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ON THE LOGICS RELATED TO A. ARRUDA'S SYSTEM V1

Four logics $\mathbf{I_0}$, $\mathbf{I_1}$, $\mathbf{I_2}$, and $\mathbf{I_3}$ related to A. Arruda's system **V1** are considered. For each of them the semantics of descriptions of states in the style of E. K. Vojshvillo [2] is constructed, the question of characterizability by means of finite logical matrix is investigated and Gentzen-type sequent version is presented.

Definition 1. The language $\mathcal{L}$ is standard propositional language with alphabet $\langle \mathcal{S}, \&, \vee, \supset, \neg,), (\rangle$, where $\mathcal{S} = \{S_0, S_1, S_2, \ldots\}$ is the set of all propositional letters of $\mathcal{L}$. Let $\mathcal{F}$ be the set of all formulæ of $\mathcal{L}$.

Definition 2. Let $\mathbf{Cl}_{\&\vee\supset}$ be the set of all classical tautologies from $\mathcal{F}$ which do not contain negation $\neg$.

Definition 3. The logic $\mathbf{I_0}$ is the smallest subset of $\mathcal{F}$ closed on the *modus ponens* and the rule of substitution such that $\mathbf{Cl}_{\&\vee\supset} \subseteq \mathbf{I_0}$ and for $A, B \in \mathcal{F}$:

(1) $\neg(S_0 \supset S_0) \supset A \in \mathbf{I_0}$,

(2) if $A \notin \mathcal{S}$ then
$(A \supset \neg(S_0 \supset S_0)) \supset \neg A \in \mathbf{I_0}$, and $(A \supset B) \supset ((B \supset \neg A) \supset \neg A) \in \mathbf{I_0}$,

(3) if $A \notin \mathcal{S}$ then
$(A \supset (\neg A \supset \neg(S_0 \supset S_0)) \in \mathbf{I_0}$, and $((B \supset A) \supset A) \supset (\neg A \supset B) \in \mathbf{I_0}$.

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The definition of $\mathbf{I_1}$ (resp. $\mathbf{I_2}$) is obtained from the definition of $\mathbf{I_0}$ simply by avoiding the restriction on A in the clause (2) (resp. in the clause (3)) and replacing $\mathbf{I_0}$ by $\mathbf{I_1}$ (resp. by $\mathbf{I_2}$).

Note that $\mathbf{I_1}$ is a set of all provable in **V1** formulae which do not contain any occurences of "classical propositional letters" (in terms of [1]) provided that $\mathcal{S}$ is a set of all "Vasil'jev's propositional letters" in **V1** (in terms of [1]).

To obtain the definition of $\mathbf{I_3}$ add to the definition of $\mathbf{I_0}$ (and then replace $\mathbf{I_0}$ by $\mathbf{I_3}$) the clause:

(4) $A \supset (\neg A \supset ((B \supset \neg B) \supset \neg B)) \in \mathbf{I_3}$.

Definition 4. A *description of state* is a mapping of the set $\{S_0, \neg S_0, S_1, \neg S_1, S_2, \neg S_2, \ldots\}$ into the set $\{0,1\}$. Let DS be the set of all descriptions of state.

Definition 5. Let $v \in \mathrm{DS}$. Then

v is *complete*	iff	for each $i \in \mathbb{N}$: $v(S_i) = 1$ or $v(\neg S_i) = 1$.
v is *consistent*	iff	for each $i \in \mathbb{N}$: $v(S_i) = 0$ or $v(\neg S_i) = 0$.
v is *quasi-complete*	iff	either $v(S_i) = 0$ and $v(\neg S_i) = 0$ for each $i \in \mathbb{N}$, or $v(S_i) = 1$ or $v(\neg S_i) = 1$ for each $i \in \mathbb{N}$.

Definition 6. For each $v \in \mathrm{DS}$, a mapping $|\ |_v \colon \mathcal{F} \to \{0,1\}$ is specified as follows:

(a) for each $i \in \mathbb{N}$: $|S_i|_v = v(S_i)$ and $|\neg S_i|_v = v(\neg S_i)$;

(b) for each $A \notin \mathcal{S}$: $|\neg A|_v = 1$ iff $|A|_v = 0$;

(c) for each $A, B \in \mathcal{F}$:

$$\begin{aligned} |A \mathbin{\&} B|_v = 1 &\quad \text{iff} \quad |A|_v = 1 \text{ and } |B|_v = 1; \\ |A \vee B|_v = 1 &\quad \text{iff} \quad |A|_v = 1 \text{ or } |B|_v = 1; \\ |A \supset B|_v = 1 &\quad \text{iff} \quad |A|_v = 0 \text{ or } |B|_v = 1. \end{aligned}$$

It is known that a formula is classical tautology iff $|A|_v = 1$ for every complete and consistent v in DS. Similar propositions can be proved for the systems under consideration.

Theorem 1. $A \in \mathbf{I_0}$ *iff for each* $v \in \mathrm{DS}$*:* $|A|_v = 1$.

Theorem 2. $A \in \mathbf{I_1}$ *iff for each complete* $v \in \mathrm{DS}$*:* $|A|_v = 1$.

Theorem 3. $A \in \mathbf{I_2}$ *iff for each consistent* $v \in \mathrm{DS}$*:* $|A|_v = 1$.

THEOREM 4. $A \in \mathbf{I_3}$ *iff for each quasi-complete* $v \in \mathrm{DS}$*:* $|A|_v = 1$.

DEFINITION 7. Let $\mathfrak{M}_0 = \langle \{0, 1, \mathrm{t}, \mathrm{f}\}, \{1\}, \&^0, \vee^0, \supset^0, \neg^0 \rangle$ is logical matrix operations of which are defined by the following tableaux:

$x \&^0 y$	1	0	t	f
1	1	0	1	0
0	0	0	0	0
t	1	0	1	0
f	0	0	0	0

$x \vee^0 y$	1	0	t	f
1	1	1	1	1
0	1	0	1	0
t	1	1	1	1
f	1	0	1	0

$x \supset^0 y$	1	0	t	f
1	1	0	1	0
0	1	1	1	1
t	1	0	1	0
f	1	1	1	1

x	$\neg^0 x$
1	0
0	1
t	1
f	0

Let $\mathfrak{M}_1 = \langle \{0, 1, \mathrm{t}\}, \{1\}, \&^1, \vee^1, \supset^1, \neg^1 \rangle$ and $\mathfrak{M}_2 = \langle \{0, 1, \mathrm{f}\}, \{1\}, \&^2, \vee^2, \supset^2, \neg^2 \rangle$ are submatrices of $\mathfrak{M}_0$ (where $\&^1$ and $\&^2$ are the results of corresponding narrowing of $\&^0$; similarly for all other operations in $\mathfrak{M}_1$ and $\mathfrak{M}_2$).

DEFINITION 8. An *evaluation* of $\mathcal{F}$ in the matrix $\mathfrak{M}_i$ (for $i = 0,\ 1,\ 2$) is a mapping v from $\mathcal{F}$ into a carrier of the matrix $\mathfrak{M}_i$ such that $v(\neg A) = \neg^i v(A)$ and $v(A \circ B) = v(A) \circ^i v(B)$ where $\circ \in \{\&, \vee, \supset\}$.

DEFINITION 9. An evaluation v of $\mathcal{F}$ in $\mathfrak{M}_i$ ($i = 0,\ 1,\ 2$) is *quasi-complete* iff either $v(S_i) \neq \mathrm{t}$ for every $i \in \mathbb{N}$, or $v(S_i) \neq \mathrm{f}$ for every $i \in \mathbb{N}$.

Then the following theorems can be proved by means of the modification of Henkin's method.

THEOREM 5. *For* $i = 0, 1, 2$*:*

$A \in \mathbf{I}_{\boldsymbol{i}}$ *iff for each evaluation* v *of* $\mathcal{F}$ *in* $\mathfrak{M}_i$*:* $|A|_v = 1$.

THEOREM 6. $A \in \mathbf{I_3}$ *iff for each quasi-complete evaluation* v *of* $\mathcal{F}$ *in* $\mathfrak{M}_0$*:* $|A|_v = 1$.

Sequent calculus $\mathbf{GI_0}$ can be obtained from Gentzen's **LK** (see [3]) simply by avoiding the rules for quantifiers (with corresponding modification of language) and replacing the rules

$$\frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}\ \mathrm{NES} \qquad\qquad \frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ \mathrm{NEA}$$

by the rules

$$\frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}\ \mathrm{NES}' \qquad\qquad \frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ \mathrm{NEA}' \qquad \text{where } A \notin \mathcal{S}$$

respectively (with corresponding modification of the definition of deduction). The calculus $\mathbf{GI_1}$ (respectively $\mathbf{GI_2}$) is obtained from $\mathbf{GI_0}$ when NES$'$ is replaced by NES (respectively NEA$'$ is replaced by NEA) with corresponding modification of the definition of deduction. $\mathbf{GI_3}$ is $\mathbf{GI_0}$ extended by a set of basic sequent of the form $S_n, \neg S_n \to S_m, \neg S_m$ where $S_n, S_m \in \mathcal{S}$ (with corresponding modification of the definition of deduction).

Cut-elimination theorem can be proved for each $\mathbf{GI}_{\boldsymbol{i}}$ ($i \in \{0,1,2,3\}$) using the method presented in [3].

Theorem 7. *For $i = 0, 1, 2, 3$:*

$$A \in \mathbf{I}_{\boldsymbol{i}} \quad \textit{iff} \quad \textit{the sequent} \to A \textit{ is deducible in } \mathbf{GI}_{\boldsymbol{i}}.$$

References

[1] Arruda, A.I., "On the imaginary logic of N. A. Vasil'ev", in *Proceedings of Fourth Latin-American Symposium on Mathematical Logic*, North-Holland, 1979.

[2] Vojshvillo, E. K., *Philosophical and Methodological Aspects of Relevant Logic*, Moscow, 1988 (in Russian).

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