Alexander S. Karpenko

JAŚKOWSKI’S CRITERION
AND THREE-VALUED PARACONSISTENT LOGICS

Abstract. A survey is given of three-valued paraconsistent propositional logics connected with Jaśkowski’s criterion for constructing paraconsistent logics. Several problems are raised and four new matrix three-valued paraconsistent logics are suggested.

From the paper of Jaśkowski [14, p. 145] we can extract the following criterion for constructing paraconsistent logic PL:

a) PL does not verify the implicational law of overfilling

\[ p \rightarrow (\neg p \rightarrow q); \]

b) PL is would be rich enough to enable practical inference;

c) PL has would have an intuitive justification.

The second condition means for us that PL verifies *modus ponens* and at least BCI-logic:

(I) \[ p \rightarrow p, \]

(B) \[ (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \]

(C) \[ (p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)). \]

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The third condition means that in three-valued PL restrictions of the unary operation \( \lnot \) and the binary operations \( \supset, \lor, \land \) to the subset \{0, 1\} coincide with the classical logical operations: negation, implication, disjunction and conjunction. Now let us consider some implications and negations:

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In the above mentioned paper, Jaśkowski (with a reference to J. Słupecki) gives the first example of a matrix three-valued paraconsistent logic with the following operations: \( \rightarrow_{J} \) and \( \lnot_{J} \). But the thesis

\[
(Luk) \quad p \rightarrow (\neg p \rightarrow (\neg \neg p \rightarrow q)),
\]

which was already known to J. Łukasiewicz, holds in this logic. This was the reason for Jaśkowski to reject this logic.

It is really surprising that Jaśkowski did not take as negation the involution \(~\) from Łukasiewicz’s three-valued logic \( L_3 \) with initial operations \( \{\lnot, ~\} \) [17]. The most famous three-valued paraconsistent logic which was constructed independently in many works is the one with \( \rightarrow_{J}, ~, \) and \( \lor \) as max, \( \land \) as min (see [24], [4], [7, p. 214], [21]). Let us denote this logic by \( A_{1} \).

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1. We have also a first-order paraconsistent logic introduced by N.C.A. da Costa in 1964. Cf. also Rozonoe’s [21].
Now let us consider other three-valued paraconsistent logics. B. Sobociński [23] axiomatized three-valued matrix logic with operations → and ∼. It turns out that this logic (S_1) is the implication-negation fragment of RM [19]. In [9] we have a full axiomatization of the three-valued case of RM, namely, RM_3.

The situation in as follows: to relevant logic R [1] the two following axioms are added

\[ (\neg A \land B) \to (A \to B), \]
\[ A \lor (A \to B). \]

A. Avron [5] proved that A_1 and RM_3 are identical (see also [6]):

\[ p \to q = (p \to q) \land \neg q \to \neg p, \]
\[ p \to q = q \lor (p \to q). \]

D. Batens [7, p. 201] considered another three-valued paraconsistent logic: Heyting’s three-valued implication →_H with involution ∼. But Batens rejects this logic (let us denote it by B_1) because adding disjunction to it yields several unpleasant consequences.

Note that the implication of S_1 is relevant, the implication of B_1 intuitionistic, whereas the implication of A_1 classical.

Now I want to attract readers attention to a different famous three-valued paraconsistent logic, namely P_1 [17] with operations →_Se and [. Here operations \lor and \land are defined by means of →_Se and [, where \lor is not max, \land is not min. For the first time truth-tables for these operations appeared in [10], where they were used for the refutation of some tautologues of C_2 which are invalid in the paraconsistent logic C_1 of N. C. A. da Costa. See also [11], where P_1 was called as F. The logic P_1 was also independently found by C. Mortensen in 1979, who called it C_0,1 (see [18, p. 299]). See also A. Arruda’s system V_1 in [2] and in [25].

Only in 1997 E. K. Vojshvillo and J-Y. Béziau [8] discovered independently that in P_1 from [\ A and [ A follows B. So, P_1 contains the formula (Łuk). About unusual properties of P_1 see [15].

Let us note that, if in the full P_1 the operation [ is replaced by the operation ∼ then we have Mortensen’s paraconsistent logic C_0,2 [18] which is a generalization of da Costa’s logic C_1.

Now we consider the following two three-valued paraconsistent logics: Priest’s logic LP [20], and D’Ottaviano’s logic J_3 [12]. The first is Kleene’s three-valued logic \{-K, ∼, ∨, ∧\} [16] with two designated truth-values.
F. Asenjo [3] was the first to propose this logic. It is well known that such logic verifies all tautologies of classical propositional logic $C_2$. So we have there the law of noncontradiction and the law $p \rightarrow (\neg p \rightarrow q)$. But G. Priest defines a relation of logical consequence such that $B$ does not follow from \{A, \neg A\}, and as a consequence *modus ponens* is invalid. The second is the logic $A_1$ with the extra connective $\Diamond$. The functional properties $J_3$ are the same as those of Łukasiewicz’s three-valued logic \{→, L, ¬\} [17], but with the two designated truth-values. D’Ottaviano suggests two axiomatizations of $J_3$ and one of them is rather unusual: it is an extension of from $C_2$ with the operations $→, [\ ]$, $∧$, $\sim$ (see especially in [13, ch. IX]). So we once more have the law of noncontradiction and the law $p \rightarrow (\neg p \rightarrow q)$. Then the question arises, why do we criticize these laws?

At last, we can suggest four new three-valued paraconsistent logics: \{→, [\ ], \neg S, [\ ], \neg H, [\ ], \neg L, [\ ]\}. But all these logics as well as $P_1$ verify the formula (Łuk).

In connection with the formula (Łuk) the problem arises of making more precise the notion of paraconsistent logic. In a usual way, a logic is paraconsistent iff from $A$ and $\neg A$ does not follow an arbitrary $B$. Now D. Batens suggests to restrict this notion: A logic with the formula (Łuk) is not strictly paraconsistent, i.e., for some $A$: $B$ is derivable from $A$ and $\neg A$.

Incidentally, E. K. Vojshvillo suggests the following generalization of the notion of paraconsistency: A logic is paraconsistent, if it does not contain a finite set of formulas from which an arbitrary formula $B$ is derivable.

We still have another problem. Although Johanson’s minimal logic is paraconsistent in the usual sense, it verifies the formula $p \rightarrow (\neg p \rightarrow \neg q)$. (Jaśkowski pointed out that Kolmogorov’s logic has the same properties [14, p. 146]). For details, see [8], where new definitions of paraconsistent logic are given.

**References**


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