) are those CLuN-models that are not eliminated by the sufficient information strategy.

**Definition.** \( M \) is an ALCuN3-model of \( \Gamma \) iff (i) \( M \) is a CLuN-model of \( \Gamma \) and (ii) \( M \) is not eliminated by the sufficient information strategy.

**Definition.** \( \Gamma \models_{\text{ACLuN3}} A \) iff \( A \) is true in all ALCuN3-models of \( \Gamma \).

### 4.3. The Minimally Abnormal Sufficient Information Strategy of ALCuN4

#### 4.3.1. Semantics

For ALCuN4, it appears advisable to start from the semantics. The central difference with the ALCuN3-semantics is that a stronger selection of CLuN-models occurs: all ALCuN4-models of \( \Gamma \) are ALCuN3-models of \( \Gamma \), but the converse does not always hold. If, e.g., \( \text{DEK}\{p,q\} \) is the only minimal DEK-consequence of \( \Gamma \), then, unlike for ALCuN3-models of \( \Gamma \), either \( p \land \neg p \) or \( q \land \neg q \) is false in any ALCuN4-model of \( \Gamma \).

**Definition.** \( M \) is an ALCuN4-model of \( \Gamma \) iff (i) \( M \) is a CLuN-model of \( \Gamma \) and (ii) there is no CLuN-model \( M' \) such that \( \text{Ab}_{\text{U}}(M') \subset \text{Ab}_{\text{U}}(M) \) and (iii) \( M \) is not eliminated by the sufficient information strategy.

**Definition.** \( \Gamma \models_{\text{ACLuN4}} A \) iff \( A \) is true in all ALCuN4-models of \( \Gamma \).

#### 4.3.2. Proof theory

The format of proofs is as for ALCuN3, except that no lines are deleted in ALCuN4-proofs, but that there may be tentative lines, indicated with a mark ‘OUT’. Marked lines are not considered as occurring in the proof and
may not be relied upon for adding further lines. After each step, the marks are updated, viz. removed or added.

The updating of the marks is governed by an integrity criterion, as presented in [5]. The intuitive idea is as follows. Suppose that $A$ is derived on one or more lines the fifth element of which is not empty. $A$ is considered as derived (at a stage of the proof) and the lines become a full part of the proof if $A$ comes out true under any maximally normal ‘interpretation’ of the least DEK-formulas (at that stage).

As the integrity criterion will look at combinations of factors of DEK-formulas, it is useful to remark that some DEK-formulas that occur unconditionally in a proof may be disregarded. Suppose that a Gödel-numbering (or some other ordering) of formulas is given. Where $A$ and $B$ are DEK-formulas, the following definitions can be given:

**Definition.** $A \prec B$ iff either (i) $A \vdash \text{CLuN} B$ and $B \not\vdash \text{CLuN} A$, or (ii) $A$ and $B$ are CLuN-equivalent and the Gödel-number of $A$ is smaller than the Gödel number of $B$.

**Definition.** $A$ is a least DEK-formula (at a stage of the proof) if it occurs unconditionally in the proof and no DEK-formula $B$, such that $B \prec A$, occurs unconditionally in the proof.

If $\text{DEK}(\Gamma \cup \{Px\})$ and $\text{DEK}(\Gamma \cup \{Py\})$ occur unconditionally in the proof and the Gödel-number of the former is smaller than that of the latter, then at best the former will be a least DEK-formula. Neither of them is a least DEK-formula if $\text{DEK}(\Gamma \cup \{P\alpha\})$ also occurs unconditionally in the proof. Clearly, if one disjunct of each least DEK-formula is true, then all DEK-formulas are true (at that stage).

Let $\Phi^*_s$ be the set of all sets that contain one factor out of each least DEK-formula (at stage $s$ of the proof). $\Phi^*_s$ may contain redundant elements for two different reasons. The first is related to the individual variables. Where neither $x$ nor $y$ occurs free in $A(z)$, $\exists(A(x) & \sim A(x))$ is CLuN-equivalent to $\exists(A(y) & \sim A(y))$. But $A(x)$ may be a factor of some least DEK-formula and $A(y)$ of another. Hence, $\Phi^*_s$ may contain $\{Px, Py\}$, or may contain both $\{Px, p\}$ and $\{Py, p\}$. To reduce these, $\Phi^*_s$ is defined from $\Phi^*_s$ by relettering all open formulas in the members of $\Phi^*_s$ in such a way that the free variables occur always in the same order (for all formulas, the first occurring free variable is always $x_1$, the second always $x_2$, etc.). The second reason for redundant elements is that the same factor may occur in different least DEK-formulas. If $\text{DEK}\{p, q\}$ and $\text{DEK}\{p, r\}$ are the least
DEK-formulas, \( *\Phi_s = \circ \Phi_s = \{ \{ p \}, \{ p, r \}, \{ p, q \}, \{ q, r \} \} \). Of these \{ p, r \} and \{ p, q \} are redundant: both \( DEK\{ p, q \} \) and \( DEK\{ p, r \} \) are true if \( p \& \sim p \) is true; there is no need that also \( r \& \sim r \) or \( q \& \sim q \) be true. So, let \( \Phi_s \) be obtained from \( \circ \Phi_s \) by eliminating elements from it that are proper supersets of other elements. The members of \( \Phi_s \) are sets of formulas, such that, if \( \exists (A \& \sim A) \) is true for all members \( A \) of such a set, then all \( DEK \)-formulas that occur unconditionally in the proof are true. To see this, it is sufficient to realize that, if \( A \) and \( B \) are different formulas (and not reletterings of each other with respect to the individual variables), then \( \exists (A \& \sim A) \) and \( \exists (B \& \sim B) \) are \( CLuN \)-independent formulas — remember that \( CLuN \) does not spread inconsistencies.

**Definition.** Where \( A \) is the second element of line \( j \), line \( j \) fulfills the integrity criterion (at stage \( s \)) iff (i) the intersection of some member of \( \Phi_s \) and of the fifth element of line \( j \) is empty, and (ii) for each \( \varphi \in \Phi_s \) there is a line \( k \) such that the intersection of \( \varphi \) and of the fifth element of line \( k \) is empty and \( A \) is the second element of line \( k \).

As a (very) simple illustration, consider:

\[
\begin{array}{cccc}
(j) & DEK\{ p, q, r \} & \emptyset \\
(j + 1) & A & \ldots & \{ p, q \} \\
(j + 2) & A & \ldots & \{ q, r \} \\
(j + 3) & A & \ldots & \{ p, r \}
\end{array}
\]

If \( (j) \) is the only least \( DEK \)-formula in the proof, \( \Phi_s = \{ \{ p \}, \{ q \}, \{ r \} \} \) and lines \((j+1) - (j+3)\) fulfill the integrity criterion. They also fulfill the integrity criterion if the second element of line \( (j) \) is \( DEK\{ p, q, r, s \} \).

Let us now turn to the \( ACLuN4 \)-rules.

**RU** As for \( ACLuN3 \).

**RC** If \( A \) (resp. \( \sim A \)) occurs as the second element of a line in the proof at depth zero (i.e. not depending on any hypothesis), then you may derive \( \neg \sim A \) (resp. \( \neg A \)) provided that, at that stage, \( A \) behaves consistently. The fifth element of the new line is the union of the fifth element of the line on which \( A \) (resp. \( \sim A \)) occurs and \( \{ A \} \).

**RQ+** A mark is added to a line that does not fulfill the integrity criterion, and to all lines derived from it.

**RQ−** If a line fulfills the integrity criterion and is marked, the mark is removed.
Definition. \( A \) is finally derived at some line in an ACLuN4-proof iff \( A \) is the second element of that line and any (possibly infinite) extension of the proof can be further extended in such a way that the line is unmarked.

Definition. \( \Gamma \vdash_{\text{ACLuN4}} A \) (\( A \) is an ACLuN4-consequence of \( \Gamma \)) iff \( A \) is finally derived at some line of an ACLuN2-proof from \( \Gamma \).

5. Some comparisons

5.1. The difference between ACLuN3 and ACLuN4

Here is an example of an ACLuN3-proof.\(^7\)

1. \( s \& q \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
2. \( t \lor p \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
3. \( \sim q \lor t \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
4. \( \sim p \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
5. \( r \lor \sim s \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
6. \( p \lor \sim q \) \hspace{1cm} \text{PREM} \hspace{0.5cm} \emptyset
7. \( s \) \hspace{1cm} 1 \hspace{0.5cm} \emptyset
8. \( q \) \hspace{1cm} 1 \hspace{0.5cm} \emptyset
9. \( \sim \sim s \) \hspace{1cm} 7 \hspace{0.5cm} \{s\}
10. \( \sim \sim q \) \hspace{1cm} 8 \hspace{0.5cm} \{q\} \hspace{0.5cm} \text{OUT}
11. \( t \) \hspace{1cm} 3, 8 \hspace{0.5cm} \{q\} \hspace{0.5cm} \text{OUT}
12. \( \sim \sim t \) \hspace{1cm} 11 \hspace{0.5cm} \{q, t\} \hspace{0.5cm} \text{OUT}
13. \( r \) \hspace{1cm} 5, 9 \hspace{0.5cm} \{s\}
14. \( \sim \sim r \) \hspace{1cm} 13 \hspace{0.5cm} \{r, s\}
15. \( \sim p \) \hspace{1cm} 4 \hspace{0.5cm} \{p\} \hspace{0.5cm} \text{OUT}
16. \( t \) \hspace{1cm} 2, 15 \hspace{0.5cm} \{p\} \hspace{0.5cm} \text{OUT}
17. \( \sim \sim t \) \hspace{1cm} 16 \hspace{0.5cm} \{p, t\} \hspace{0.5cm} \text{OUT}
18. \( (p \& \sim p) \lor \sim q \) \hspace{1cm} 4, 6 \hspace{0.5cm} \emptyset
19. \( (p \& \sim p) \lor (q \& \sim q) \) \hspace{1cm} 8, 18 \hspace{0.5cm} \emptyset

Line 9 is a typical conditional derivation. From \( s \) we derive \( \sim s \), as \( s \) is reliable at that stage of the proof. We mention \( s \) as the fifth element of line 9. Line 10 is also a conditional derivation. From \( q \) we derive \( \sim q \), as \( q \) is reliable at that stage of the proof. At a later stage, viz. after writing down line 19, it is discovered that the consistent behaviour of \( p \) is connected with the consistent behaviour of \( q \), and thus that \( p \) becomes \( \Gamma \)-unreliable.

\(^7\) I omit the names for the (derivable) natural deduction rules.
The deletion rule forces us to remove all lines the fifth element of which contains \( q \): lines 10, 11 and 12 are marked ‘out’ and do not longer belong to the proof. By similar reasoning, lines 15, 16 and 17 are removed from the proof at stage 19 (because \( p \) is also \( \Gamma \)-unreliable). It is easy to see that all formulas that occur on unmarked lines are finally \( \text{ACLuN3} \)-derivable from the premises, whereas those that occur on marked lines are not.

The picture looks rather different if we regard the above proof as an \( \text{ACLuN4} \)-proof. If we apply the integrity criterion to the proof, we see that \( \Phi_{19} = \{ \{ p \}, \{ q \} \} \). Line 10 should be marked at stage 19: \( \neg \neg q \) is not derived at some line the fifth element of which does not contain \( q \). Line 15 should also be marked at stage 19: \( \neg p \) is not derived at some line the fifth element of which does not contain \( p \). However, lines 11, 12, 16 and 17 should be unmarked, as they fulfil the integrity criterion at stage 19 of the proof. It is easily seen that all formulas that occur on unmarked lines are finally \( \text{ACLuN4} \)-derivable from the premises, whereas those that occur on marked lines are not. Hence, the \( \text{ACLuN4} \)-consequence set of \( \Gamma \) is richer (it contains \( t \) and \( \neg \neg t \)) than the \( \text{ACLuN3} \)-consequence set of \( \Gamma \).

5.2. Differences between \( \text{ACLuN1/2} \) and \( \text{ACLuN3/4} \)

It should be noticed that we can give alternative characterizations of \( \text{ACLuN1} \)- and \( \text{ACLuN2} \)-models, by means of the definitions given in section 4.

Definition. \( M \) is an \( \text{ACLuN1} \)-model of \( \Gamma \) iff \( M \) is a \( \text{CLuN} \)-model of \( \Gamma \) that is minimally abnormal with respect to the \( \Gamma \)-reliable formulas.

Definition. \( M \) is an \( \text{ACLuN2} \)-model of \( \Gamma \) iff \( M \) is a \( \text{CLuN} \)-model of \( \Gamma \) that is minimally abnormal with respect to both the \( \Gamma \)-reliable as the \( \Gamma \)-unreliable formulas.

As the sufficient information strategy leads to a much weaker selection of \( \text{CLuN} \)-models than the minimal abnormality strategy with respect to \( \Gamma \)-reliable formulas, there will be in general less \( \text{ACLuN1/2} \)-models of \( \Gamma \) than \( \text{ACLuN3/4} \)-models of \( \Gamma \), and hence more \( \text{ACLuN1/2} \)-consequences from \( \Gamma \) than \( \text{ACLuN3/4} \)-consequences. Let me illustrate this with a few examples:

(i) From \( \Gamma = \{ p, \neg p \lor q \} \), \( q \) is (finally) derivable both with \( \text{ACLuN1/2} \) as with \( \text{ACLuN3/4} \) (as \( p \) is \( \Gamma \)-reliable and verified by all \( \text{CLuN} \)-models of \( \Gamma \), whereas \( \neg p \) is not).
(ii) Let $\Gamma = \{(p \& \sim p) \lor q\}$. Then $\Gamma \vdash_{\text{ACLuN1/2}} q$. However $q$ is neither derivable by ACLuN3 nor ACLuN4: $p$ is reliable with respect to $\Gamma$, but neither $p$ nor $\sim p$ is verified by all CLuN-models of $\Gamma$.

(iii) From $\Gamma = \{p \supset q, \sim q\}$, $\sim p$ is (finally) derivable both with ACLuN1/2 as with ACLuN3/4 (as $\sim q$ is $\Gamma$-reliable and verified by all CLuN-models of $\Gamma$, whereas $q$ is not).

(iv) Let $\Gamma = \{p \supset q, p \supset \sim q\}$. Then $\Gamma \vdash_{\text{ACLuN1/2}} \sim p$. With ACLuN3/4, $\sim p$ is not derivable from $\Gamma$: $q$ is reliable with respect to $\Gamma$, but neither $q$ nor $\sim q$ is verified by all CLuN-models of $\Gamma$.

(v) Good old Tweety

Consider the following ACLuN3/4-proof:

\begin{itemize}
  \item 1. $(\forall x)(P x \supset \sim F x)$ \hspace{1cm} PREM \hspace{1cm} $\emptyset$
  \item 2. $(\forall x)(P x \supset B x)$ \hspace{1cm} PREM \hspace{1cm} $\emptyset$
  \item 3. $P t$ \hspace{1cm} PREM \hspace{1cm} $\emptyset$
  \item 4. $B a$ \hspace{1cm} PREM \hspace{1cm} $\emptyset$
  \item 5. $(\forall x)(B x \supset F x)$ \hspace{1cm} PREM \hspace{1cm} $\emptyset$
  \item 6. $P t \supset \sim F t$ \hspace{1cm} 1 \hspace{1cm} $\emptyset$
  \item 7. $P t \supset B t$ \hspace{1cm} 2 \hspace{1cm} $\emptyset$
  \item 8. $B t \supset F t$ \hspace{1cm} 5 \hspace{1cm} $\emptyset$
  \item 9. $\sim F t$ \hspace{1cm} 3, 6 \hspace{1cm} $\emptyset$
  \item 10. $B t$ \hspace{1cm} 3, 7 \hspace{1cm} $\emptyset$
  \item 11. $\sim F t$ \hspace{1cm} 9 \hspace{1cm} $\{F t\}$ \hspace{1cm} OUT
  \item 12. $\sim B t$ \hspace{1cm} 8, 11 \hspace{1cm} $\{F t\}$ \hspace{1cm} OUT
  \item 13. $F t$ \hspace{1cm} 8, 10 \hspace{1cm} $\emptyset$
  \item 14. $F t \& \sim F t$ \hspace{1cm} 9, 13 \hspace{1cm} $\emptyset$
  \item 15. $B a \supset F a$ \hspace{1cm} 5 \hspace{1cm} $\emptyset$
  \item 16. $F a$ \hspace{1cm} 4, 13 \hspace{1cm} $\emptyset$
  \item 17. $P a \supset \sim F a$ \hspace{1cm} 1 \hspace{1cm} $\emptyset$
  \item 18. $\sim F a$ \hspace{1cm} 14 \hspace{1cm} $\{F a\}$
  \item 19. $P b \supset \sim F b$ \hspace{1cm} 15, 16 \hspace{1cm} $\{F a\}$
  \item 20. $P b \supset B b$ \hspace{1cm} 2 \hspace{1cm} $\emptyset$
  \item 21. $B b \supset F b$ \hspace{1cm} 5 \hspace{1cm} $\emptyset$
  \item 22. $P b \supset (F b \& \sim F b)$ \hspace{1cm} 19, 20 \hspace{1cm} $\emptyset$
  \item 23. $\sim P b \lor (F b \& \sim F b)$ \hspace{1cm} 18, 21 \hspace{1cm} $\emptyset$
  \item 24. $(\forall x)(\sim P x \lor (F x \& \sim F x))$ \hspace{1cm} 23 \hspace{1cm} $\emptyset$
\end{itemize}
So, the $\text{ACLuN3/4}$-consequence set contains $Pt$, $Bt$, $Ft$, $\sim Ft$, $Ba$, $Fa$ and $\sim Pa$ but does not contain $(\forall x)(x \neq t \supset \sim Px)$. The reason is that neither $F\alpha$ nor $\sim F\alpha$ (for all constants $\alpha$ other than $t$ and $a$) is derivable at some stage of the proof. Hence only (the much ‘weaker’ formula) $(\forall x)(\sim Px \vee (Fx \& \sim Fx))$, which is a $\text{CLuN}$-consequence of the premises, is $\text{ACLuN3/4}$-derivable. If an accurate reconstruction of (fragments of) Default Logic is indeed possible by the procedure described in section 2, then — with respect to this specific application — $\text{ACLuN3/4}$ is a better candidate for the deductive component than $\text{ACLuN1/2}$.

6. In conclusion

Albeit the inconsistency-adaptive logics $\text{ACLuN3}$ and $\text{ACLuN4}$ were developed for a specific goal, viz. a reconstruction of (mixed) nonmonotonic logics of the default type, they certainly deserve to be studied in their own right. I list some open problems:

(i) The elimination procedure of the sufficient information strategy obviously needs further study. The proof that the sufficient information strategy always leads to a unique set of remaining $\text{CLuN}$-models, will be given in a subsequent paper.

(ii) Is the semantics adequate for the dynamic proof theory? Is another (static) formulation of the semantics of $\text{ACLuN3/4}$ possible? Of course much other meta-theoretic properties should be investigated.

(iii) Which fragments of Default Logic can be reconstructed by means of $\text{ACLuN3/4}$. Could it within certain contexts be preferable to start from a richer lower limit logic, such as $\text{CLuNs}$?

(iv) Some more strategies for inconsistency-adaptive logics should be worked out (once one has found some variants, it is not very difficult to find more). Some suggestions in this direction are made in [6].

---

8 In [10], Guido Vanackere presents the inconsistency-adaptive logic $\text{PRL}$. When preferences are given, $\text{PRL}$ ‘resolves’ inconsistencies derived from the premises by deleting the least preferred half of each inconsistency. $\text{PRL}$ is a viable tool for reconstructing mixed nonmonotonic logics, along a different road than the one followed by Batens and myself.
References


Kristof De Clercq
Centre for Logic and Philosophy of Science
University of Ghent
Kristof.declercq@rug.ac.be