Logic and Logical Philosophy
Volume 8 (2000), 65–80

Kristof De Clercq*

TWO NEW STRATEGIES FOR INCONSISTENCY-ADAPTIVE LOGICS

Abstract. In this paper I present two new strategies for inconsistency-adaptive logics: the *reliable sufficient information strategy* of **ACLuN3** and the *minimally abnormal sufficient information strategy* of **ACLuN4**. I give proof theory and semantics for both **ACLuN3** and **ACLuN4**. I also compare them with the well-known inconsistency-adaptive logics **ACLuN1** and **ACLuN2**.

1. Introduction

Inconsistency-adaptive logics are a special brand of paraconsistent logics, and were developed by Diderik Batens around 1980.[1] The best studied inconsistency-adaptive logics are Batens' **ACLuN1** and **ACLuN2** (see especially [5]). Loosely speaking, they 'oscillate' between a lower limit logic, the paraconsistent **CLuN**, and an upper limit logic, Classical Logic (**CL**): they localize the inconsistencies of a set of premises Γ, safeguard Γ for triviality by preventing specific rules of **CL** being applied 'in the neighbourhood of inconsistencies', but behave exactly like **CL** for all other derivations from Γ. They allow for inconsistencies but presuppose the consistency of all sentences 'unless and until proven otherwise'. Interpreting a set of premises

---

* Research Assistant of the Fund for Scientific Research – Flanders (Belgium)(F.W.O.).

[1] See [2], although the paper was written much earlier. For a study of the predicative version see [5]. A survey of the domain is presented in Batens [6]. For an informal description and the relation with argumentation, see [4].

'as consistently as possible', they *adapt* themselves to the *specific* inconsistencies that occur in it. In [6] it is shown that there are two different strategies to do so: the *reliabilitiy strategy* of **ACLuN1**, and the *minimal abnormality strategy* of **ACLuN2**.

For a long time, it seemed that those strategies were the only strategies to devise inconsistency-adaptive logics. An attempt to reconstruct Default Logic by means of an inconsistency-adaptive logic, brought me to develop two new strategies (that are even more cautious than reliability): the *reliable sufficient information strategy* of **ACLuN3** and the *minimally abnormal sufficient information strategy* of **ACLuN4**. After all, it would have been a logical mystery that there exist exactly two and only two strategies to devise inconsistency-adaptive logics.

ACLuN3 and **ACLuN4** are based on the paraconsistent logic **CLuN**. **CLuN** is a poor and basic paraconsistent logic. It is obtained by extending **CL** (with $\neg$ as the classical negation) with the very poor paraconsistent negation $\sim$, by means of the axiom schema $A \vee \sim A$ (semantically, $\sim$ is characterized by a negation-completeness clause only: if $v_M(A) = 0$, then $v_M(\sim A) = 1$). **CLuN** maximally *isolates* inconsistencies in that no contradiction $A \;\&\; \sim A$ entails any other contradiction (not even one for any subformula or superformula of A). For a detailed presentation of **CLuN**, as well of **ACLuN1** and **ACLuN2**, I refer to [5] and [7]. Albeit the fact that classical negation, $\neg$, is defined in **CLuN**, we only allow the occurence of paraconsistent negation, $\sim$, in the premises.

In section 2, I mention a problem concerning the reconstruction of nonmonotonic logics of the default-type by means of inconsistency-adaptive logics. In section 3, I present the *sufficient information strategy.* In section 4, I present the inconsistency-adaptive logics **ACLuN3** and **ACLuN4**. In section 5, I make some comparisons between **ACLuN1/2** and **ACLuN3/4**. I mention some open problems in section 6.

2. Inconsistency-adaptive logics as reconstruction tools for "mixed nonmonotonic logics"[2]

In [3], Diderik Batens proposes an interesting procedure for the reconstruction of mixed nonmonotonic logics. The procedure consists of two compo-

[2] In [3], Diderik Batens introduces the label "mixed nonmonotonic logics" for those (popular) nonmonotonic logics in which a deductive and preferential component are blended together (e.g. the Circumscription approach, Default Logics).

nents. A *deductive* component leads from the premises to a possibly inconsistent consequence set. Several candidates for the deductive component are evaluated, and inconsistency-adaptive logics prove most suitable in this respect. The ensuing *preferential* component is formulated in terms of models and consists itself of two parts: (i) a purely logical procedure connects a set of consistent models to the set of (possibly inconsistent) models of the premises; (ii) a selection procedure picks out the preferred models by relying on the preferences.

Batens offers a successful reconstruction of circumscription by using the inconsistency-adaptive logic **ACLuN2** for the deductive component. Circumscription minimizes (the occurrence of) abnormality predicates in a certain order. This minimization of the abnormality predicates corresponds to a purely logical step in the reconstruction procedure: the restriction to **ACLuN2**-models of the premises minimizes inconsistencies. The *order* in which abnormality predicates are minimized corresponds to the step in which the preferences come in: the selection of the preferred models by relying on the preferences. I will not go into details here, the interested reader should consult [3].

In [9] I attempt to reconstruct (fragments of)[3] Default Logic, using the general procedure of [3]. It turned out that using **ACLuN1** or **ACLuN2** for the deductive component, the yielded consequences were in a sense too strong. Let me illustrate this with an example. Consider the following default theory $T = \langle W, D\rangle$, where $W = \{(\forall x)(Px \supset {\sim} Fx), (\forall x)(Px \supset Bx), Pt, Ba\}$ and $D = \{\frac{Bx:Fx}{Fx}\}$.[4] The default theory T has one extension $E = \mathrm{Cn}(W \cup \{Fa\})$. So Tweety is a penguin, and hence a bird, that does not fly, and a is a flying bird.

For a reconstruction of this default theory, we take $\Gamma = \{(\forall x)(Px \supset {\sim} Fx), (\forall x)(Px \supset Bx), Pt, Ba, (\forall x)(Bx \supset Fx)\}$. The **CLuN**-consequence set of Γ contains the following formulas: Pt, Bt, Ft, ${\sim} Ft$, Ba, Fa, $(\forall x)({\sim} Px \vee (Fx \& {\sim} Fx))$. If we take **ACLuN1** or **ACLuN2** for the deductive component, the consequence set of Γ contains the following formulas: Pt, Bt, Ft, ${\sim} Ft$, Ba, Fa, ${\sim} Pa$, $(\forall x)(x \neq t \supset {\sim} Px)$. The last formula expresses that there are no penguins other than Tweety. Albeit this is exactly what we get following the circumscription approach of the Tweety-example,

[3] The fragment of normal defaults and those semi-normal defaults that can also be represented in a Prioritized Default Logic. See [1] and [8].

[4] Interpreting P, B, F and t as resp. 'Penguin', 'Bird', 'Fly' and 'Tweety', we obtain (a version of) the well known Tweety example.

the default theory T does not imply that Tweety is the only non-flying bird (i.e. Tweety is the only abnormal bird) nor that there are no penguins other than Tweety. This difference between the circumscription approach and Default Logic reveals different underlying intuitions behind *rules with exceptions*. The circumscription approach presupposes that anything that is not bound to be abnormal in view of the premises, is normal: as all penguins are abnormal birds (with respect to being a flyer), all birds not given to be non-flyers are supposed to be flyers (and hence non-penguins). In Default Logic, the intuition behind rules with exceptions is rather different: if there is one exception to a rule, it is plausible there will be others, so it is credulous to assume that the known exceptions to a rule are the only exceptions to that rule. As Tweety is an exception to the rule 'Birds fly', it is plausible that there will be other exceptions to the rule (e.g. other penguins), hence we do not want to conclude that there are no penguins other than Tweety.

Why is $(\forall x)(x \neq t \supset {\sim} Px)$ an **ACLuN1/2**-consequence of Γ? As $(\forall x)({\sim} Px \vee (Fx \& {\sim} Fx)$ is true in all **CLuN**-models of Γ, and **ACLuN1/2** presupposes that all formulas are consistent wherever the premises do not command inconsistency ($Ft \& {\sim} Ft$ is the only contradiction that is 'forced' by the premises), **ACLuN1/2** presupposes that $F\alpha \& {\sim} F\alpha$ is false for all α other than t. Hence, $(\forall x)(x \neq t \supset {\sim} Px)$ is true in all **ACLuN1/2**-models of Γ.

If we want to avoid consequences as $(\forall x)(x \neq t \supset {\sim} Px)$, we will have to use a logic that is 'weaker' (or 'more cautious') than the inconsistency-adaptive logics **ACLuN1** and **ACLuN2**. At the other hand, we want a logic that is *stronger* (leads to a richer consequence set) than the paraconsistent logic **CLuN** (in the example, ${\sim} Pa$ is not a **CLuN**-consequence).

3. Sufficient information strategy

3.1. Intuitive formulation

As described in section 2, attempts to reconstruct Default Logic by using an inconsistency-adaptive logic for the deductive component, forced me into a search for other, less powerful strategies for inconsistency-adaptive logics. One of the most viable candidates was the following strategy:

(S) If all **CLuN**-models of Γ verify A (respectively ${\sim} A$) and some of them falsify ${\sim} A$ (respectively A), then eliminate the **CLuN**-models that verify ${\sim} A$ (respectively A).

As an illustration, let $\Gamma = \{\sim p \vee q, p\}$. As all **CLuN**-models of Γ verify p, and some **CLuN**-models falsify $\sim p$, all **CLuN**-models that verify $\sim p$ are eliminated. As a result, all non-eliminated **CLuN**-models verify q (because all **CLuN**-models of Γ verify $\sim p \vee q$). In the next subsection I will show that this intuitive formulation (S) has two major drawbacks, so it has to be modified.

3.2. Two problems

Consider the set of premises $\Gamma = \{p, q, \sim p \vee \sim q\}$. If we apply (S), we get the following 'instructions':

(i) Because $\Gamma \vDash_{\mathbf{CLuN}} p$ and $\Gamma \nvDash_{\mathbf{CLuN}} \sim p$, all **CLuN**-models of Γ that verify $\sim p$ have to be eliminated. Hence, in all non-eliminated models $v_M(\sim p) = 0$, hence $\sim q$ is true in all of them.

(ii) Because $\Gamma \vDash_{\mathbf{CLuN}} q$ and $\Gamma \nvDash_{\mathbf{CLuN}} \sim q$, all **CLuN**-models of Γ that verify $\sim q$ have to be eliminated. Hence, in all non-eliminated models $v_M(\sim q) = 0$, hence $\sim p$ is true in all of them.

Initially, both (i) as (ii) are applicable. However, as soon as one applies either (i) or (ii), the other becomes inapplicable. From instruction (i) it follows that $\sim p$ is false in all models, hence $p \,\&\, \sim p$ is false in all of them, hence $\sim q$ has to be true in all of them. Hence instruction (ii) becomes inapplicable. By analogous reasoning, the same holds if we start by instruction (ii). It turns out that it depends merely on the accidental order in which we apply the instructions whether $\sim p$ is derivable and $\sim q$ is not, or the other way around.

The diagnosis of the trouble is that the premises do not provide sufficient information to decide which of the sentences behaves inconsistently and which consistently. As $\Gamma \vDash_{\mathbf{CLuN}} (p \,\&\, \sim p) \vee (q \,\&\, \sim q)$ but $\Gamma \nvDash_{\mathbf{CLuN}} p \,\&\, \sim p$ and $\Gamma \nvDash_{\mathbf{CLuN}} q \& \sim q$, p and q are connected with respect to their consistency (in the terminology of **ACLuN1**: both p and q are Γ-unreliable). A remedy is straightforward: (S) should be applied on Γ-reliable formulas only.

A second problem is that by applying (S) we do not reach a fixed point. Consider the set of premises $\Gamma = \{p, \sim p \vee q, r \vee \sim q\}$. As $\Gamma \vDash_{\mathbf{CLuN}} p$ and $\Gamma \nvDash_{\mathbf{CLuN}} \sim p$, all **CLuN**-models that verify $\sim p$ are eliminated. This implies that all non-eliminated models verify q (because $\sim p \vee q$ has to be true in all of them). Following (S) as it stands, no more models can be eliminated (hence it can not be deduced that all non-eliminated **CLuN**-models verify r).[5]

[5] Due to the specific formulation of (S): albeit all non-eliminated models of Γ verify q, the original **CLuN**-models of Γ do not verify q ($\Gamma \nvDash_{\mathbf{CLuN}} q$), and (S) cannot be applied.

Now we would want to apply (S) to these non-eliminated **CLuN**-models: as all non-eliminated **CLuN**-models verify q, and some of them falsify $\sim q$, we eliminate all (remaining) **CLuN**-models of Γ that verify $\sim q$. Hence all remaining **CluN**-models of Γ verify r.

3.3. Decent characterization of the Sufficient Information Strategy

The improved version of the Sufficient Information Strategy, goes as follows:

(SI) If A is reliable with respect to Γ, and all remaining (i.e. not yet eliminated) **CLuN**-models of Γ verify A (respectively $\sim A$), then eliminate the **CLuN**-models that verify $\sim A$ (respectively A). This elimination procedure should be iterated as long as it can (until no more models are eliminated).

It is important to notice that (SI) does indeed lead to a fixed point, and that (SI) leads to a *unique* set of remaining **CLuN**-models, independent of the order in which the **CLuN**-models are eliminated. Although this order depends on the arbitrary picking out of Γ-reliable formulas A, this has no impact at all on the final set of remaining (i.e. non-eliminated) **CLuN**-models of the premises.[6]

By means of (SI) we can formulate two (slightly different) inconsistency-adaptive logics, depending on the way in which Γ-unreliable formulas are interpreted.

4. The inconsistency-adaptive logics ACLuN3 and ACLuN4

4.1. Some definitions

In order to formulate the logics **ACLuN3** and **ACLuN4**, we first need some definitions. Let $DEK\{A_1, \ldots, A_n\}$ refer to $\exists(A_1 \& \sim A_1) \vee \cdots \vee \exists(A_n \& \sim A_n)$: a disjunction of (where necessary) existentially quantified contradictions. A DEK-formula is a formula of the form $DEK\{A_1, \ldots, A_n\}$, and $A_1, \ldots, A_n$ are said to be the *factors* of $DEK\{A_1, \ldots, A_n\}$. Henceforth, it will be easier to write $DEK(\Delta)$, recalling that this is a formula and hence that Δ is finite.

DEFINITION. A DEK-consequence of Γ is a DEK-formula which is **CLuN**-derivable from Γ.

DEFINITION. $DEK(\Delta)$ is a *minimal* DEK-consequence of Γ iff $\Gamma \vDash_{\mathbf{CLuN}} DEK(\Delta)$ and, for no $\Theta \subset \Delta$, $\Gamma \vDash_{\mathbf{CLuN}} DEK(\Theta)$.

[6] Proofs of these claims will have to be postponed for another paper.

DEFINITION. Let $\mathbb{U}(\Gamma) = \{A \mid A \in \Delta$ for some minimal *DEK*-consequence Δ of $\Gamma\}$. $\mathbb{U}(\Gamma)$ is the set of formulas that are (semantically) *unreliable* with respect to Γ.

To get grip on the definitions I give a simple example. Let $\Gamma = \{p \vee q, \sim p, \sim q\}$. It is obvious that $\Gamma \vdash_{\mathbf{CLuN}} (p \,\&\sim p) \vee (q \,\&\sim q)$, while neither $p \,\&\sim p$ nor $q \,\&\sim q$ is **CLuN**-derivable from Γ. Hence, $(p \,\&\sim p) \vee (q \,\&\sim q)$ is a minimal DEK-consequence of Γ. Hence, $\mathbb{U}(\Gamma) = \{p, q\}$, which means that both p and q are unreliable with respect to the set of premises Γ.

DEFINITION. Where M is a model, $\mathrm{Ab}(M) = \{A \mid v_M(\exists(A \,\&\sim A)) = 1\}$

DEFINITION. Where M is a model of a set of premises Γ, $\mathrm{Ab}_U(M) = \mathrm{Ab}(M) \cap \mathbb{U}(\Gamma)$.

DEFINITION. Where M is a model of a set of premises Γ, and $\mathscr{F}$ the set of all formulas, $\mathrm{Ab}_R(M) = \mathrm{Ab}(M) \cap (\mathscr{F} - \mathbb{U}(\Gamma))$.

It is easy to see that $\mathrm{Ab}_{\mathbb{U}}(M) \cap \mathrm{Ab}_R(M) = \emptyset$, and $\mathrm{Ab}_{\mathbb{U}}(M) \cup \mathrm{Ab}_R(M) = \mathrm{Ab}(M)$.

DEFINITION. A **CLuN**-model M of Γ is *minimally abnormal with respect to the Γ-unreliable formulas* iff there is no **CLuN**-model M' of Γ such that $\mathrm{Ab}_{\mathbb{U}}(M') \subset \mathrm{Ab}_{\mathbb{U}}(M)$.

DEFINITION. A **CLuN**-model M of Γ is *minimally abnormal with respect to the Γ-reliable formulas* iff there is no **CLuN**-model M' of Γ such that $\mathrm{Ab}_R(M') \subset \mathrm{Ab}_R(M)$.

4.2. The Reliable Sufficient Information Strategy of ACLuN3

4.2.1. Proof theory

The idea of the proof theory of **ACLuN3** is that we apply all rules of (or derivable in) **CLuN** unconditionally, whereas a conditional rule is applied on a provisional basis and on the condition that certain formulas are *reliable* (with respect to their consistent behaviour). To keep the matter algorithmic, the consistent behaviour of a formula will be determined by the stage of the proof. As a result (in accordance with **ACLuN1/2**), proofs will be dynamic in that wffs derived at some stage of the proof may not be derivable at a later stage.

Following [5], **ACLuN3**-proofs are written in a specific format. Each line in a proof consists of five elements: (i) a line number; (ii) the wff derived;

(iii) the line numbers of the wffs from which it is derived; (iv) the rule by which it is derived; and (v) the set of formulas that should be reliable in order for the wff to be derivable.

DEFINITION. A formula A *occurs unconditionally* at some line of a proof iff the fifth element of that line is empty.

DEFINITION. *A behaves consistently at a stage of a proof* iff $\exists(A \& \sim A)$ does not occur unconditionally in the proof at that stage.

DEFINITION. The consistent behaviour of A_1 is *connected to* the consistent behaviour of $A_2, \ldots, A_n$ at a stage of a proof iff $DEK(A_1, \ldots, A_n)$ occurs unconditionally in the proof at that stage whereas $DEK(\Delta)$ does not occur unconditionally in it for any $\Delta \subset \{A_1, \ldots, A_n\}$.

DEFINITION. A is reliable at a stage of a proof iff A behaves consistently at that stage and its consistent behaviour is not connected to the consistent behaviour of other formulas.

Given these definitions, proofs in **ACLuN3** are governed by an *unconditional rule*, a *conditional rule* and a *deletion rule*. An application of RU or RC to a proof at a stage produces the next stage.

RU All derivation rules of **CLuN** are unconditionally valid in any **ACLuN3**-proof. The fifth element of the new line is the union of the fifth elements of the lines mentioned in its third element.

RC If A (resp. $\sim A$) occurs as the second element of a line in the proof at depth zero (i.e. not depending on any hypothesis), then you may derive $\neg \sim A$ (resp. $\neg A$) *provided* that A is reliable at that stage of the proof. The fifth element of the new line is the union of the fifth element of the line on which A (resp. $\sim A$) occurs and $\{A\}$.

RD If C is not (any more) reliable, then delete from the proof all lines the fifth element of which contains C.

Wffs that occur unconditionally are **CLuN**-derivable from the premises (and cannot possibly be 'deleted' later). The unconditional occurence of DEK-formulas at a stage determines which formulas are reliable at that stage. Wffs that occur in the proof at a stage are derivable at that stage. Of course, we need a more stable notion, *final* derivability, that does not depend on the stage of the proof.

DEFINITION. A is finally derived at some line in an **ACLuN3**-proof iff, (i) A is the second element of the line and (ii) where Δ ($\subseteq \emptyset$) is the fifth element of the line, any extension of the proof can be furter extended in such a way that it contains a line that has A as its second element and Δ as its fifth element.

Definition. $\Gamma \vdash_{\mathbf{ACLuN3}} A$ (A is an **ACLuN3**-consequence of Γ) iff A is finally derived at some line in an **ACLuN3**-proof.

4.2.2. Semantics

The **ACLuN3**-semantics is obtained from the **CLuN**-semantics by defining, for each Γ, a subset of the **CLuN**-models of Γ. The idea is that any Γ defines a set of (semantically) unreliable formulas, and that the **ACLuN3**-models of Γ are those **CLuN**-models that are not eliminated by the sufficient information strategy.

Definition. M is an **ACLuN3**-model of Γ iff (i) M is a **CLuN**-model of Γ and (ii) M is not eliminated by the sufficient information strategy.

Definition. $\Gamma \vDash_{\mathbf{ACLuN3}} A$ iff A is true in all **ACLuN3**-models of Γ.

4.3. The Minimally Abnormal Sufficient Information Strategy of ACLuN4

4.3.1. Semantics

For **ACLuN4**, it appears advisable to start from the semantics. The central difference withe the **ACLuN3**-semantics is that a stronger selection of **CLuN**-models occurs: all **ACLuN4**-models of Γ are **ACLuN3**-models of Γ, but the converse does not always hold. If, e.g., $DEK\{p,q\}$ is the only minimal DEK-consequence of Γ, then, unlike for **ACLuN3**-models of Γ, either $p \mathbin{\&} {\sim} p$ or $q \mathbin{\&} {\sim} q$ is false in any **ACLuN4**-model of Γ.

Definition. M is an **ACLuN4**-model of Γ iff (i) M is a **CLuN**-model of Γ and (ii) there is no **CLuN**-model M' such that $\mathrm{Ab}_{\mathbb{U}}(M') \subset \mathrm{Ab}_{\mathbb{U}}(M)$ and (iii) M is not eliminated by the sufficient information strategy.

Definition. $\Gamma \vDash_{\mathbf{ACLuN4}} A$ iff A is true in all **ACLuN4**-models of Γ.

4.3.2. Proof theory

The format of proofs is as for **ACLuN3**, except that no lines are deleted in **ACLuN4**-proofs, but that there may be tentative lines, indicated with a mark 'OUT'. Marked lines are not considered as occurring in the proof and may not be relied upon for adding further lines. After each step, the marks are updated, viz. removed or added.

The updating of the marks is governed by an *integrety criterion*, as presented in [5]. The intuitive idea is as follows. Suppose that A is derived on one or more lines the fifth element of which is not empty. A is considered as derived (at a stage of the proof) and the lines become a full part of the proof if A comes out true under any maximally normal 'interpretation' of the least *DEK*-formulas (at that stage).

As the integrity criterion will look at combinations of factors of *DEK*-formulas, it is useful to remark that some *DEK*-formulas that occur unconditionally in a proof may be disregarded. Suppose that a Gödel-numbering (or some other ordering) of formulas is given. Where A and B are *DEK*-formulas, the following definitions can be given:

DEFINITION. $A \prec B$ iff either (i) $A \vdash_{\mathbf{CLuN}} B$ and $B \nvdash_{\mathbf{CLuN}} A$, or (ii) A and B are **CLuN**-equivalent and the Gödel-number of A is smaller than the Gödel number of B.

DEFINITION. A is a *least DEK-formula* (at a stage of the proof) if it occurs unconditionally in the proof and no *DEK*-formula B, such that $B \prec A$, occurs unconditionally in the proof.

If $DEK(\Gamma \cup \{Px\})$ and $DEK(\Gamma \cup \{Py\})$ occur unconditionally in the proof and the Gödel-number of the former is smaller than that of the latter, then at best the former will be a least *DEK*-formula. Neither of them is a least *DEK*-formula if $DEK(\Gamma \cup \{Pa\})$ also occurs unconditionally in the proof. Clearly, if one disjunct of each least *DEK*-formula is true, then all *DEK*-formulas are true (at that stage).

Let ${}^*\Phi_s$ be the set of all sets that contain one factor out of each least *DEK*-formula (at stage s of the proof). ${}^*\Phi_s$ may contain redundant elements for two different reasons. The first is related to the individual variables. Where neither x nor y occurs free in $A(z)$, $\exists(A(x) \,\&\, {\sim} A(x))$ is **CLuN**-equivalent to $\exists(A(y) \,\&\, {\sim} A(y))$. But $A(x)$ may be a factor of some least *DEK*-formula and $A(y)$ of another. Hence, ${}^*\Phi_s$ may contain $\{Px, Py\}$, or may contain both $\{Px, p\}$ and $\{Py, p\}$. To reduce these, ${}^\circ\Phi_s$ is defined from ${}^*\Phi_s$ by relettering all open formulas in the members of ${}^*\Phi_s$ in such a way that the free variables occur always in the same order (for all formulas, the first occurring free variable is always x_1, the second always x_2, etc.). The second reason for redundant elements is that the same factor may occur in different least *DEK*-formulas. If $DEK\{p, q\}$ and $DEK\{p, r\}$ are the least

DEK-formulas, ${}^*\Phi_s = {}^\circ\Phi_s = \{\{p\},\{p,r\},\{p,q\},\{q,r\}\}$. Of these $\{p,r\}$ and $\{p,q\}$ are redundant: both $DEK\{p,q\}$ and $DEK\{p,r\}$ are true if $p \mathbin{\&} {\sim} p$ is true; there is no need that also $r \mathbin{\&} {\sim} r$ or $q \mathbin{\&} {\sim} q$ be true. So, let Φ_s be obtained from ${}^\circ\Phi_s$ by eliminating elements from it that are proper supersets of other elements. The members of Φ_s are sets of formulas, such that, if $\exists(A \mathbin{\&} {\sim} A)$ is true for all members A of such a set, then all DEK-formulas that occur unconditionally in the proof are true. To see this, it is sufficient to realize that, if A and B are different formulas (and not reletterings of each other with respect to the individual variables), then $\exists(A \mathbin{\&} {\sim} A)$ and $\exists(B \mathbin{\&} {\sim} B)$ are **CLuN**-independent formulas — remember that **CLuN** does not spread inconsistencies.

DEFINITION. Where A is the second element of line j, line j *fulfils* the integrity criterion (at stage s) iff (i) the intersection of some member of Φ_s and of the fifth element of line j is empty, and (ii) for each $\varphi \in \Phi_s$ there is a line k such that the intersection of φ and of the fifth element of line k is empty and A is the second element of line k.

As a (very) simple illustration, consider:

(j)	$DEK\{p,q,r\}$			$\emptyset$
$(j+1)$	A	...	...	$\{p,q\}$
$(j+2)$	A	...	...	$\{q,r\}$
$(j+3)$	A	...	...	$\{p,r\}$

If (j) is the only least DEK-formula in the proof, $\Phi_s = \{\{p\},\{q\},\{r\}\}$ and lines $(j+1)-(j+3)$ fulfil the integrity criterion. They also fulfil the integrity criterion if the second element of line (j) is $DEK\{p,q,r,s\}$.

Let us now turn to the **ACLuN4**-rules.

RU As for **ACLuN3**.

RC If A (resp. ${\sim} A$) occurs as the second element of a line in the proof at depth zero (i.e. not depending on any hypothesis), then you may derive $\neg{\sim} A$ (resp. $\neg A$) *provided* that, at that stage, A *behaves consistently.* The fifth element of the new line is the union of the fifth element of the line on which A (resp. ${\sim} A$) occurs and $\{A\}$.

RQ+ A mark is added to a line that does not fulfil the integrity criterion, and to all lines derived from it.

RQ− If a line fulfils the integrity criterion and is marked, the mark is removed.

DEFINITION. A is finally derived at some line in an **ACLuN4**-proof iff A is the second element of that line and any (possibly infinite) extension of the proof can be further extended in such a way that the line is unmarked.

DEFINITION. $\Gamma \vdash_{\mathbf{ACLuN4}} A$ (A is an **ACLuN4**-consequence of Γ) iff A is finally derived at some line of an **ACLuN2**-proof from Γ.

5. Some comparisons

5.1. The difference between ACLuN3 and ACLuN4

Here is an example of an **ACLuN3**-proof.[7]

1.	$s \,\&\, q$	PREM	$\emptyset$	
2.	$t \vee p$	PREM	$\emptyset$	
3.	$\sim q \vee t$	PREM	$\emptyset$	
4.	$\sim p$	PREM	$\emptyset$	
5.	$r \vee \sim s$	PREM	$\emptyset$	
6.	$p \vee \sim q$	PREM	$\emptyset$	
7.	s	1	$\emptyset$	
8.	q	1	$\emptyset$	
9.	$\neg \sim s$	7	$\{s\}$	
10.	$\neg \sim q$	8	$\{q\}$	OUT
11.	t	3, 8	$\{q\}$	OUT
12.	$\neg \sim t$	11	$\{q, t\}$	OUT
13.	r	5, 9	$\{s\}$	
14.	$\neg \sim r$	13	$\{r, s\}$	
15.	$\neg p$	4	$\{p\}$	OUT
16.	t	2, 15	$\{p\}$	OUT
17.	$\neg \sim t$	16	$\{p, t\}$	OUT
18.	$(p \,\&\, \sim p) \vee \sim q$	4, 6	$\emptyset$	
19.	$(p \,\&\, \sim p) \vee (q \,\&\, \sim q)$	8, 18	$\emptyset$	

Line 9 is a typical conditional derivation. From s we derive $\neg \sim s$, as s is reliable at that stage of the proof. We mention s as the fifth element of line 9. Line 10 is also a conditional derivation. From q we derive $\neg \sim q$, as q is reliable at that stage of the proof. At a later stage, viz. after writing down line 19, it is discovered that the consistent behaviour of p is connected with the consistent behaviour of q, and thus that p becomes Γ-unreliable.

[7] I omit the names for the (derivable) natural deduction rules.

The deletion rule forces us to remove all lines the fifth element of which contains q: lines 10, 11 and 12 are marked 'OUT' and do not longer belong to the proof. By similar reasoning, lines 15, 16 and 17 are removed from the proof at stage 19 (because p is also Γ-unreliable). It is easy to see that all formulas that occur on unmarked lines are finally **ACLuN3**-derivable from the premises, whereas those that occur on marked lines are not.

The picture looks rather different if we regard the above proof as an **ACLuN4**-proof. If we apply the integrity criterion to the proof, we see that $\Phi_{19} = \{\{p\}, \{q\}$. Line 10 should be marked at stage 19: $\neg \sim q$ is not derived at some line the fifth element of which does not contain q. Line 15 should also be marked at stage 19: $\neg p$ is not derived at some line the fifth element of which does not contain p. However, lines 11, 12, 16 and 17 should be unmarked, as they fulfil the integrity criterion at stage 19 of the proof. It is easily seen that all formulas that occur on unmarked lines are finally **ACLuN4**-derivable from the premises, whereas those that occur on marked lines are not. Hence, the **ACLuN4**-consequence set of Γ is richer (it contains t and $\neg \sim t$) than the **ACLuN3**-consequence set of Γ.

5.2. Differences between ACLuN1/2 and ACLuN3/4

It should be noticed that we can give alternative characterizations of **ACLuN1**- and **ACLuN2**-models, by means of the definitions given in section 4.

Definition. M is an **ACLuN1**-model of Γ iff M is a **CLuN**-model of Γ that is minimally abnormal with respect to the Γ-*reliable* formulas.

Definition. M is an **ACLuN2**-model of Γ iff M is a **CLuN**-model of Γ that is minimally abnormal with respect to both the Γ-*reliable* as the Γ-*unreliable* formulas.

As the sufficient information strategy leads to a much weaker selection of **CLuN**-models than the minimal abnormality strategy with respect to Γ-reliable formulas, there will be in general less **ACLuN1/2**-models of Γ than **ACLuN3/4**-models of Γ, and hence more **ACLuN1/2**-consequences from Γ than **ACLuN3/4**-consequences. Let me illustrate this with a few examples:

(i) From $\Gamma = \{p, \sim p \vee q\}$, q is (finally) derivable both with **ACLuN1/2** as with **ACLuN3/4** (as p is Γ-reliable and verified by all **CLuN**-models of Γ, whereas $\sim p$ is not).

(ii) Let $\Gamma = \{(p \,\&\, {\sim} p) \vee q\}$. Then $\Gamma \vdash_{\mathbf{ACLuN1/2}} q$. However q is neither derivable by **ACLuN3** nor **ACLuN4**: p is reliable with respect to Γ, but neither p nor ${\sim} p$ is verified by all **CLuN**-models of Γ.

(iii) From $\Gamma = \{p \supset q, {\sim} q\}$, ${\sim} p$ is (finally) derivable both with **ACLuN1/2** as with **ACLuN3/4** (as ${\sim} q$ is Γ-reliable and verified by all **CLuN**-models of Γ, whereas q is not).

(iv) Let $\Gamma = \{p \supset q, p \supset {\sim} q\}$. Then $\Gamma \vdash_{\mathbf{ACLuN1/2}} {\sim} p$. With **ACLuN3/4**, ${\sim} p$ is not derivable from Γ: q is reliable with respect to Γ, but neither q nor ${\sim} q$ is verified by all **CLuN**-models of Γ.

(v) GOOD OLD TWEETY

Consider the following **ACLuN3/4**-proof:

1.	$(\forall x)(Px \supset {\sim} Fx)$	PREM	$\emptyset$	
2.	$(\forall x)(Px \supset Bx)$	PREM	$\emptyset$	
3.	Pt	PREM	$\emptyset$	
4.	Ba	PREM	$\emptyset$	
5.	$(\forall x)(Bx \supset Fx)$	PREM	$\emptyset$	
6.	$Pt \supset {\sim} Ft$	1	$\emptyset$	
7.	$Pt \supset Bt$	2	$\emptyset$	
8.	$Bt \supset Ft$	5	$\emptyset$	
9.	${\sim} Ft$	3, 6	$\emptyset$	
10.	Bt	3, 7	$\emptyset$	
11.	$\neg Ft$	9	$\{Ft\}$	OUT
12.	$\neg Bt$	8, 11	$\{Ft\}$	OUT
13.	Ft	8, 10	$\emptyset$	
14.	$Ft \,\&\, {\sim} Ft$	9, 13	$\emptyset$	
13.	$Ba \supset Fa$	5	$\emptyset$	
14.	Fa	4, 13	$\emptyset$	
15.	$Pa \supset {\sim} Fa$	1	$\emptyset$	
16.	$\neg {\sim} Fa$	14	$\{Fa\}$	
17.	$\neg Pa$	15, 16	$\{Fa\}$	
18.	$Pb \supset {\sim} Fb$	1	$\emptyset$	
19.	$Pb \supset Bb$	2	$\emptyset$	
20.	$Bb \supset Fb$	5	$\emptyset$	
21.	$Pb \supset Fb$	19, 20	$\emptyset$	
22.	$Pb \supset (Fb \,\&\, {\sim} Fb)$	18, 21	$\emptyset$	
23.	${\sim} Pb \vee (Fb \,\&\, {\sim} Fb)$	22	$\emptyset$	
24.	$(\forall x)({\sim} Px \vee (Fx \,\&\, {\sim} Fx))$	23	$\emptyset$	

So, the **ACLuN3/4**-consequence set contains Pt, Bt, Ft, $\sim Ft$, Ba, Fa and $\neg Pa$ but does not contain $(\forall x)(x \neq t \supset \sim Px)$. The reason is that neither $F\alpha$ nor $\sim F\alpha$ (for all constants α other than t and a) is derivable at some stage of the proof. Hence only (the much 'weaker' formula) $(\forall x)(\sim Px \vee (Fx \,\&\, \sim Fx))$, which is a **CLuN**-consequence of the premises, is **ACLuN3/4**-derivable. If an accurate reconstruction of (fragments of) Default Logic is indeed possible by the procedure described in section 2, then — with respect to this specific application — **ACLuN3/4** is a better candidate for the deductive component than **ACLuN1/2**.[8]

6. In conclusion

Albeit the inconsistency-adaptive logics **ACLuN3** and **ACLuN4** were developed for a specific goal, viz. a reconstruction of (mixed) nonmonotonic logics of the default type, they certainly deserve to be studied in their own right. I list some open problems:

(i) The elimination procedure of the *sufficient information* strategy obviously needs furter study. The proof that the sufficient information strategy always leads to a unique set of remaining **CLuN**-models, will be given in a subsequent paper.

(ii) Is the semantics adequate for the dynamic proof theory? Is another (static) formulation of the semantics of **ACLuN3/4** possible? Of course much other meta-theoretic properties should be investigated.

(iii) Which fragments of Default Logic can be reconstructed by means of **ACLuN3/4**. Could it within certain contexts be preferable to start from a richer lower limit logic, such as **CLuNs**?

(iv) Some more strategies for inconsistency-adaptive logics should be worked out (once one has found some variants, it is not very difficult to find more). Some suggestions in this direction are made in [6].

[8] In [10], Guido Vanackere presents the inconsistency-adaptive logic **PRL**. When preferences are given, **PRL** 'resolves' inconsistencies derived from the premises by deleting the least preferred half of each inconsistency. **PRL** is a viable tool for reconstructing mixed nonmonotonic logics, along a different road than the one followed by Batens and myself.

References

[1] Antoniou, G., *Nonmonotonic Reasoning*. Cambridge, Mass., MIT Press, 1996.

[2] Batens, D., "Dynamic Dialectical logics", in: G. Priest, R. Routley & J. Norman (eds.), *Paraconsistent Logic. Essays on the Inconsistent*. München, Philosophia Verlag, 1989, 187–217.

[3] Batens, D., "Inconsistency-Adaptive Logics and the Foudation of Non-Monotonic Logics", *Logique et Analyse* 145, Mars 1994, 57–94.

[4] Batens, D., "Functioning and teaching of adaptive logics", in: J. Van Benthem, F. H. Van Eemeren, R. Grootendorst and F. Veltman (eds.), *Logic and Argumentation*. Amsterdam, North-Holland, 1996, pp. 241–254.

[5] Batens, D., "Inconsistency-Adaptive Logics", in: Ewa Orłowska (ed.), *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*. Heidelberg, New York, Springer, 1998, 445–472.

[6] Batens, D., "A survey of inconsistency-adaptive logics". In D. Batens, C. Mortensen, G. Priest, J. P. Van Bendegem (eds.): *Frontiers of Paraconsistent Logic*. Baldock, Research Studies Press, King's College Publications, 2000, pp. 49–73.

[7] Batens, D., "Towards the unification of inconsistency handling mechanisms". In this volume pp. 5–31.

[8] Brewka, G., *Nonmonotonic Reasoning: Logical Foundations of Commonsense*. Cambridge University Press, 1991.

[9] De Clercq, K., "Reconstructie van een aantal niet-monotone logica's aan de hand van een inconsistentie-adaptive logica". Unpublished dissertation (in Dutch).

[10] Vanackere, G., "Preferences as inconsistency-resolvers: the inconsistency-adaptive Logic PRL". In this volume pp. 47–63.

Kristof De Clercq
Centre for Logic and Philosophy of Science
University of Ghent
`Kristof.declercq@rug.ac.be`