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JAŚKOWSKI'S CRITERION AND THREE-VALUED PARACONSISTENT LOGICS*

Abstract. A survey is given of three-valued paraconsistent propositional logics connected with Jaśkowski's criterion for constructing paraconsistent logics. Several problems are raised and four new matrix three-valued paraconsistent logics are suggested.

From the paper of Jaśkowski [14, p. 145] we can extract the following criterion for a constructing paraconsistent logic **PL**:

a) **PL** does not verify the implicational law of overfilling

$$p \to (\neg p \to q);$$

b) **PL** is would be rich enough to enable practical inference;

c) **PL** has would have an intuitive justification.

The second condition means for us that **PL** verifies *modus ponens* and at least **BCI**-logic:

(I) $$p \to p\,,$$

(B) $$(q \to r) \to ((p \to q) \to (p \to r))\,,$$

(C) $$(p \to (q \to r)) \to (q \to (p \to r))\,.$$

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The third condition means that in three-valued **PL** restrictions of the unary operation $\neg$ and the binary operations $\supset$, $\vee$, $\wedge$ to the subset $\{0, 1\}$ coincide with the classical logical operations: negation, implication, disjunction and conjunction. Now let us consider some implications and negations:

$\to_{\mathrm{J}}$	0	½	1
0	1	1	1
*½	0	½	1
*1	0	½	1

$\to_{\mathrm{S}}$	0	½	1
0	1	1	1
*½	0	½	1
*1	0	0	1

$\to_{\mathrm{Se}}$	0	½	1
0	1	1	1
*½	0	1	1
*1	0	1	1

$\to_{\mathrm{H}}$	0	½	1
0	1	1	1
½	0	1	1
*1	0	½	1

$\to_{\mathrm{L}}$	0	½	1
0	1	1	1
½	½	1	1
*1	0	½	1

$\to_{\mathrm{K}}$	0	½	1
0	1	1	1
½	½	½	1
*1	0	½	1

p	$\neg_{\mathrm{J}}\, p$	$\sim p$	$\lceil p$	$\rceil p$	$\Diamond p$
0	1	1	1	1	0
½	0	½	1	0	1
1	½	0	0	0	1

In the above mentioned paper, Jaśkowski (with a reference to J. Słupecki) gives the first example of a matrix three-valued paraconsistent logic with the following operations: $\to_{\mathrm{J}}$ and $\neg_{\mathrm{J}}$. But the thesis

$$p \to (\neg\, p \to (\neg\,\neg\, p \to q))\,, \tag{Łuk}$$

which was already known to J. Łukasiewicz, holds in this logic. This was the reason for Jaśkowski to reject this logic.

It is really surprising that Jaśkowski did not take as negation the involution $\sim$ from Łukasiewicz's three-valued logic $\mathbf{Ł_3}$ with initial operations $\{\to_{\mathrm{L}}, \sim\}$ [17]. The most famous three-valued paraconsistent logic which was constructed independently in many works is the one with $\to_{\mathrm{J}}$, $\sim$, and $\vee$ as max, $\wedge$ as min (see [24], [4], [7, p. 214], [21][1]). Let us denote this logic by $\mathbf{A_I}$.

[1] We have also a first-order paraconsistent logic introduced by N. C. A. da Costa in 1964. Cf. also Rozonoer's [21].

Now let us consider other three-valued paraconsistent logics. B. Sobociński [23] axiomatized three-valued matrix logic with operations $\to_{\mathrm{S}}$ and $\sim$. It turns out that this logic ($\mathbf{S_1}$) is the implication-negation fragment of **RM** [19]. In [9] we have a full axiomatization of the three-valued case of **RM**, namely, **RM3**.

The situation in as follows: to relevant logic **R** [1] the two following axioms are added

$$(\neg A \wedge B) \to (A \to B),$$
$$A \vee (A \to B).$$

A. Avron [5] proved that $\mathbf{A_1}$ and **RM3** are identical (see also [6]):

$$p \to_{\mathrm{S}} q = (p \to_{\mathrm{J}} q) \wedge {\sim} q \to_{\mathrm{J}} {\sim} p),$$
$$p \to_{\mathrm{J}} q = q \vee (p \to_{\mathrm{S}} q).$$

D. Batens [7, p. 201] considered another three-valued paraconsistent logic: Heyting's three-valued implication $\to_{\mathrm{H}}$ with involution $\sim$. But Batens rejects this logic (let us denote it by $\mathbf{B_1}$) because adding disjunction to it yields several unpleasant consequences.

Note that the implication of $\mathbf{S_1}$ is relevant, the implication of $\mathbf{B_1}$ intuitionistic, whereas the implication of $\mathbf{A_1}$ classical.

Now I want to attract readers attention to a different famous three-valued paraconsistent logic, namely $\mathbf{P_1}$ [17] with operations $\to_{\mathrm{Se}}$ and $\lceil$. Here operations $\vee$ and $\wedge$ are defined by means of $\to_{\mathrm{Se}}$ and $\lceil$, where $p \vee q$ is not max, $p \wedge q$ is not min. For the first time truth-tables for these operations appeared in [10], where they were used for the refutation of some tautologues of $\mathbf{C_2}$ which are invalid in the paraconsistent logic $\mathbf{C_1}$ of N. C. A. da Costa. See also [11], where $\mathbf{P_1}$ was called as **F**. The logic $\mathbf{P_1}$ was also independently found by C. Mortensen in 1979, who called it $\mathbf{C_{0,1}}$ (see [18, p. 299]). See also A. Arruda's system **V1** in [2] and in [25].

Only in 1997 E. K. Vojshvillo and J-Y. Béziau [8] discovered independently that in $\mathbf{P_1}$ from $\lceil A$ and $\lceil \lceil A$ follows B. So, $\mathbf{P_1}$ contains the formula (Łuk). About unusual properties of $\mathbf{P_1}$ see [15].

Let us note that, if in the full $\mathbf{P_1}$ the operation $\lceil$ is replaced by the operation $\sim$ then we have Mortensen's paraconsistent logic $\mathbf{C_{0,2}}$ [18] which is a generalization of da Costa's logic $\mathbf{C_1}$.

Now we consider the following two three-valued paraconsistent logics: Priest's logic **LP** [20], and D'Ottaviano's logic $\mathbf{J_3}$ [12]. The first is Kleene's three-valued logic $\{\to_{\mathrm{K}}, \sim, \vee, \wedge\}$ [16] with two designated truth-values.

F. Asenjo [3] was the first to propose this logic. It is well known that such logic verifies all tautologies of classical propositional logic $\mathbf{C_2}$. So we have there the law of noncontradiction and the law $p \to (\neg\, p \to q)$. But G. Priest defines a relation of logical consequence such that B does not follow from $\{A, \neg\, A\}$, and as a consequence *modus ponens* is invalid. The second is the logic $\mathbf{A_1}$ with the extra connective $\diamond$. The functional properties $\mathbf{J_3}$ are the same as those of Łukasiewicz's three-valued logic $\{\to_{\mathrm{L}}, \sim\}$ [17], but with the two designated truth-values. D'Ottaviano suggests two axiomatizations of $\mathbf{J_3}$ and one of them is rather unusual: it is an extension of from $\mathbf{C_2}$ with the operations $\to_{\mathrm{J}}$, $\rceil$, $\wedge$, $\sim$ (see especially in [13, ch. IX]). So we once more have the law of noncontradiction and the law $p \to (\neg\, p \to q)$. Then the question arises, why do we criticize these laws?

At last, we can suggest four new three-valued paraconsistent logics: $\{\to_{\mathrm{J}}, \lceil\}$, $\{\to_{\mathrm{S}}, \lceil\}$, $\{\to_{\mathrm{H}}, \lceil\}$, and $\{\to_{\mathrm{L}}, \lceil\}$. But all these logics as well as $\mathbf{P_1}$ verify the formula (Łuk).

In connection with the formula (Łuk) the problem arises of making more precise the notion of paraconsistent logic. In a usual way, a logic is paraconsistent iff from A and $\neg\, A$ does not follow an arbitrary B. Now D. Batens suggests to restrict this notion: A logic with the formula (Łuk) is not strictly paraconsistent, i.e., for some A: B is derivable from A and $\neg\, A$.

Incidentally, E. K. Vojshvillo suggests the following generalization of the notion of paraconsistency: A logic is paraconsistent, if it does not contain a finite set of formulas from which an arbitrary formula B is derivable.

We still have another problem. Although Johanson's minimal logic is paraconsistent in the usual sense, it verifies the formula $p \to (\neg\, p \to \neg\, q)$. (Jaśkowski pointed out that Kolmogorov's logic has the same properties [14, p. 146]). For details, see [8], where new definitions of paraconsistent logic are given.

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